Covariant Self-Force Regularization of a Particle Orbiting a Schwarzschild Black Hole

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The covariant structure of the self-force of a particle in a general curved background has been elucidated in the cases of scalar, electromagnetic, and gravitational charges. Specifically, what we need is the part of the self-field that is non-vanishing off and within the past light cone of the particle’s location, the so-called tail. The radiation reaction force in the absence of external fields is entirely contained in the tail. In this paper, we develop the mathematical tools necessary for the regularization and propose a practical method to calculate the self-force of a particle orbiting a Schwarzschild black hole.

§1. Introduction

For a particle possessing a scalar,1) electromagnetic,2) or gravitational charge,3) 4) the field configuration of the corresponding type varies in time as it moves around a black hole. To the lowest order in the charge, the particle motion follows a geodesic in the black hole background in the absence of external force fields. However, one part of the time-varying field becomes radiation near the future null infinity or future horizon and carries energy-momentum away from the system, and another part of it is scattered by the background curvature and comes back to the location of the particle. In this manner, the motion of the particle is affected at the next order. The force exerted by the back-scattered self-field is called the “local reaction force” or simply the “self-force”. To establish a calculational strategy for this force is our ultimate goal.

It is noted that we can consider the local reaction force to consist of two parts: the part that describes the loss of the energy-momentum of the particle and the part that contributes only to the shift of conserved quantities. In the case of geodesic motion in the Schwarzschild background, this division is unambiguous, because the geodesic motion is completely determined by the two conserved quantities, the energy and the z-component of the angular momentum. (A geodesic can be assumed to be on the equatorial plane without loss of generality.) In the case of the Kerr background, however, it is unclear if these two parts can be identified uniquely, because of the

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presence of a conserved quantity called the Carter constant, which has no explicit relation to the energy-momentum of the system.

When we attempt to calculate the reaction force on a point charge (particle), we encounter the problem that the force diverges. Hence, it is necessary to extract the physically meaningful finite part of the force. Because the force is a vector by definition, with respect to a background space-time, and because any vector depends on the choice of coordinates in a covariant manner, the finite reaction force should be given covariantly.

The covariant structure of the reaction force is investigated in the scalar case in Ref. 1), in the electromagnetic case in Ref. 2), and in the gravitational case in Refs. 3) and 4). In these investigations, it was found that the divergent part of the force can be described solely in terms of local geometrical quantities, whereas the finite part that contributes to the equation of motion was found to be given by the tail part, which is due to the curvature scattering of the self-field.

Because the tail part depends non-locally on the geometry of the background space-time, it is almost impossible to calculate it directly. However, for a certain class of space-times, such as Schwarzschild/Kerr geometries, there is a way to calculate the full field generated by a point charge. Considering a field point slightly off the particle trajectory, it is then possible to obtain the tail part by subtracting the locally given divergent part from the full field. Thus, denoting the field by $s\phi$ for the scalar ($s = 0$), electromagnetic ($s = 1$) and gravitational ($s = 2$) cases, with space-time indices suppressed, the reaction force is schematically given by

$$F_\alpha(\tau_0) = \lim_{x \to z(\tau_0)} F_\alpha[s\phi^{\text{tail}}](x),$$

$$F_\alpha[s\phi^{\text{tail}}](x) = F_\alpha[s\phi^{\text{full}}](x) - F_\alpha[s\phi^{\text{dir}}](x), \quad [x \neq z(\tau)]$$

where $z$ is the orbit of the particle with proper time $\tau$, and $\tau_0$ is the proper time at the orbital point at which we calculate the force. The expression $s\phi^{\text{tail}}$ represents the tail field induced by the particle, which is regular in the coincidence limit $x \to z(\tau)$, $s\phi^{\text{full}}$ for the full field and $s\phi^{\text{dir}}$ for the direct part, as defined in Refs. 1) – 4). Both $s\phi^{\text{full}}$ and $s\phi^{\text{dir}}$ diverge in the coincidence limit, $x \to z(\tau)$. The quantity $F_\alpha[...]$ is a tensor operator on the field and is defined as

$$F_\alpha[s\phi] = \begin{cases} 
qP_{\alpha}^{\beta}\nabla_\beta\phi, & (s = 0) \\
epP_{\alpha}^{\beta}(\phi_{\gamma;\beta} - \phi_{\beta;\gamma})V^\gamma, & (s = 1) \\
-mP_{\alpha}^{\beta}(\phi_{\beta;\gamma} - \frac{1}{2}g_{\beta\gamma}\phi_{\epsilon;\delta})V^\gamma V^\delta & (s = 2) 
\end{cases}$$

where $P_{\alpha}^{\beta} = \delta_{\alpha}^{\beta} + V_{\alpha}V^\beta$ is the projection tensor, with $V^\alpha$ being an appropriate extension of the four velocity $v^\alpha(\tau_0)$ off the orbital point.

In practice, it is a non-trivial task to carry out the subtraction of the direct part, which we call the subtraction problem. In this paper we propose a method to carry out this subtraction procedure covariantly.

It should be noted, however, that solving the subtraction problem is not enough when treating the gravitational case. In the scalar or electromagnetic case, the
reaction force is a gauge-invariant quantity. In contrast, the reaction force in the gravitational case does depend on the gauge choice. Therefore, one has to fix the gauge appropriately and evaluate the full metric perturbation and its direct part in the same gauge before calculating the force. We call this the \textit{gauge problem}, which seems to be very difficult. We do not discuss the possible solution of the gauge problem in this paper, leaving it for a future work. Instead, we only calculate the direct part of the linear gravitational force under the harmonic gauge condition. We should emphasize that our calculation is based on the local, covariant expression of the direct part. This contrasts with the case of the method employed by Barack and Ori,\textsuperscript{15),16) in which the extraction of the direct part is performed after the harmonic expansion of the Green function, and hence the covariant structure is not manifest.

In this paper, as a first step, we consider the case in which the background can be approximated by a Schwarzschild black hole and use the Boyer-Lindquist coordinates,

\begin{equation}
    ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\end{equation}

We use notation in which \(x = \{t, r, \theta, \phi\}\) represents a field point and \(z(\tau_0) = z_0 = \{t_0, r_0, \theta, \phi\}\) an orbital point.

The paper is organized as follows. In \S\,2, we describe basics of the problem we consider. In \S\,3, we describe our strategy for the regularization, which we call the \textit{mode decomposition regularization}. In \S\,4, focusing on the scalar case, we present our method for the mode-decomposition of the direct part of the self-force. In doing so, we obtain a useful mathematical formula, Eq. (4.31), for the harmonic decomposition. In \S\,5, summing over the azimuthal modes \(m\), we compare our result with that of Barack and Ori,\textsuperscript{15),16) who took a quite different approach to obtain the direct part. We find that the results are identical. We conclude that the two methods are equally powerful on the Schwarzschild background. In our opinion, however, our method has the advantage that it can be easily generalized to the Kerr background which must be used when treating orbits off the equatorial plane. In \S\,6, we discuss the strengths and weaknesses of our method and conclude the paper by pointing out some issues to be considered in the future. Technical details and the mode decomposition regularization of the electromagnetic and gravitational cases are treated in the appendices.

\section{Basics}

There are two important problems involved in the regularization. One is determining how to implement a regularization method in Eq. (1.1). Both \(s_{\phi}^{\text{full}}(x)\) and \(s_{\phi}^{\text{dir}}(x)\) diverge in the coincidence limit \(x \to z_0\), and such divergent quantities are difficult to treat in actual calculations, particularly in numerical computations. We discuss this problem in \S\,2.1. The other problem is determining how to evaluate the direct part of the field, which we discuss in \S\,2.2.
2.1. Infinite series expansion

We consider the case of a particle in a geodesic orbit in the Schwarzschild space-time. Even in the Newtonian limit, the integration of the orbit involves an elliptic function. For this reason, numerical computations are necessary at some stage of deriving the self-force. However, there is a divergence to be regularized, and this is difficult to do numerically. The idea for overcoming this difficulty is to replace the divergence by an infinite series, each term of which is finite.

Suppose we have a unique decomposition method applicable to $F_\alpha[s\phi^{\text{full}}](x)$, $F_\alpha[s\phi^{\text{dir}}](x)$ and $F_\alpha[s\phi^{\text{tail}}](x)$ as

$$F_\alpha[s\phi^{\text{full}}](x) = \sum_A F_\alpha^A[s\phi^{\text{full}}](x),$$  \hspace{1cm} (2.1)

$$F_\alpha[s\phi^{\text{dir}}](x) = \sum_A F_\alpha^A[s\phi^{\text{dir}}](x),$$  \hspace{1cm} (2.2)

$$F_\alpha[s\phi^{\text{tail}}](x) = \sum_A F_\alpha^A[s\phi^{\text{tail}}](x).$$  \hspace{1cm} (2.3)

Because of the uniqueness of the decomposition, we have

$$F_\alpha^A[s\phi^{\text{tail}}](x) = F_\alpha^A[s\phi^{\text{full}}](x) - F_\alpha^A[s\phi^{\text{dir}}](x).$$  \hspace{1cm} (2.4)

At this stage, we assume that each term of the infinite series is finite. Then it is possible to take the coincidence limit $x \to z_0$, because $F_\alpha^A[s\phi^{\text{tail}}](x)$ is guaranteed to be finite. Therefore we have

$$F_\alpha(\tau_0) = \sum_A F_\alpha^A[s\phi^{\text{tail}}](z_0).$$  \hspace{1cm} (2.5)

This approach itself does not justify the use of a numerical method in the regularization calculation. However, because of the convergence of the infinite sum (2.5), we expect that the sum of a finite number of terms in (2.5) gives an approximate value of the self-force $F_\alpha(\tau_0)$.

However, there is a very delicate problem encountered in this approach. The exact decomposition calculation usually requires global analytic structure of the field, so that we can uniquely define each term in the infinite series. On the other hand, the regularization scheme is derived through just the local analysis of the field.\(^1\)\(^-\)\(^4\) Thus, the direct part is defined only in the local neighborhood of the particle, and we have an ambiguity in the definition of the direct part. Because of this ambiguity, each term in the infinite series expansion is no longer unique but depends on the global extension of the direct part we adopt. Nevertheless, the final result for the self-force should be unique.

In this paper, we present a decomposition method based on the spherical harmonic series expansion. Although we have no explicit proof of the uniqueness of the resulting regularization counterterms for the self-force, the fact that our result is identical to that of Barack and Ori\(^15\),\(^16\) strongly supports the validity of both methods.
2.2. Direct part of the scalar field

The derivation of the direct part $\phi^{\text{dir}}(x)$ is one of the main steps in the regularization calculation. The direct part of the scalar field is obtained by integrating the direct part of the retarded Green function with the source charge. Here, we focus on the scalar case. The electromagnetic and gravitational cases are treated in the same manner. The details of this treatment are given in Appendix B.

