Parameters of Black Holes in Sources with Periodic Variability

Oldřich SEMERÁK and Vladimír KARAS
Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, CZ-180 00 Prague 8, Czech Republic
E-mail (O.S.): oldrich.semerak@mff.cuni.cz

and
Fernando DE FELICE
Department of Physics "G. Galilei", University of Padova, Via Marzolo 8, I-35131 Padova, Italy

(Received 1999 May 17; accepted 1999 June 22)

Abstract

We propose a way to deduce the parameters of accreting black holes. The method employs the properties of the spectral features observed in radiation from an accretion disk. It is applicable to sources which exhibit periodic modulations of variability, provided: (i) the gravitational field is determined by the black hole and described by the Kerr metric; (ii) a thin accretion disk of negligible mass lies in the equatorial plane of the hole; (iii) a secondary object (also with negligible mass) moves on a slightly inclined almost circular orbit around the black hole and passes periodically through the disk; (iv) the collisions result in observable photometric and spectroscopic features (temporal variability of the radiation flux and of spectral-line profiles produced in the disk), which show frequencies of the orbital motion and of latitudinal oscillations; (v) one can measure the width of the spectral line from hot spots arising in the disk due to collisions with the orbiter, and/or detect predicted low-frequency oscillations induced in the disk. These frequencies and the line width provide enough information to determine in physical units three parameters characterizing the source: the mass and angular momentum of the central black hole, and radius of the orbit of the secondary.

Key words: Accretion disks — Black holes — X-Rays: sources

1. Introduction

There is ever-growing theoretical and observational evidence that the dark masses present in some galactic nuclei and X-ray binaries are black holes (Rees 1998). The standard interpretation of observational data employs a black hole and an accretion disk or a torus (Kato et al. 1998). A number of ways have been suggested of how to deduce the properties of putative holes. The first rough estimates of their mass were based on energy considerations and limits implied by the shortest time-scales of variability. More precise methods became possible with modern observational techniques like HST, VLBI, and X-ray satellites. Our present paper is relevant for sources with a rotating black hole surrounded by a thin accretion disk in the equatorial plane and a much less massive secondary orbiter passing periodically through the disk. First, we briefly recall the relevant spectral properties which have been discussed as possible observational signatures of a black hole (we do not mention other approaches which involve different physical mechanisms, e.g., gravitational waves).

In galactic nuclei, the existence and size of the dark central mass are deduced from a profile of the surface brightness of the nuclei and from the spatial distribution and dynamics of surrounding gas and stars (Kormendy, Richstone 1995). Current optical and radio studies still probe only scales above $10^4$ gravitational radii of the nucleus, which is insufficient to resolve any imprints of relativistic effects in the innermost regions around the centre, in particular to deduce the rate of its rotation. However, X-ray observations of the fluorescence iron Kα line (Nandra et al. 1997ab), especially its broad and variable profile skewed to lower energies, indicate that the Doppler and gravitational redshifts play a role, and that emission takes place very near to a rotating black hole. Several ways have been proposed to estimate the parameters of the black holes in active galactic nuclei which rely on the source variability: for example, limits on the black-hole mass and its rate of rotation can be imposed from time delays between variations of the emission-line strength and of the continuum, and from temporal changes of the observed emission lines (e.g., Blandford, McKee 1982; Stella 1990; Bromley et al. 1997).

The evidence for stellar-mass black holes comes from observations of galactic X-ray binaries where they are indicated by low luminosity of one of the components, temporal variability, spectral features (mainly the presence of relatively strong ultrasoft component and of the hard X-ray tail), and the minimum mass of the dark component, as estimated from the mass function (e.g., Charles
It has also been argued (Narayan et al. 1997) that black-hole X-ray binaries could be distinguished by a larger variation in luminosity between the bright and faint states than is expected in sources with neutron stars. This way of identifying black holes follows from the low radiative efficiency of advection-dominated accretion flows around black holes. Present observational evidence offers several objects of this type which show relatively stable periodic modulation (van der Klis 1997).

In the present contribution it is assumed that the disk remains in the equatorial plane and the modulating source is represented by a blob which keeps its identity for several orbital periods. A specific scheme for the signal modulation can be described in words as follows: The orbiter intersects the disk periodically and pulls gaseous matter out its plane. This material temporarily obscures the disk, mainly at the radius of intersection, and affects the radiation flux and emission-line profiles formed in that region and characterized by the observed widths. Phenomenologically, a spot forms in the disk and then orbits around the center, surviving several orbital periods. Radiation of the spot modulates the observed radiation at the Keplerian frequency, which is different from the frequency of collisions with the disk when the central black hole rotates. Both ingredients of the model, i.e., local physics of collisions with the disk and the observed signal from orbiting spots, have been described in the literature (recently, Ivanov et al. 1998; Karas 1997). In the usual terminology the blob represents a hot spot; however, any perturbation of the disk surface emissivity (either bright, or less prominent due to obscuration) suffices for the purpose of a phenomenological description.

Subsequent paragraphs describe the technical details of the above-mentioned scenario. We will also show a predicted variable spectral feature involving relevant time-scales in a simplified situation.

2. Assumptions

Consider a system of a rotating black hole, an equatorial thin disk, and a secondary orbiter (which may be a low-mass black hole — see Syer et al. 1991; Vokrouhlický, Karas 1993). If both the disk and the secondary have negligible masses with respect to the central hole, the gravitational field is determined solely by the central black hole. In this situation, the motion of the secondary is governed by the Kerr metric, which in usual notation (Misner et al. 1973) acquires the form

$$ds^2 = -\Delta\Sigma^{-1}dt^2 + A\Sigma^{-1}\sin^2\theta (d\phi - \omega_K dt)^2 + \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2, \tag{1}$$

where Boyer–Lindquist spheroidal coordinates \((t, r, \theta, \phi)\) and geometerized units (in which \(c = G = 1\), \(c\) being the speed of light in vacuum and \(G\) the gravitational constant) are used, \(M\) and \(a\) denote the mass and specific rotational angular momentum of the centre, \(\Delta = r^2 - 2Mr + a^2, \Sigma = r^2 + a^2\cos^2\theta, \Delta = (r^2 + a^2) - \Delta \sin^2\theta, \omega_K = 2Mar/\Delta\).