The direct part of the retarded Green function $G^{\text{dir}}$ is given in a covariant manner as

$$G^{\text{dir}}(x, x') = -\frac{1}{4\pi} \theta[\Sigma(x), x'] \sqrt{\Delta(x, x')} \delta\left(\sigma(x, x')\right),$$

where $\sigma(x, x')$ is the bi-scalar of half the squared geodesic distance, $\Delta(x, x')$ is the generalized van Vleck-Morette determinant, $\Sigma(x)$ is an arbitrary space-like hypersurface containing $x$, and $\theta[\Sigma(x), x'] = 1 - \theta[x', \Sigma(x)]$ is equal to 1 when $x'$ lies in the past of $\Sigma(x)$ and vanishes when $x'$ lies in the future. We summarize the basic properties of the bi-scalars $\sigma(x, x')$ and $\Delta(x, x')$ in Appendix A.

The physical meaning of the direct part can be understood by considering the factor $\theta[\Sigma(x), x'] \delta(\sigma(x, x'))$ in Eq. (2.6). Because $\sigma(x, x')$ describes the geodesic distance between $x$ and $x'$, the direct part of the Green function becomes non-zero only when $x'$ lies on the past lightcone of $x$. Hence the direct part describes the effect of the waves propagating directly from $x'$ to $x$ without scattered by the background curvature.

For the actual evaluation of the direct part, several methods have been proposed. In Refs. 7) and 9), the direct part of the field is calculated by picking up the limiting contribution in the full Green function from the light cone as

$$\phi^{\text{dir}}(x) = \lim_{\epsilon \to +0} \int_{\tau_{\text{ret}}(x) - \epsilon}^{\infty} d\tau G^{\text{full}}(x, z(\tau)) S(\tau),$$

where $G^{\text{full}}$ is the retarded Green function, $S(\tau)$ is the scalar charge density, and $\tau_{\text{ret}}(x)$ is the retarded time defined by the past lightcone condition of the field point $x$ as

$$\theta[\Sigma(x), z(\tau_{\text{ret}})] \delta(\sigma(x, z(\tau_{\text{ret}}))) = 0.$$  

A number of works have been made using this approach. 7), 9) However, the calculation becomes rather cumbersome when we apply this method to a general orbit.

In Ref. 6), the direct part is evaluated using the local bi-tensor expansion technique. Using the bi-tensor, the direct part is expanded around the particle location as

$$\phi^{\text{dir}}(x) = q \left[ \frac{1}{\sigma_{\alpha}(x, z(\tau_{\text{ret}})) v^\alpha(\tau_{\text{ret}})} \right] + O(y^2),$$

where $\mu, \nu, \ldots$ are the indices of the field point $x$, $\alpha, \beta, \ldots$ are the indices of the orbital point $z$, and $v^\alpha(\tau)$ is the orbital four velocity at $z(\tau)$. The order of
the local expansion is represented by powers of $y$, which is linear in the coordinate difference between the field point $x$ and the orbital point $z_0$. Because the full force is quadratically divergent, we must carry out a local bi-tensor expansion of the full field through $O(y)$.

By evaluating the local coordinate values of the relevant bi-tensors, we obtain the local expansion of the full force in a given coordinate system (see Appendix A). As described in Ref. 6), this can be done in a systematic manner, and it is possible to obtain the explicit form of the divergence for a general orbit. However, the problem is determining how to decompose it into an appropriate infinite series. This is done in §4.

§3. Mode decomposition regularization

We call the regularization calculation using the spherical harmonic expansion the “mode decomposition regularization”. In this section, we briefly describe the regularization procedure in this approach.

The harmonic decomposition is defined by the analytic structure of the field on a two-sphere. However, because both the direct field and the full field diverge on a sphere that include the particle location, the mode decomposition is ill-defined on it. Therefore, we perform a harmonic decomposition of the direct and full fields on a sphere that does not include the orbit, but is sufficiently close to it. The steps in the mode decomposition regularization are as follows.

1) We evaluate both the full field and the direct field at

$$x = \{t, r, \theta, \phi\},$$

where we do not take the coincidence limit of either $t$ or $r$.

2) We decompose the full force and direct force into infinite harmonic series as

$$F_\alpha[s_\phi^{full}](x) = \sum_{\ell m} F_\alpha^{\ell m}[s_\phi^{full}](x),$$

$$F_\alpha[s_\phi^{dir}](x) = \sum_{\ell m} F_\alpha^{\ell m}[s_\phi^{dir}](x),$$

where the functions $F_\alpha[s_\phi^{full/dir}](x)$ are expanded in terms of the spherical harmonics $Y_{\ell m}(\theta, \phi)$, with coefficients dependent on $t$ and $r$.\(^\ast\) For the direct part, the harmonic expansion is carried out by extending the locally defined direct force to the entire two-sphere in a way that correctly reproduces the divergent behavior around the orbital point $z_0$, up to finite term.

3) We subtract the direct part from the full part in each $\ell, m$ mode, obtaining

$$F^{\ell m}_\alpha[s_\phi^{tail}] = (F^{\ell m}_\alpha[s_\phi^{full}] - F^{\ell m}_\alpha[s_\phi^{dir}]).$$

\(^\ast\) In the rigorous treatment, the angular components of the force are expanded as $F_\alpha = C_{\ell m}Y_{A,\ell m}$, where $Y_{A,\ell m} (A = \theta, \phi)$ are the vector spherical harmonics. We also note that $F_\alpha[s_\phi^{\ell m}](x)$ and $F_\alpha^{\ell m}[s_\phi](x)$ are different, because the tensorial property of the operator $F_\alpha[...]$ depends on the spin $s$ of the field $s_\phi$. 

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The text is a continuation of the discussion on the covariant self-force regularization of a particle in a gravitational field, focusing on the mode decomposition regularization technique. The authors describe how to represent the local expansion of the full force using powers of $y$, which is linear in the coordinate difference between the field point $x$ and the orbital point $z_0$. Since the full force diverges quadratically, a local bi-tensor expansion is necessary. The local coordinate values of the relevant bi-tensors are evaluated to obtain the local expansion of the full force in a given coordinate system. This can be done systematically, allowing for the explicit form of the divergence to be obtained for a general orbit. However, the challenge lies in decomposing the divergence into an appropriate infinite series, which is addressed in §4.

The §3. Mode decomposition regularization section introduces the mode decomposition regularization calculation using the spherical harmonic expansion. The authors define the harmonic decomposition in terms of the analytic structure of the field on a two-sphere. However, since both the direct field and the full field diverge on a sphere including the particle location, the mode decomposition is ill-defined on it. Therefore, the authors perform a harmonic decomposition of the direct and full fields on a sphere that does not include the orbit but is sufficiently close to it. The steps in the mode decomposition regularization are as follows:

1. Evaluate both the full field and the direct field at $x = \{t, r, \theta, \phi\}$, without taking the coincidence limit of either $t$ or $r$.

2. Decompose the full force and direct force into infinite harmonic series as

$$F_\alpha[s_\phi^{full}](x) = \sum_{\ell m} F_\alpha^{\ell m}[s_\phi^{full}](x),$$

$$F_\alpha[s_\phi^{dir}](x) = \sum_{\ell m} F_\alpha^{\ell m}[s_\phi^{dir}](x),$$

where the functions $F_\alpha[s_\phi^{full/dir}](x)$ are expanded in terms of the spherical harmonics $Y_{\ell m}(\theta, \phi)$, with coefficients dependent on $t$ and $r$.\(^\ast\) For the direct part, the harmonic expansion is carried out by extending the locally defined direct force to the entire two-sphere in a way that correctly reproduces the divergent behavior around the orbital point $z_0$, up to finite term.

3. Subtract the direct part from the full part in each $\ell, m$ mode, obtaining

$$F^{\ell m}_\alpha[s_\phi^{tail}] = (F^{\ell m}_\alpha[s_\phi^{full}] - F^{\ell m}_\alpha[s_\phi^{dir}]).$$

\(^\ast\) In the rigorous treatment, the angular components of the force are expanded as $F_\alpha = C_{\ell m}Y_{A,\ell m}$, where $Y_{A,\ell m} (A = \theta, \phi)$ are the vector spherical harmonics. We also note that $F_\alpha[s_\phi^{\ell m}](x)$ and $F_\alpha^{\ell m}[s_\phi](x)$ are different, because the tensorial property of the operator $F_\alpha[...]$ depends on the spin $s$ of the field $s_\phi$. 

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The text concludes with a detailed explanation of the mode decomposition regularization technique, emphasizing the importance of decomposing the divergence into an appropriate infinite series. The authors discuss the steps involved in this process, which include evaluating the local field at the particle's location and decomposing both the full and direct forces into harmonic series. They then subtract the direct part from the full part to obtain the tail part, highlighting the differences in the representation of the full and direct forces. The method is crucial for understanding the behavior of a particle in a gravitational field, particularly in the context of self-force regularization.
Then, we take the coincidence limit, \( x \to z_0 \). Here we note that the order of the procedures can be reversed, i.e., taking the coincidence limit first and then subtracting, provided that the mode coefficients are finite in the coincidence limit.

4) Finally, taking the sum over the modes, we obtain the self-force as

\[
F_\alpha(\tau_0) = \sum_{\ell m} F^\ell_m [s_{\phi_{\text{tail}}}](z_0). \tag{3.5}
\]

It should be noted that because of the divergence of the full force and direct force along a time-like orbit, the mode coefficients of the full force and the direct force are not uniquely defined when we take the coincidence limit in 3). However, the tail force is regular along the orbit, \(^1\)–\(^4\) and it is uniquely defined. Therefore, we believe that the non-uniqueness of the direct force does not cause a problem as long as the coincidence limit is taken consistently for the full force and the direct force.

§4. Decomposition of the direct part

The advantage of using (2.9) is that we have a systematic method for evaluating the direct part, which we describe in §4.1. In §4.2, we describe our method for the harmonic decomposition of the direct part.

4.1. Local coordinate expansion

Though we have a covariant form of the local bi-tensor expansion of the direct part, it is not useful for the derivation of its infinite series expansion until we evaluate it in a specific coordinate system. Here we discuss the method to evaluate the bi-tensors in a general regular coordinate system.

Before considering the local expansion in a given coordinate system, we calculate the derivative of (2.9) and derive the direct part of the force with the local bi-tensor expansion using the equal-time condition,\(^*\)

\[
0 = \left[ d \sigma(x, z(\tau)) \right]_{\tau = \tau_{\text{eq}}(x)}. \tag{4.1}
\]

We define the extension of the four-velocity off the orbit by

\[
V^\alpha(x) := \bar{g}_{\alpha\bar{\alpha}}(x, z_{\text{eq}}) v^\alpha_{\text{eq}}, \tag{4.2}
\]

where \( \bar{g}_{\alpha\bar{\alpha}} \) is the parallel displacement bi-vector, \( z_{\text{eq}} = z(\tau_{\text{eq}}(x)) \) and \( v^\alpha_{\text{eq}} = dz^{\bar{\alpha}}/d\tau |_{\tau = \tau_{\text{eq}}(x)} \). Using the formulas in Refs. 2) and 3), we have

\[
F_\alpha[\phi_{\text{dir}}](x) = q \bar{g}_{\alpha\bar{\alpha}}(x, z_{\text{eq}}) \frac{1}{e^3 \kappa}.
\]

\(^*\) Actually, the equal-time condition is not essential for the evaluation of the direct part. However, the dependence on the spin of the field can be seen clearly under the equal-time condition, as described in Appendix B.
\[ \times \left\{ \sigma;\bar{\alpha}(x, z_{eq}) + \frac{1}{3} \epsilon^2 R_{\bar{\alpha}\bar{\beta}\gamma\bar{\delta}}(z_{eq}) \nu^{\bar{\beta}}_{eq} \sigma^{;\gamma}(x, z_{eq}) \nu^{\bar{\delta}}_{eq} \right\} + O(y), \quad (4.3) \]

\[ \epsilon = \sqrt{2\sigma(x, z_{eq})}, \quad (4.4) \]

\[ \kappa = \sqrt{-\sigma_{\bar{\alpha}\bar{\beta}}(x, z_{eq}) \nu^{\bar{\alpha}}_{eq} \nu^{\bar{\beta}}_{eq}} \]

\[ = 1 + \frac{1}{6} R_{\bar{\alpha}\bar{\beta}\gamma\bar{\delta}}(z_{eq}) \nu^{\bar{\alpha}}_{eq} \sigma^{;\gamma}(x, z_{eq}) \nu^{\bar{\delta}}_{eq} \sigma^{;\bar{\delta}}(x, z_{eq}) + O(y^3). \quad (4.5) \]

The bi-tensors necessary for the evaluation of the direct force (4.3) are \( \sigma(x, \bar{x}) \)
and \( g_{a\bar{\alpha}}(x, \bar{x}) \), which satisfy

\[ \sigma(x, \bar{x}) = \frac{1}{2} g^{\alpha\beta} \sigma_{\alpha\beta}(x, \bar{x}) = \frac{1}{2} g^{\alpha\bar{\beta}} \sigma_{\bar{\alpha}\bar{\beta}}(x, \bar{x}), \quad (4.6) \]

\[ \lim_{x \rightarrow \bar{x}} \sigma_{\alpha\beta}(x, \bar{x}) = \lim_{x \rightarrow \bar{x}} \sigma_{\bar{\alpha}\bar{\beta}}(x, \bar{x}) = 0, \quad (4.7) \]

\[ \bar{g}_{a\bar{\alpha};\beta}(x, \bar{x}) g^{\beta\gamma}(x) \sigma_{\gamma\bar{\beta}}(x, \bar{x}) = 0, \quad (4.8) \]

\[ \bar{g}_{a\bar{\alpha};\beta}(x, \bar{x}) g^{\beta\bar{\gamma}}(\bar{x}) \sigma_{\bar{\gamma}\bar{\beta}}(x, \bar{x}) = 0, \quad (4.9) \]

\[ \lim_{x \rightarrow \bar{x}} \bar{g}_{a\bar{\alpha}} = \delta_{a\bar{\alpha}}. \quad (4.10) \]

In addition, we need the generalized van Vleck-Morette determinant,

\[ \Delta(x, \bar{x}) = \det\left(-\bar{g}^{a\bar{\alpha}}(x, \bar{x}) \sigma_{a\bar{\alpha}}(x, \bar{x})\right). \quad (4.11) \]

We consider the local coordinate expansion of these bi-tensors around the coincidence limit \( x \rightarrow \bar{x} \), assuming that we have no coordinate singularity at \( \bar{x} \).

In the coincidence limit, the effect of the curvature is small, and we know the exact forms of half the geodesic distance bi-scalar and the parallel displacement bi-vector in the locally Cartesian coordinates to be

\[ \sigma(x, \bar{x}) = \frac{1}{2} \eta_{a\beta}(x^a - \bar{x}^a)(x^\beta - \bar{x}^\beta) + O(|x - \bar{x}|^3), \]

\[ \bar{g}_{a\bar{\alpha}}(x, \bar{x}) = \eta_{a\bar{\alpha}} + O(|x - \bar{x}|). \quad (4.12) \]

Therefore, in a general regular coordinate system, \( \sigma(x, \bar{x}) \) and \( \bar{g}_{a\bar{\alpha}}(x, \bar{x}) \) can be expanded as

\[ \sigma(x, \bar{x}) = \frac{1}{2} g_{a\beta}(\bar{x}) y^{a\beta} + \sum_{n=3,4,...} \frac{1}{n!} A_{a\alpha_1 \alpha_2 ... \alpha_n}(\bar{x}) y^{a_1 \alpha_2 ... \alpha_n}, \quad (4.13) \]

\[ \bar{g}_{a\bar{\alpha}}(x, \bar{x}) = g_{a\bar{\alpha}}(\bar{x}) + \sum_{n=1,2,...} \frac{1}{n!} B_{a\alpha_1 \beta_1 \beta_2 ... \beta_n}(\bar{x}) y^{\alpha_1 \beta_1 \beta_2 ... \beta_n}, \quad (4.14) \]

where

\[ y^{a_1 \alpha_2 ...} = (x^a - \bar{x}^a)(x^{\alpha_2 - \bar{x}^{\alpha_2}}) ... \cdot \quad (4.15) \]

To calculate the reaction force to a monopole particle, it is enough to know the expansion coefficients for \( n = 3 \) and \( 4 \) in Eq. (4.13) and \( n = 1 \) and \( 2 \) in Eq. (4.14).*

* In the calculation of the direct force given below, not only Eq. (4.13) but also Eq. (4.14) turns out to be necessary. This is a result of our choice of the off-world line extension of the direct force, i.e., the parallel-propagation extension (4.2).
For a general metric, from (4.6) and (4.9), we have
\[ A_{\alpha\beta\gamma} = \frac{3}{2} g_{(\alpha\beta,\gamma)} , \]  
(4.16)
\[ A_{\alpha\beta\gamma\delta} = 2 g_{(\alpha\beta,\gamma\delta)} - g_{\mu\nu} \Gamma^\mu_{(\alpha\beta} \Gamma^\nu_{\gamma\delta)} , \]  
(4.17)
\[ B_{\alpha\beta|\gamma} = \Gamma_{\beta|\alpha\gamma} , \]  
(4.18)
\[ B_{\alpha\beta|\gamma\delta} = \frac{1}{2} \left( \Gamma_{\beta|\alpha\gamma,\delta} + \Gamma_{\beta|\alpha\delta,\gamma} - g_{\mu\nu} \Gamma^\mu_{(\alpha\gamma} \Gamma^\nu_{\beta\delta)} - g_{\mu\nu} \Gamma^\mu_{(\alpha\delta} \Gamma^\nu_{\beta\gamma)} \right) . \]  
(4.19)

The explicit evaluation of these coefficients in the Boyer-Lindquist coordinates for the Schwarzschild metric is given in Appendix A.

The local expansion of the force (4.3) in the Boyer-Lindquist coordinates is quite tedious, though systematic. We carried out this calculation using Maple(R), an algebraic calculation program, and obtained 100 pages output. However, most of the terms yield a vanishing contribution to the harmonic coefficients in the coincidence limit \( x \rightarrow z_0 \). Below, we focus on the terms that are non-vanishing in this limit.

4.2. Harmonic decomposition of the direct part

Without loss of generality, we may assume that the particle is located at \((\theta_0, \phi_0) = (\pi/2, 0)\) at time \( t_0 \). Because the full force is calculated in the form of a Fourier-harmonic expansion, and the Fourier modes are independent of the spherical harmonics, we can take the field point to lie on the hypersurface \( t = t_0 \) in the full force. Hence we can set \( t = t_0 \) before we perform the local coordinate expansion of the direct force. That is, we consider the local coordinate expansion of the direct force at a point \( \{t_0, r, \theta, \phi\} \) near the particle position \( \{t_0, r_0, \pi/2, 0\} \).

The local expansion of the direct force in the Boyer-Lindquist coordinates can be carried out in such a way that it consists of terms of the form
\[ \frac{R^{n_1} \Theta^{n_2} \phi^{n_3}}{\xi^{2n_4+1}} , \]  
(4.20)
where \( n_1, n_2, n_3 \) and \( n_4 \) are non-negative integers, and
\[ \xi := \sqrt{2} r_0 \left( a - \cos \tilde{\theta} + \frac{b}{2} (\phi - \phi')^2 \right)^{1/2} , \]  
(4.21)
\[ R := r - r_0 , \quad \Theta := \theta - \frac{\pi}{2} , \]  
(4.22)
with \( a, b \) and \( \phi' \) defined by
\[ a := 1 + \frac{1}{2} \frac{r_0^2}{r_0^2} \frac{r_0^2}{r_0^2 + L^2} \frac{r_0^2}{r_0^2 + L^2} \frac{L^2}{r_0^2} \frac{r_0^2}{r_0^2 + L^2} \frac{E^2 R^2}{r_0^2} , \]  
(4.23)
\[ b := \frac{L^2}{r_0^2} , \]  
(4.24)
\[ \phi' := - \frac{L}{r_0^2 + L^2} u_r R , \]  
(4.25)

where \( E := -g_{tt} dt/d\tau, \quad L := g_{\phi\phi} d\phi/d\tau \), and \( u_r := g_{rr} dr/d\tau \), and \( \tilde{\theta} \) is the angle between \((\theta, \phi)\) and \((\pi/2, \phi')\).
There are two apparently different terms in the covariant form of the direct force given by Eq. (4.3): the first term in the curly brackets exhibiting quadratic divergence, and the second term, proportional to the curvature tensor, that appears to be finite in the coincidence limit. In the local coordinate expansion, the second term gives terms of the forms \( \sim R/\xi \) and \( \phi/\xi \). As shown in §5, the harmonic coefficients of \( R/\xi \) vanish in the coincidence limit, while those of \( \phi/\xi \) are finite but give no contribution to the final result when the infinite harmonic modes are summed up after the coincidence limit is taken. Hence, we can focus on the first term. In passing, it is worthwhile to note that because the direct force possesses dependence on the spin of the field only through this second term (see Appendix B), the harmonic coefficients of the direct force, which are subtracted from the full force, are independent of the spin of the field.

Let us focus on the first term in the curly brackets of Eq. (4.3). Because the orbit is always on the equatorial plane, the force is symmetric under the transformation \( \theta \rightarrow \pi - \theta \), which implies that there is no term proportional to odd powers of \( \Theta \). Therefore we only need to consider the case in which \( n_2 \) is an even number in the general form given by Eq. (4.20). Then the factor \( \Theta^n R/\xi \) can be eliminated by expressing \( \Theta^2 \) in terms of \( \xi, R \) and \( \phi \), and we are left with terms of the form

\[
\frac{R^{n_1} \phi^{n_3}}{\xi^{2n_4+1}}.
\] (4.26)

Explicitly, we find

\[
F^\text{dir}_t = q \left( \mathcal{E} u_r \frac{R}{\xi^3} + \mathcal{E} \mathcal{L} \frac{\phi}{\xi^3} \right)
- \frac{1}{2} \left( \frac{r_0^2}{r_0^2} - 2M \right) \mathcal{E} u_r \frac{1}{\xi} + \frac{2}{2} \left( \frac{r_0^2}{r_0^2} - 2M \right) \mathcal{E} \mathcal{L} u_r \frac{\phi^2}{\xi^3}
- \frac{3}{2} \left( \frac{r_0^2}{r_0^2} - 2M \right) \mathcal{E} \mathcal{L} u_r \frac{\phi^4}{\xi^5} \right),
\] (4.27)

\[
F^\text{dir}_r = q \left( \frac{\mathcal{L}^2}{r_0(r_0 - 2M) \xi^3} \frac{R}{\xi^3} - \frac{r_0^2 \mathcal{L}^2}{(r_0 - 2M)^2 \xi^3} \frac{R}{\xi^3} - \mathcal{L} u_r \frac{\phi}{\xi^3} \right)
- \frac{1}{2} \left( \frac{r_0^2}{r_0^2} + \frac{L^2}{r_0^2} \right) \frac{1}{\xi} + \frac{1}{2} \left( \frac{r_0^2}{r_0^2} - 2M \right) \frac{1}{\xi}
+ \frac{1}{2} \left( \frac{3r_0^2}{r_0^2} + 4L^2 \right) \frac{\phi^2}{\xi^3} - \frac{2}{2} \left( \frac{3r_0^2}{r_0^2} + 4L^2 \right) \frac{\phi^4}{\xi^5}
+ \frac{2}{2} \left( \frac{3r_0^2}{r_0^2} + 4L^2 \right) \frac{\phi^4}{\xi^5} \right),
\] (4.28)

\[
F^\text{dir}_\theta = 0.
\] (4.29)

\(^{\text{\ast}}\) This is also a result of the specific off-worldline extension chosen for the four-velocity. It is valid for the parallel-propagation extension, but it does not hold, in general, for other extensions.
\[ F^{\text{dir}}_{\phi} = q \left( -\mathcal{L}_{ur} \frac{R}{\xi^3} - (r_0^2 + \mathcal{L}^2) \phi \frac{1}{\xi^3} + \frac{1}{2} \frac{(r_0 - 2M)\mathcal{L}_{ur}}{r_0^2} \frac{1}{\xi} - \frac{1}{2} \frac{(r_0 - 2M)(r_0^2 + 4\mathcal{L}^2)\mathcal{L}_{ur}}{r_0^2} \frac{\phi^2}{\xi^3} + \frac{3}{2} \frac{(r_0 - 2M)(r_0^2 + \mathcal{L}^2)\mathcal{L}_{ur}}{r_0^2} \frac{\phi^4}{\xi^5} \right), \]  

where \( F^{\text{dir}}_{\alpha} = F_{\alpha}[\phi^{\text{dir}}] \). The absence of \( F^{\text{dir}}_{\theta} \) is due to the symmetry of the background; the orbit remains on the equatorial plane even under the action of the self-force. In the above, we have discarded the terms of the forms \( \sim R/\xi \) and \( \phi/\xi \). As mentioned above, and as shown in §5, such terms give no contribution to the final force.

We now must perform the harmonic decomposition of the components of the direct force given above. To do so, we note the following important fact. Apart from a trivial multiplicative factor of \( R^{n_1} \), which is independent of the spherical coordinates, the terms to be expanded in the spherical harmonics are of the forms \( \sim \phi^{n_3}/\xi^{2n_4+1} \), and \( (\phi - \phi')^{n_3}/\xi^{2n_4+1} \). To the order of accuracy we need (In fact, only the leading order accuracy is necessary, as discussed in Appendix C.), the factor \( (\phi - \phi')^{n_3} \) can be eliminated by replacing it with an equivalent \( \phi \)-derivative operator of degree \( n_3 \) acting on \( \xi^{2n_3-2n_4-1} \), which is further converted into a polynomial in \( m \) after the harmonic expansion of \( \xi^{2n_3-2n_4-1} \). Thus, the only basic formula we need is the harmonic expansion of \( \xi^{2p-1} \), where \( p \) is an integer. Detailed derivation of this expansion is given in Appendix D. Note that, apart from the term \( b(\phi - \phi')^2/2 \) in \( \xi \), with respect to which we expand \( \xi \) in a convergent infinite series, \( \xi^{2p-1} \) is defined over the whole sphere to allow for a straightforward harmonic decomposition. The result to the leading order in the coincidence limit \( a \rightarrow 1 + 0 \) is

\[
\left( \frac{\xi}{\sqrt{2}r_0} \right)^{2p-1} = \left( a - \cos \tilde{\theta} + \frac{b}{2}(\phi - \phi')^2 \right)^{p-1/2}
\]

\[ = 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D^{p-1/2}_{\ell m}(a) Y_{\ell m}(\theta, \phi) Y^{*}_{\ell m}(\theta', \phi'), \]  

\[ D^{p-1/2}_{\ell m}(a) \rightarrow \begin{cases} 
1 & \text{for } p + 1/2 < 0 \\
\frac{1}{\sqrt{1 + b}} \frac{1}{\Gamma(p+1/2)} (a - 1)^{p+1/2} & \text{for } p + 1/2 = 0 \\
\frac{1}{\sqrt{1 + b}} \frac{(-1)^{\ell}2^{p+1/2}}{\Gamma(p+1/2)} & \text{for } p + 1/2 > 0 
\end{cases} \]

\[ \times \sum_{n=0}^{\infty} \frac{\Gamma(p+1/2 + n)}{\Gamma(p+\ell + 1/2 + n)} \frac{1}{\Gamma(p+n+\ell+3/2)} \frac{1}{n!} \left( \frac{-m^2b}{1+b} \right)^n. \]  

We note that, although what we need here is only the results for the case of integer \( p \), the above formula is valid for any \( p \) (except for \( p = -1/2 \)), and the \( n \) summation in \( D^{p-1/2}_{\ell m} \) for \( p + 1/2 > 0 \) does not realize a simple form, but it converges.

After the decomposition, we can take the radial coincidence limit \( r \rightarrow r_0 \) (followed by the angular coincidence limit if desired). The basic properties of the resulting mode coefficients in the coincidence limit are discussed in §5. Here we briefly
Y. Mino, H. Nakano and M. Sasaki explain the reason that the terms proportional to $R/\xi$ and $\phi/\xi$ give no contribution to the final result. The term $\sim R/\xi$ corresponds to $R$ times the term in the case of $p = 0$, for which $D^{p-1/2}_{\ell m}$ is finite in the limit $a \to 1$ (i.e. $R \to 0$). Hence, all of its coefficients vanish in the radial coincidence limit. The term $\sim \phi/\xi$ can be replaced by $(\phi - \phi')/\xi$, which is equivalent to $\partial_\phi \xi$ in the coincidence limit. This corresponds to the term in the case $p = 1$ multiplied by $m$. Hence, all the harmonic coefficients become odd functions of $m$, and their sum over $m$ for each $\ell$ vanishes in the angular coincidence limit. As a result, non-vanishing contributions come only from the terms $R/\xi^3$, $\phi/\xi^3$, $1/\xi$, $\phi^2/\xi^3$ and $\phi^4/\xi^5$.

§5. Regularization counterterms

In this section, we present the mode decomposition of the direct force given by Eqs. (4.27)–(4.30) and compare the resulting regularization counterterms with those obtained by Barack and Ori\textsuperscript{16}) in their mode-sum regularization scheme (MSRS).\textsuperscript{9)}

Barack and Ori define the regularization counterterms as

$$\lim_{x \to z_0} F_{\alpha \ell}^{\text{dir}} = A_\alpha L + B_\alpha + C_\alpha/L + O(L^{-2}), \quad (5.1)$$

$$D_\alpha = \sum_{\ell=0}^{\infty} \left[ \lim_{x \to z_0} F_{\alpha \ell}^{\text{dir}} - A_\alpha L - B_\alpha - C_\alpha/L \right], \quad (5.2)$$

where $F_{\alpha \ell}^{\text{dir}}$ is the multipole $\ell$-mode of $F_{\alpha \ell}^{\text{dir}}$, $L = \ell + 1/2$, and $A_\alpha$, $B_\alpha$ and $C_\alpha$ are independent of $L$. Here, the $A_\alpha$ term is to subtract the quadratic divergence, the $B_\alpha$ term the linear divergence, and the $C_\alpha$ term the logarithmic divergence. The $D_\alpha$ term is the remaining finite contribution of the direct force to be subtracted. As shown in Appendix C, we find $C_\alpha = D_\alpha = 0$, in agreement with the results of Barack and Ori.\textsuperscript{9)} We also find that the $A_\alpha$ and $B_\alpha$ terms are identical with the corresponding terms in their results for a general geodesic orbit,\textsuperscript{15),16} as given below.

The direct part of the force to be considered has the form given by Eq. (4·26), which can be rewritten as

$$\frac{R^{n_1}(\phi - \phi')^{n_3}}{\xi^{2n_4+1}}, \quad (5.3)$$

where $n_1$, $n_3$ and $n_4$ are non-negative integers. Because the highest order of divergence is quadratic, it is sufficient to consider the cases $n_1 + n_3 - 2n_4 = -1$, 0 and 1.\textsuperscript{*}

We first note that

$$\frac{\phi - \phi'}{\xi^{2n_4+1}} = -\frac{1}{2n_4 - 1} \frac{1}{\ell^2} \frac{1}{\ell} \frac{\partial}{\partial \phi} \left( \frac{1}{\xi^{2n_4+1}} \right) + O(y^{-2n_4+2}). \quad (5.4)$$

\textsuperscript{*}Although we may further restrict $n_4$ to the range $0 \leq n_4 \leq 2$ from the explicit form of the direct force in Eqs. (4·27)–(4·30), we choose not to do so, because it turns out to be unnecessary in the following analysis.
Using this equation recursively, we obtain

\[
\frac{(\phi - \phi')^{n_3}}{\xi^{2n_4+1}} \propto \frac{\partial^{n_3}}{\partial \phi^{n_3}} \xi^{2n_3-2n_4-1} + O(y^{n_3-2n_4+1}).
\]

In the context of the harmonic decomposition, we can replace the derivative $\partial/\partial \phi$ by $i m$. Hence, instead of Eq. (5.3), we may consider the terms of the form

\[
m^{n_3} R^{n_1} \xi^{2n_3-2n_4-1} + O(y^{n_1+n_3-2n_4+1}).
\]  

(5.5)

In Eq. (5.5), we have indicated by $O(y^{n_1+n_3-2n_4+1})$ the presence of correction terms of $O(y^2)$ relative to the original form (5.3). In terms of the regularization parameters $A_\alpha$, $B_\alpha$, $C_\alpha$ and $D_\alpha$, this implies that the terms with $n_1 + n_3 - 2n_4 = -1$ contribute to $A_\alpha$ and $C_\alpha$, the terms with $n_1 + n_3 - 2n_4 = 0$ to $B_\alpha$ and $D_\alpha$, and the terms with $n_1 + n_3 - 2n_4 = 1$ to $D_\alpha$. However, as shown in Appendix C, by a general argument, we can show that both $C_\alpha$ and $D_\alpha$ vanish. Therefore we need not consider the $O(y^2)$ corrections in Eq. (5.5) and therefore can focus on its leading behavior in the coincidence limit. Keeping this fact in mind, we now investigate which cases of the form (5.5) contribute to the regularization parameters $A_\alpha$ and $B_\alpha$. For this purpose, we set $n_1 + n_3 - 2n_4 = q$, where $q = -1$, 0 or 1. Then, comparing Eq. (5.5) with Eq. (4.31), we find that it is convenient to separately consider the following two cases.

1. The case $2p = 2n_3 - 2n_4 = n_3 - n_1 + q \leq -2$.

In this case, the harmonic coefficients of Eq. (5.5) behave as

\[
\sim R^{n_1+2n_3-2n_4+1} = R^{n_3+q+1}.
\]

Because $n_3 \geq 0$, the harmonic coefficients are non-vanishing in the limit $R \to 0$ only if $n_3 = 0$ and $q = -1$. This means that $n_1 = 2n_4 - 1$ ($\geq 0$). Therefore only the terms of the form $R^{2n_4-1}/\xi^{2n_4+1}$ ($n_4 \geq 1$) give finite coefficients, and they contribute to $A_\alpha$.

2. The case $2p = 2n_3 - 2n_4 = n_3 - n_1 + q \geq 0$.

In this case, because the harmonic coefficients of $\xi^{2n_3-2n_4-1}$ are finite in the limit $R \to 0$, we must have $n_1 = 0$, and hence $n_3 = 2n_4 + q$. Therefore, since $D_{\ell m}^{p-1/2}$ is an even function of $m$, the harmonic coefficients are odd functions of $m$ if $q$ is odd, i.e. if $q = -1$ or 1. When the sum over $m$ is taken, the result vanishes in the angular coincidence limit if $q$ is odd because of the symmetry property of $|Y_{\ell m}(\theta, \phi)|^2$ under $m \to -m$. Thus only the cases $q = 0$ and $n_3 = 2n_4$ remain. The corresponding terms are of the form $\sim (\phi - \phi')^{2n_4}/\xi^{2n_4+1}$, and they contribute to $B_\alpha$.

From the above results, and noting that $\phi' \propto R$, we obtain the equality

\[
\frac{(\phi - \phi')^{2n}}{\xi^{2n+1}} = \frac{(\phi^2 - 2\phi\phi' + \phi'^2)^{n}}{\xi^{2n+1}} = \frac{\phi^{2n}}{\xi^{2n+1}},
\]  

(5.6)

which holds with respect to its contributions to the regularization parameters. Thus, to summarize, the non-vanishing contributions are from the terms either of the form $\sim R^{2n+1}/\xi^{2n+3}$ or of the form $\sim \phi^{2n}/\xi^{2n+1}$, where $n$ is a non-negative integer. Terms of the former type contribute to $A_\alpha$ and those of the latter type to $B_\alpha$. 


5.1. The $A$-term

The $A$-term describes the quadratic divergent terms of the direct force. Thus we consider the most divergent terms in Eqs. (4.27)–(4.30),

$$\sim \frac{R}{\xi^3} \quad \text{and} \quad \sim \frac{\phi}{\xi^3}. \quad (5.7)$$

Because $\phi/\xi^3 = (\phi - \phi')/\xi^3 + \phi'/\xi^3$, we can replace $\phi/\xi^3$ by $\phi'/\xi^3$, due to the result given in the above discussion. Hence, we can focus on the form $\sim R/\xi^3$. The essential point here is that this is odd in $R$. This leads to harmonic coefficients proportional to $\text{sign}(R)$. Using the formula (D.4), we obtain

$$A_t = \text{sign}(R) \frac{q^2 r_0 - 2M}{r_0} \frac{u_r}{1 + L^2/r_0^2}, \quad (5.8)$$

$$A_r = -\text{sign}(R) \frac{q^2 r_0}{r_0} \frac{r_0 - 2M}{1 + L^2/r_0^2}, \quad (5.9)$$

$$A_\phi = 0. \quad (5.10)$$

These $A$-terms vanish when we average these quantities in both limits $R \to \pm 0$. There could be correction terms of $O(y_0)$ contributing to the $C$ and $D$-terms. However, as shown in Appendix C, in fact these do not exist.

5.2. The $B$-term

The linearly divergent terms are described by the $B$-term. These terms are of the form

$$\frac{\phi^{2n}}{\xi^{2n+1}}, \quad (5.11)$$

in Eqs. (4.27)–(4.30). The Legendre coefficients are given by the formula (D.15). We find

$$B_t = -\frac{(r_0 - 2M)E u_r}{2r_0} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} (-1)^n \frac{(2n+1)^2 \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)} \frac{L^{2n}}{r_0^{2n+1}}, \quad (5.12)$$

$$B_r = \frac{(r_0 - 2M)u_r^2}{2r_0} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} (-1)^n \frac{(2n+1)^2 \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)} \frac{L^{2n}}{r_0^{2n+1}}$$

$$-\frac{1}{2r_0} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} (-1)^n \frac{(-2n+1) \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)} \frac{L^{2n}}{r_0^{2n+1}}, \quad (5.13)$$

$$B_\phi = \frac{(r_0 - 2M)\mathcal{L} u_r}{2r_0^2} \left( \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} (-1)^n \frac{(2n+1)^2 \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)} \frac{L^{2n}}{r_0^{2n+1}} ight)$$

$$-\frac{r_0^2}{\mathcal{L}^2} \sum_{n=0}^{\infty} \frac{(2(n+1))!}{2^{2(n+1)} ((n+1)!)^2} (-1)^n$$

$$\times \frac{(2n+1)^2 \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+1)} \frac{L^{2(n+1)}}{r_0^{2(n+1)+1}}. \quad (5.14)$$
The above may be expressed in terms of the hypergeometric functions as

\[ B_t = -\frac{(r_0 - 2M)\mathcal{E}u_r}{2r_0^3} F\left(\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}\right), \quad (5.15) \]

\[ B_r = \frac{(r_0 - 2M)u_r^2}{2r_0^3} F\left(\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}\right) \]
\[ -\frac{1}{2r_0^2} \left( F\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{L^2}{r_0^2}\right) + \frac{L^2}{2r_0^2} F\left(\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}\right) \right), \quad (5.16) \]

\[ B_\phi = \frac{(r_0 - 2M)\mathcal{L}u_r}{16r_0^3} \left( 8F\left(\frac{3}{2}, \frac{3}{2}; 1; -\frac{L^2}{r_0^2}\right) \right. \]
\[ -4F\left(\frac{3}{2}, \frac{3}{2}; 2; -\frac{L^2}{r_0^2}\right) + \frac{9L^2}{r_0^2} F\left(\frac{5}{2}, \frac{5}{2}; 3; -\frac{L^2}{r_0^2}\right) \right) \quad (5.17) \]

The above results for the \( A \) and \( B \)-terms are identical to the corresponding results obtained by Barack and Ori in a quite different fashion.\(^{15}, 16, *\)

\section{Conclusion and discussion}

Our final goal is to establish a method for calculating the local gravitational reaction force on a point particle orbiting a Kerr black hole. We have pointed out in §1 that there are two problems, the ‘subtraction problem’ and the ‘gauge problem’. In this paper, we have only studied a possible approach to the subtraction problem. We introduced a regularization method that utilizes the spherical-harmonic decomposition and derived the direct part of the self-force, which turns out to be independent of the spin \( s \) of the field under consideration. The harmonic decomposition of this direct part was carried out, and the regularization counterterms for the self-force were derived for a general geodesic orbit. We found that our result is identical to the result obtained by Barack and Ori\(^{16}\) in their mode-sum regularization scheme (MSRS).\(^9\)

To compare with the MSRS, we have derived the regularization counterterms, which are obtained by summing the harmonic coefficients over \( m \). However, when we extend our method to the Kerr background, it may be necessary to carry out the regularization before performing the \( m \)-summation. In this sense, the formulas derived in Appendix D, where no summation over \( m \) is assumed, may be still useful in the Kerr case.

It is worthwhile to point out that the gauge problem in the gravitational case seems far more difficult to treat than the subtraction problem. What we know at the moment is that the gravitational self-force is described by the tail part of the metric perturbation induced by a particle.\(^3, 4\) However this is valid only in the harmonic

\(^{*}\) Here we give the values of \( B_\alpha \) expressed in terms of generalized hypergeometric functions, while in Ref. 15) these are given in terms of the two complete elliptic integrals \( K \) and \( E \). The relation between the two expressions can be revealed by changing the variables and using the formulas in Ref. 18).
gauge, while the full metric perturbation can be obtained only in the Regge-Wheeler gauge or in the radiation gauge where the identification of the tail part is highly non-trivial. A prescription to identify the tail part of the metric perturbation is proposed in Ref. 10), but it must be verified. The gauge problem for the non-radiative monopole and dipole components of the metric perturbation, which can not be obtained in the Teukolsky formalism seems to stand as an additional serious obstacle. Possible solutions of the gauge problem are under investigation.\footnote{Recently some progress has been made with respect to this in Ref. 17).}

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Appendix A

--- Bi-Tensors and Local Coordinate Expansion ---

Bi-tensors are tensors that depend on two distinct field points, say, $x^\alpha$ and $\bar{x}^{\bar{\alpha}}$. For our purpose, we consider half the squared geodesic interval bi-scalar $\sigma(x, \bar{x})$ and the geodesic parallel displacement bi-vector $\bar{g}_{\alpha\bar{\alpha}}(x, \bar{x})$, which satisfy

\[ \sigma(x, \bar{x}) = \frac{1}{2} g^{\alpha\bar{\beta}} \sigma_{\alpha}(x, \bar{x})\sigma_{\bar{\beta}}(x, \bar{x}) = \frac{1}{2} \bar{g}^{\alpha\bar{\beta}} \sigma_{\alpha}(x, \bar{x})\sigma_{\bar{\beta}}(x, \bar{x}), \]  
\[ \lim_{x \to \bar{x}} \sigma_{\alpha}(x, \bar{x}) = \lim_{\bar{x} \to x} \sigma_{\bar{\alpha}}(x, \bar{x}) = 0, \]  
\[ \bar{g}_{\alpha\bar{\alpha}}(x, \bar{x}) g^{\bar{\alpha}\bar{\gamma}}(x)\sigma_{\bar{\gamma}}(x, \bar{x}) = 0, \quad \bar{g}_{\alpha\bar{\alpha}}(x, \bar{x}) g^{\bar{\beta}\gamma}(\bar{x})\sigma_{\gamma}(x, \bar{x}) = 0, \]  
\[ \lim_{x \to \bar{x}} \bar{g}_{\alpha\bar{\alpha}} = \delta_{\alpha\bar{\alpha}}. \]  

We also need the generalized van Vleck-Morette determinant bi-scalar,

\[ \Delta(x, \bar{x}) = \det(-\bar{g}^{\alpha\bar{\alpha}}(x, \bar{x})\sigma_{\bar{\alpha}}\bar{\beta}(x, \bar{x})). \]

We consider the local expansion of these bi-tensors around the coincidence limit $x \to \bar{x}$.
In calculating the local expansion of the field and its derivatives in a covariant way, the following formulas are useful: \(^3\)

\[
\sigma_{\alpha\beta}(x, z) = g_{\alpha\beta}(z) - \frac{1}{3} R_{\alpha}^{\gamma \beta \delta}(z) \sigma_{\gamma \delta}(x, z) + O(\epsilon^3),
\]

\[
\sigma_{\mu\nu}(x, z) = -\bar{g}_{\mu}^{\alpha}(x, z) \left( g_{\alpha\beta}(z) + \frac{1}{6} R_{\alpha\gamma\beta\delta}(z) \sigma_{\gamma\delta}(x, z) \right) + O(\epsilon^3),
\]

\[
\bar{g}_{\mu\nu}^{\alpha\beta}(x, z) = -\frac{1}{2} \bar{g}_{\mu\nu}^{\gamma\delta}(x, z) R_{\alpha}^{\gamma \beta \delta}(z) \sigma_{\beta\delta}(x, z) + O(\epsilon^2),
\]

\[
\bar{g}_{\mu\nu}^{\gamma\alpha}(x, z) = -\frac{1}{2} \bar{g}_{\mu\nu}^{\gamma\delta}(x, z) \bar{g}_{\delta\gamma}(x, z) R_{\alpha}^{\gamma \beta \delta}(z) \sigma_{\beta\delta}(x, z) + O(\epsilon^2).
\]

For the calculation of \(\Delta(x, \bar{x})\), we use the result of the covariant expansion given in Ref. 2). We have

\[
\sigma_{\alpha\beta}(x, \bar{x}) = -\bar{g}_{\alpha\beta}(x, \bar{x}) \left( g_{\alpha\beta}(\bar{x}) + \frac{1}{6} R_{\alpha}^{\gamma \beta \delta}(\bar{x}) \sigma_{\gamma\delta}(x, \bar{x}) \right) + O(|x - \bar{x}|^3).
\]

Then, for a vacuum background, we obtain

\[
\Delta(x, \bar{x}) = 1 + O(|x - \bar{x}|^3).
\]

The local expansion of these bi-tensors in the background coordinates is derived from the formulas (4.13) and (4.14), and the expansion coefficients expressed in terms of the ordinary derivatives of the background metric are given in Eqs. (4.16)–(4.19). In the Schwarzschild background, the non-vanishing components of these coefficients are

\[
A_{trtr} = A_{trt} = A_{rtt} = -\frac{M}{r^2},
\]

\[
A_{rrrr} = -\frac{3M}{(r - 2M)^2},
\]

\[
A_{r\theta\theta} = A_{\theta r\theta} = A_{\theta\theta r} = r,
\]

\[
A_{r\phi\phi} = A_{\phi r\phi} = A_{\phi\phi r} = r \sin^2 \theta,
\]

\[
A_{\theta\phi\phi} = A_{\phi\theta\phi} = A_{\phi\phi\theta} = r^2 \sin \theta \cos \theta,
\]

\[
A_{tttt} = -\frac{M^2(r - 2M)}{r^5},
\]

\[
A_{tttt} = A_{trtr} = A_{trt} = A_{rtt} = A_{rrtt} = -\frac{M(4r - 5M)}{3r^3(r - 2M)},
\]

\[
A_{tt\theta\theta} = A_{t\theta t\theta} = A_{\theta t\theta t} = A_{\theta\theta tt} = A_{\theta\theta t\theta} = -\frac{M(r - 2M)}{3r^2},
\]

\[
A_{tt\phi\phi} = A_{t\phi t\phi} = A_{\phi tt\phi} = A_{\phi\phi t\phi} = A_{\phi\phi\phi t} = -\frac{M(r - 2M)}{3r^2} \sin^2 \theta,
\]

\[
A_{rrrr} = \frac{M(8r - M)}{r(r - 2M)^3},
\]
\[ A_{rr\theta} = A_{r\theta r} = A_{\theta r r} = A_{r\theta r} = A_{\theta r r} = -\frac{M}{3(r - 2M)}, \]
\[ A_{rr\phi} = A_{r\phi r} = A_{\phi r r} = A_{r\phi r} = A_{\phi r r} = -\frac{M}{3(r - 2M)} \sin^2 \theta, \]
\[ A_{\theta\theta \phi} = -r(r - 2M), \]
\[ A_{\phi \phi \phi} = -r(r - 2M) \sin^4 \theta - r^2 \sin^2 \theta \cos^2 \theta, \]
\[ A_{r \phi \phi} = A_{\phi r \phi} = A_{r \phi \phi} = A_{\phi r \phi} = A_{\phi \phi r} = A_{\phi \phi r} = r \sin \theta \cos \theta, \]
\[ B_{tt|r} = -B_{t r}|t = B_{r t}|t = -\frac{M}{r^2}, \]
\[ B_{rr|r} = -\frac{M}{(r - 2M)^2}, \]
\[ B_{\theta \theta |r} = -B_{r \theta |\theta} = B_{\theta r}|\theta = r, \]
\[ B_{\phi \phi |r} = -B_{r \phi}|\phi = B_{r \phi}|\phi = r \sin^2 \theta, \]
\[ B_{\theta \phi |\phi} = -B_{\phi \theta}|\phi = B_{\phi \theta}|\phi = r^2 \sin \theta \cos \theta, \]
\[ B_{tt|tt} = -\frac{M^2(r - 2M)}{r^5}, \]
\[ B_{tt|rr} = \frac{M(2r - 3M)}{r^3(r - 2M)}, \]
\[ B_{tr|tr} = B_{tr|rt} = -\frac{M(r - 3M)}{r^3(r - 2M)}, \]
\[ B_{rt|tr} = B_{rt|rt} = \frac{M(r - M)}{r^3(r - 2M)}, \]
\[ B_{rr|tt} = \frac{M^2}{r^3(r - 2M)}, \]
\[ B_{tt|\theta \theta} = B_{t \theta |\theta t} = B_{\theta t}|\theta t = B_{\theta t}|\theta t = \frac{M(r - 2M)}{2r^2}, \]
\[ B_{t \phi |\phi t} = B_{t \phi |\phi t} = B_{\phi t}|\phi t = B_{\phi t}|\phi t = \frac{M(r - 2M)}{2r^2} \sin^2 \theta, \]
\[ B_{rr|rr} = \frac{M(2r - M)}{r(r - 2M)^3}, \]
\[ B_{rr|\theta \theta} = -1, \]
\[ B_{r \theta |r \theta} = B_{r \theta |r \theta} = -\frac{M}{2(r - 2M)}, \]
\[ B_{\theta \theta |r \theta} = B_{\theta r |r \theta} = -\frac{2r - 3M}{2(r - 2M)}, \]
\[ B_{rr|\phi \phi} = -\sin^2 \theta, \]
\[ B_{r\phi|r\phi} = B_{r\phi|\phi r} = -\frac{M}{2(r - 2M)} \sin^2 \theta, \]
\[ B_{\phi r|r\phi} = B_{\phi r|\phi r} = -\frac{2r - 3M}{2(r - 2M)} \sin^2 \theta, \]
\[ B_{\theta\theta|\phi\phi} = -r(r - 2M), \]
\[ B_{\theta\phi|\phi\theta} = -r^2 \cos^2 \theta, \]
\[ B_{\theta\phi|\phi\theta} = -r(r - M) \sin^2 \theta, \]
\[ B_{\phi\theta|\phi\theta} = -r^2 \cos^2 \theta + Mr \sin^2 \theta, \]
\[ B_{\phi\phi|\phi\phi} = -r^2 \sin^2 \theta, \]
\[ B_{\phi\phi|\phi\phi} = -r(r - 2M) \sin^4 \theta - r^2 \sin^2 \theta \cos^2 \theta, \]
\[ B_{r\theta|\phi\phi} = B_{\theta\phi|\phi\theta} = -B_{\phi\theta|\phi\theta} = B_{\phi\phi|\phi\phi} = -B_{r\phi|\phi\phi} = -B_{r\phi|\phi\phi} = -B_{r\phi|\phi\phi} = -B_{r\phi|\phi\phi} = -B_{r\phi|\phi\phi} = -r \sin \theta \cos \theta. \]

**Appendix B**

The Direct Part of the Electromagnetic and Gravitational Self-Force

In this appendix, we present the direct parts of the vector and tensor fields. The direct part of the field is obtained by integrating the direct part of the Green function \( sG_{\{A\}}^{\text{dir}} \), as in the scalar case. We find

\[ sG_{\{A\}}^{\text{dir}}(x, x') = \frac{\theta[\Sigma(x), x'] u_{\{A\}}(x, x') \delta\left(\sigma(x, x')\right)}{4\pi}, \quad \text{(B.1)} \]

\[ s u_{\{A\}}(x, x') = \begin{cases} \sqrt{\Delta(x, x')}, & (s = 0) \\ \sqrt{\Delta(x, x') \bar{g}_{\mu\nu}(x, x')}, & (s = 1) \\ \sqrt{\Delta(x, x') \bar{g}_{\mu\nu}(x, x') \bar{g}_{\mu'\nu'}(x, x')}, & (s = 2) \end{cases} \quad \text{(B.2)} \]

where the suffix \( \{A\} \) represents the space-time indices appropriate to the spin \( s \) of the field.

From the above Green functions, we obtain the direct part of the field, which is expanded around the particle position, as

\[ s\theta_{\{A\}}^{\text{dir}}(x) = \begin{cases} q \left[ \frac{1}{\sigma_{\alpha}(x, z(\tau_{\text{ret}})) v^\alpha(\tau_{\text{ret}})} \right] + O(y^2), & \text{(scalar)} \\ e \left[ \frac{1}{\bar{g}_{\mu\alpha}(x, z(\tau_{\text{ret}})) v^\alpha(\tau_{\text{ret}})} \right] + O(y^2), & \text{(vector)} \\ 4Gm \left[ \frac{\bar{g}_{\mu\alpha}(x, z(\tau_{\text{ret}})) \bar{g}_{\mu'\beta}(x, z(\tau_{\text{ret}})) v^\alpha(\tau_{\text{ret}}) v^\beta(\tau_{\text{ret}})}{\sigma_{\gamma}(x, z(\tau_{\text{ret}})) v^\gamma(\tau_{\text{ret}})} \right] + O(y^2), & \text{(tensor)} \end{cases} \quad \text{(B.3)} \]

Therefore, we have

\[ F_{\alpha}[s\theta_{\{A\}}^{\text{dir}}](x) = c\bar{g}_{\alpha \tilde{\alpha}}(x, z_{\text{eq}}) \frac{1}{\varepsilon^3 K}. \]
\[
\times \left\{ \sigma_{\tilde{\alpha}}(x, z_{eq}) + \hbar \epsilon^2 R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}(z_{eq}) v_{eq}^\tilde{\beta}(x, z_{eq}) v_{eq}^\tilde{\gamma} \right\} \\
+ O(y), \quad (B.4)
\]

\[
\epsilon = \sqrt{2\sigma(x, z_{eq})}, \quad (B.5)
\]

\[
\kappa = \sqrt{-\sigma_{\tilde{\alpha}\tilde{\beta}}(x, z_{eq}) v_{eq}^\tilde{\alpha} v_{eq}^\tilde{\beta}} \\
= 1 + \frac{1}{6} R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}}(z_{eq}) v_{eq}^\tilde{\alpha} \sigma_{\tilde{\beta}}(x, z_{eq}) v_{eq}^\tilde{\gamma} \sigma_{\tilde{\delta}}(x, z_{eq}) + O(y^3), \quad (B.6)
\]

where \(c\) and \(h\) depend on the spin of the field as

\[
(c, h) = \begin{cases} 
(q^2, 1/3), & \text{(scalar)} \\
(-e^2, -2/3), & \text{(vector)} \\
(Gm^2, -11/3). & \text{(tensor)} 
\end{cases} \quad (B.7)
\]

Here, the extension of the four-velocity \(v^\alpha(\tau_0)\) necessary to define the projection tensor \(P_{\alpha\beta}\), as mentioned in the line following Eq. (1.3), is chosen such that

\[
V^\alpha(x) = \bar{g}^\alpha_{\tilde{\alpha}}(x, z_{eq}) v_{\tilde{\alpha}}(\tau_{eq}(x)). \quad (B.8)
\]

It is noted that when we consider the mode decomposition regularization for the self-force, the direct part calculated in Eqs. (4.27)–(4.30) is independent of spin.

---

**Appendix C**

*Basic Properties of the Mode Coefficients*

In this appendix, we examine the general properties of the mode coefficients for the terms that appear in the local coordinate expansion of the direct force given in Eqs. (4.27)–(4.30). We show that the \(C\) and \(D\)-terms of the regularization counter-terms vanish in the limit of coincidence with the particle position.

We first express \(\xi\) in the form

\[
\xi^2 = \xi_0^2 + \mathcal{L}^2 (\phi - \phi')^2, \quad (C.1)
\]

where \(\xi_0\) is defined by

\[
\xi_0 = \sqrt{2r_0(a - \cos \tilde{\vartheta})^{1/2}}. \quad (C.2)
\]

In terms of \(\xi_0\), \(R\) and \(\phi\), all the terms that contribute to the direct force \(F_{\alpha}^{\text{dir}}\) have the form

\[
F_{\alpha}^{\text{dir}} \sim \frac{R^p(\phi - \phi')^q \xi_0^{2r}}{\xi_3^3} \left( \frac{R^2}{\xi_0^2} \right)^m \left( \frac{\phi^2}{\xi_0^2} \right)^n, \quad (C.3)
\]

where we have replaced possible factors of the form \(\Theta^{2k}\) by polynomials in \(\xi_0^2\), \(R\) and \(\phi\), and \(m, n, p, q\) and \(r\) are non-negative integers satisfying

\[
m \geq 0, \quad n \geq 0, \quad 1 \leq p + q + 2r \leq 3. \quad (C.4)
\]
Covariant Self-Force Regularization of a Particle

This is because the highest order of the divergence in the direct force is quadratic, and it is sufficient to focus only on terms of order up to \( O(y^0) \) in the local coordinate expansion.

Let us analyze the coincidence limit in detail. By using a modified version of Eq. (5.4) with \( L^2 = 0 \) but by taking account of \( O(y^2) \) corrections, we can further reduce the above to the form

\[
(\partial_\phi)^{q+2n} \xi_0^{2q+2r-2m-3} R^{2m+p}.
\]  

Note that the \( O(y^2) \) corrections only change the original \( q \) to \( q + 2 \), and hence it is enough to consider the above form.

We can decompose (C.5) into a spherical harmonic series by using the formula

\[
C_\ell = \frac{2\ell + 1}{2} \int_{-1}^{1} \frac{P_\ell(\mu)}{\sqrt{a - \mu}} d\mu = \sqrt{2} \left( a - \sqrt{a^2-1} \right)^{\ell+1/2}.
\]  

This quantity \( C_\ell \) is equal to the special case of \( C_\ell^p(a) \) with \( p = -1/2 \) given on Eq. (D.3). Introducing the variable \( z \) by \( e^z = a + \sqrt{a^2-1} \) (and hence \( z > 0 \)), Eq. (C.6) can be re-expressed as

\[
C_\ell(z) = \frac{2\ell + 1}{2} \int_{-1}^{1} \frac{P_\ell(\mu)}{\sqrt{\cosh z - \mu}} d\mu = e^{-Lz},
\]  

where \( L = \ell + 1/2 \). Note that \( z \propto |R| \).

As discussed in §5, odd powers of the operator \( \partial_\phi \) give harmonic coefficients that vanish after summing over \( m \). Hence, we only need to consider the case in which \( q \) is an even, non-negative integer. Then it is straightforward to show that the operator \( \partial^2_\phi \) in the harmonic decomposition of (C.5) gives the factor \( L^2 \) in the coincidence limit of the angular coordinates followed by the sum over \( m \). In general, \( \partial^{2n}_\phi \) gives rise to a polynomial of degree \( n \) in \( L^2 \). Explicitly, we have

\[
\sum_{m=-\ell}^{\ell} m^{2n} \left| Y_{\ell m} \left( \frac{\pi}{2}, 0 \right) \right|^2 = \frac{L}{2\pi} \sum_{p=0}^{n} \lambda_p^{(n)} L^{2p},
\]  

where the factor \( \lambda_p^{(n)} \) is independent of \( L \) and \( \lambda_p^{(n)} = \Gamma(n+1/2)/\sqrt{\pi} \Gamma(n+1) \). The derivation of this formula is given in Appendix C of Ref. 14).

Thus we can replace \( (\partial_\phi)^{q+2n} \) in (C.5) by \( L^{2j} \), with \( 1 \leq j \leq q/2 + n \). (Note that \( q \) is even.) With this replacement, let us consider the following three cases for the powers of \( \xi_0 \) separately.

1. \( 2N + 1 := 2m + 3 - (2q + 2r + 2n) > 1 \) (\( \xi_0^{-(2N+1)} \); \( N \geq 1 \)).

In this case, to obtain the harmonic decomposition of \( \xi_0^{-(2N+1)} \), we simply apply \( [d/d(cosh z)]^N = [d/sinh zdz]^N \):

\[
\int_{-1}^{1} \frac{P_\ell(\mu)}{(\cosh z - \mu)(2n+1)^2} d\mu \propto \left[ \frac{d}{sinh zdz} \right]^n e^{-Lz}.\]  

(C.9)
Taking account of the general form (C.5), this gives rise to harmonic coefficients in the form

\[
\sim R^{2m+p} L^{2j} \frac{L^k}{z^{2N-k}} \left( 1 + \sum_{i \geq 1} c_i z^{2i} \right) e^{-Lz},
\]

(C.10)

where \(1 \leq k \leq N\). Because \(z \sim |R|\) and

\[
2m + p = 2N + 2q + 2r + 2n - 2 + p \\
\geq 2N + q + (q + 2r + p) - 2 \\
\geq 2N + q - 1 \geq 2N - 1,
\]

the only term that remains in the limit \(R \sim \pm z \to 0\) is the \(k = 1\) term, and this implies that \(2m + p\) is odd. Because we know that the leading divergence is quadratic, this gives a harmonic coefficient proportional to \(\text{sign}(R) L\), which contributes to \(A_\mu\).

(2) \(2m + 3 - (2q + 2r + 2n) = 1 (\xi_0^{-1})\).

In this case, the harmonic coefficients are non-vanishing only if \(2m + p = 0\). Because the result is independent of \(L\), it contributes to \(B_\mu\), i.e. the linearly divergent term.

(3) \(2N + 1 := 2q + 2r + 2n - (2m + 3) > 1 (\xi_0^{2N+1}; N \geq 1)\).

In this case, because the harmonic coefficients of \(\xi_0^{2N+1}\) are finite in the limit \(z \to 0\), we must have \(m = p = 0\), which implies \(q + 2r = 2\). Going back to the original form (C.5), we can then replace \(n\) by \(n + q/2\), and then the term of interest takes the form

\[
\partial_\phi^{2n} \xi_0^{2n-1}. \quad (n \geq 1)
\]

(C.11)

Using the formula (D.4), we obtain the coefficients of the Legendre decomposition of \(\xi_0^{2n-1}\) as

\[
\xi_0^{2n-1} \bigg|_L = \frac{\kappa_n}{(L^2 - 1^2)(L^2 - 2^2) \cdots (L^2 - n^2)},
\]

(C.12)

where \(\kappa_n = (-1)^n [(2n - 1)!!] r_0^{2n-1}\), and we have introduced the notation

\[
\cdots \bigg|_L
\]

to denote the \(L = \ell + 1/2\) mode coefficient of the Legendre expansion in the coincidence limit.\(^\ast\) Therefore, together with Eq. (C.8), we have

\[
\partial_\phi^{2n} \xi_0^{2n-1} \bigg|_L = \frac{(-1)^n \kappa_n}{(L^2 - 1^2)(L^2 - 2^2) \cdots (L^2 - n^2)} \sum_{p=0}^{n} \lambda_p^{(n)} L^{2p},
\]

\[
= (-1)^n \kappa_n \lambda_n^{(n)} + \sum_{p=0}^{n-1} \left[ \frac{\nu_p}{(L^2 - 1^2)(L^2 - 2^2) \cdots (L^2 - (n-p)^2)} \right],
\]

(C.13)

\(^\ast\) When the \(m\)-sum is non-trivial, \(\cdots \bigg|_L\) is defined to be the coefficient after summing over \(m\).
where $\nu_p$ is independent of $L$. We thus find that the first term on the right-hand side, which is independent of $L$, contributes to $B_\mu$, while the rest seems to give non-vanishing contributions to $D_\mu$. However, Eq. (C.12) tells us that they are simply the Legendre coefficients of positive powers of $\xi_0$, and hence they vanish after the sum over $\ell$ is taken.

In each of the above three cases, nothing contributes to $C_\mu$. Thus, to summarize, we find the $C$ and $D$-terms vanish and the $A$-term is proportional to $\text{sign}(R)$.

### Appendix D

#### Mathematical Formulas for Mode Decomposition

In this appendix, we give formulas necessary for the harmonic decomposition of $\xi^{2n-1}$. For this purpose, we introduce a dimensionless version of $\xi$ as

$$\tilde{\xi} = \left( a - \cos \tilde{\theta} + \frac{b}{2} (\phi - \phi')^2 \right)^{1/2}.$$ 

Here, as defined in the text, $\tilde{\theta}$ is the angle between $(\theta, \phi)$ and $(\theta', \phi')$, and $a > 1$. In the following, we do not restrict $n$ to be an integer. Our strategy is to consider a series expansion of $\tilde{\xi}$ by treating $b$ as the expansion parameter.

First, we consider the harmonic expansion of $\tilde{\xi}_0^{2p}$, where $\tilde{\xi}_0 = (a - \cos \tilde{\theta})^{1/2}$:

$$\tilde{\xi}_0^{2p} = (a - \cos \tilde{\theta})^p = 2\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell C_{\ell}^p(a) Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') . \quad (D.1)$$

The expansion coefficients $C_{\ell}^p$ are obtained by the Legendre integration as

$$C_{\ell}^p(a) = \int_{-1}^1 d\mu (a - \mu)^p P_\ell(\mu) , \quad (D.2)$$

where $P_\ell(\mu)$ is the Legendre function of the first kind. As $a > 1$, we can expand $(a - \mu)^p$ in powers of $\mu$. Then we obtain

$$
C_{\ell}^p(a) = \frac{1}{2^{p+1}} \frac{1}{\Gamma(-p)} \sum_{k=0}^\infty \frac{\Gamma(k-p/2 + \ell/2) \Gamma(k-p/2 + \ell/2 + 1)}{\Gamma(k+\ell+3/2)} \frac{1}{k!} \left( \frac{1}{a^2} \right)^k \\
= \sqrt{\pi} 2^{-\ell+1} a^{p-\ell} \frac{\Gamma(-p+\ell)}{\Gamma(-p) \Gamma(\ell+3/2)} F \left( -\frac{p+\ell}{2}, -\frac{p+\ell+1}{2}, \ell + \frac{3}{2}; \frac{1}{a^2} \right) \\
= - \frac{1}{p+1} a^{-p-1} \left( \frac{a^2 - 1}{2a} \right)^{p+1} F \left( \frac{p+\ell}{2} + 1, \frac{p+\ell+1}{2} + \frac{3}{2}, p+2; 1 - \frac{1}{a^2} \right) \\
+ (-1)^\ell 2^{p+1} a^{p-\ell} \frac{\Gamma(p+1)^2}{\Gamma(p+1) \Gamma(p+\ell+2)} \\
\times F \left( -\frac{p}{2} + \frac{\ell}{2}, -\frac{p}{2} + \frac{\ell}{2} + \frac{1}{2}, -p; 1 - \frac{1}{a^2} \right) . \quad (D.3)
$$

In the coincidence limit $a \to 1+0$, we have two qualitatively different types of leading
behavior, depending on the value of $p$, as

$$C^p_\ell(a) \rightarrow \begin{cases} \frac{1}{-p-1}(a-1)^{p+1}, & \text{(for } p + 1 < 0) \\ \frac{1}{-p-1}(a-1)^{p+1} \frac{\Gamma(p+1)^2}{\Gamma(p-\ell+1)\Gamma(p+\ell+2)}. & \text{(for } p + 1 > 0) \end{cases} \quad (D.4)$$

It should be noted that the divergent behavior persists in the mode coefficients for $p + 1 < 0$.

Next, we consider the harmonic expansion of $(\phi - \phi')(a - \cos \bar{\theta})^p$. We take into account only the leading order behavior in the coincidence limit. Hence we have the basic formula that converts a power of $(\phi - \phi')$ into $\phi$-derivatives,

$$(\phi - \phi')(a - \cos \bar{\theta})^p = \frac{1}{p+1} \frac{\partial}{\partial \phi} (a - \cos \bar{\theta})^{p+1} + O(y^{2p+3}). \quad (D.5)$$

Using this formula recursively, we obtain

$$(\phi - \phi')^n(a - \cos \bar{\theta})^p = \sum_{k=0}^{[n/2]} a^{(n)}_k \partial^{n-2k} (a - \cos \bar{\theta})^{p+n-k}, \quad (D.6)$$

where $[n/2]$ denotes the maximum integer not exceeding $n/2$, and $a^{(n)}_k$ satisfies the recurrence relation

$$a^{(n+1)}_k = \frac{1}{p+n-k+1} a^{(n)}_k - (n-2k+2)a^{(n)}_{k-1}, \quad (D.7)$$

$$a^{(0)}_k = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, \cdots \end{cases}. \quad (D.8)$$

This is solved to give

$$a^{(n)}_k = \frac{\Gamma(p+1)\Gamma(n+1)}{\Gamma(p+n-k+1)\Gamma(n-2k+1)} (-1)^k \frac{1}{2^k k!}. \quad (D.9)$$

Because each $\phi$-derivative is interpreted as giving one factor of $im$ to the harmonic coefficients in Eq. (D.1), we obtain the mode decomposition as

$$(\phi - \phi')^n(a - \cos \bar{\theta})^p = 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C^{(n,p)}_{\ell m}(a) Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta', \phi'), \quad (D.10)$$

$$C^{(n,p)}_{\ell m}(a) = \sum_{k=0}^{[n/2]} (im)^{n-2k} a^{(n)}_k C^{p+n-k}_{\ell}(a). \quad (D.11)$$

With the formulas (D.3) and (D.11), we can now write down the harmonic expansion of $\tilde{\xi}^{2p}$ to leading order in the local expansion around $(\theta', \phi')$. The result is

$$\tilde{\xi}^{2p} = \left( a - \cos \bar{\theta} + \frac{b}{2}(\phi - \phi')^2 \right)^p.$$
\[
= \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-n+1)} \frac{1}{n!} \left( \frac{b}{2} \right)^n (\phi - \phi')^{2n}(a - \cos \tilde{\theta})^{p-n}
\]

\[
= 2\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} D^p_{\ell m}(a) Y_{\ell m}(\theta, \phi) Y^*_{\ell m}(\theta', \phi'),
\]

where

\[
D^p_{\ell m}(a) = \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-n+1)} \frac{1}{n!} \left( \frac{b}{2} \right)^n C^{(2n,p-n)}_{\ell m}(a)
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\Gamma(p+1)\Gamma(2n+1)}{\Gamma(p+n-k+1)\Gamma(2n-2k+1)} \times (-1)^{k} b^{n} \left( -m^2 b \right)^{n-k} C^{p+n-k}_{\ell}(a).
\]

The double sum in the last equation with respect to \(k\) and \(n\) can be simplified by introducing \(\bar{n} := n - k\) and summing over \(\bar{n}\) and \(k\), which can now be taken independently. Then we have

\[
D^p_{\ell m}(a) = \sum_{\bar{n}=0}^{\infty} \left( \sum_{k=0}^{\bar{n}} \frac{\Gamma(2\bar{n}+2k+1)}{\Gamma(\bar{n}+k+1)} \frac{1}{k!} \left( \frac{b}{4} \right)^k \right) \times \frac{\Gamma(p+1)}{\Gamma(p+\bar{n}+1)\Gamma(2\bar{n}+1)} \left( -m^2 b \right)^{\bar{n}} C^{p+\bar{n}}_{\ell}(a)
\]

\[
= \frac{1}{\sqrt{1+b}} \sum_{n=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p+n+1)} \frac{1}{n!} \left( -\frac{m^2 b}{2(1+b)} \right)^n C^{p+n}_{\ell}(a).
\]

Using Eq. (D.4), the leading behavior of the mode coefficients (D.14) in the coincidence limit \(a \to 1 + 0\) becomes

\[
D^p_{\ell m}(a) \to \begin{cases} 
\frac{1}{\sqrt{1+b}} \frac{1}{p-1} (a-1)^{p+1}, & \text{(if } p + 1 < 0) \\
\frac{(-1)^{\ell} 2^{p+1}}{\sqrt{1+b}} \times \sum_{n=0}^{\infty} \frac{\Gamma(p+1)\Gamma(p+n+1)}{\Gamma(p+n-\ell+1)\Gamma(p+n+\ell+2)} \frac{1}{n!} \left( -\frac{m^2 b}{1+b} \right)^n. & \text{(if } p + 1 > 0) 
\end{cases}
\]

Replacing \(p\) in the above by \(p - 1/2\), we have the formula presented in the text, Eq. (4:31).

References

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