The world-line of the secondary is very close to a geodesic in the Kerr field, provided that the dynamical effect of passages through the disk is only weak, that tidal interaction is negligible (the secondary is assumed to be much smaller than the typical curvature radius of the field around), and that gravitational radiation is ignored. The trajectory undergoes three secular changes due to the interaction with the disk (Syer et al. 1991; Vokrouhlický, Karas 1993): a long-term decrease of the semi-major axis (spiralling towards the centre due to energy dissipation in collisions with the disk), circularization (the orbit becomes spherical, \(r = \text{const}\)), and gradual tilting of the orbit into the equatorial plane (the orbit declines to the disk). Since the time-scale for circularization is shorter than, or of the same order as, the time-scale necessary to drag the orbit into the disk, one can assume that the secondary follows a nearly equatorial spherical geodesic at late stages of evolution of the hole-disk-secondary system. The secondary remains on this type of orbit for a relatively long time. Let us note that we ignore the gravity of the disk, itself; the case of a massive disk was treated recently by Vokrouhlický and Karas (1998).

The resulting system is characterized by two angular frequencies: that of the azimuthal revolution of the orbiter around a Kerr black hole \(\omega\), and that of its latitudinal oscillations about the equatorial plane \(\Omega_{\phi}\). Both frequencies are in principle measurable at infinity provided that the passages of the orbiter produce sufficiently strong modulation of the disk radiation (Karas, Vokrouhlický 1994 illustrated, by Fourier analysis of simulated photometric data, how these two peaks can be recognized in the power spectrum).

The relevant parameters of a quasi-circular orbit precessing with low amplitude about the equatorial plane of the Kerr spacetime are given in next section. The relations derived in section 4 provide parameters of the central black hole in terms of four observables: the three frequencies \(\Omega_{\phi}, \omega_\perp, \omega_\|\), and \(\kappa\), as they are introduced in the text) and the spectral-line width. The equations can be easily solved if the secondary is not too close to the black hole (cf. Appendix).

3. Parameters of the Precessing Orbit

For a general bound geodesic in Kerr spacetime, the frequencies of the azimuthal and latitudinal motion can be given in terms of elliptic integrals (Karas, Vokrouhlický 1994). The relevant expressions acquire a simpler form in the case of spherical geodesics (Wilkins 1972) and simplify still further for almost equatorial or-
bits. The azimuthal angular velocity, \( \omega = d\phi/dt \) (with respect to a distant observer at rest), is then approximated by the Keplerian circular frequency,

\[
\omega_\pm = (a + 1/y_\pm)^{-1},
\]

where \( y_\pm = y(\omega_\pm) = \pm \sqrt{M/r^3} \) and the upper/lower sign corresponds to the prograde/retrograde orbit. We keep both cases for completeness, but only prograde trajectories (plus sign) are considered later (the accretion disk is more likely to be corotated with the central black hole, and the interaction with the disk also makes the secondary eventually corotate).

The proper angular frequency \( |\Omega| \) of small latitudinal harmonic oscillations about the equatorial plane is given, for a spherical orbit with steady radial component of acceleration, by (de Felice, Usseglio-Tomasset 1996; Semerák, de Felice 1997),

\[
\Omega^2 = (u^t/r)^2 \left\{ \Delta \omega^2 + 2y_\pm^2 [a - (r^2 + a^2) \omega^2] \right\}. \tag{3}
\]

Here, \( \omega = \text{const} \), and the time component of the four-velocity, given by \( (u^t)^2 = -g_{tt} - 2g_{tr} \omega - g_{t\phi} \omega^2 \), is approximated by the equatorial value

\[
(u^t)^2 = 1 - (2M/r)(1 - a\omega^2 - (r^2 + a^2) \omega^2). \tag{4}
\]

The frequency of collisions with the disk is twice the proper frequency \( |\Omega| \); the corresponding value measured by a distant observer is \( |\Omega_\infty| = |\Omega|/u^t \). From equations (2)–(3), the explicit expression for a free orbit is

\[
\Omega^2_\infty = \Omega^2 \left[ 1 - 4a^2 y_\pm (r^2/2) \right] = \Omega^2 \left[ 1 - 4a^2 y_\pm (r^2/2) \right]. \tag{5}
\]

We again mention that the frequency of the latitudinal oscillations stands in the dispersion relation for corrugated waves in accretion disks. It has been suggested that \( \Omega_\infty \) could be detected in some QPO sources (Kato 1990).

The two equations (2) and (5) are not enough to fix the three unknown \( M \), \( a \), and \( r \). Another piece of information can be provided by the width of the emission line which is formed and modulated at the radius of successive collisions between the secondary and the disk. A stationary disk produces the well-known double-born line profile, which is more or less pronounced according to the inclination angle of the source. Radiation from each element in the disk experiences a frequency shift (Fanton et al. 1997),

\[
\nu_\text{obs} = u^t \frac{\omega_\pm}{y_\pm} \left[ 1 - \frac{2M}{r} + y_\pm (a + \sqrt{\Delta} \varepsilon_\phi) \right], \tag{6}
\]

where \( \varepsilon_\phi \) is an emission direction cosine (an azimuthal component of the unit vector along the direction of emission of a given photon, measured in the emitter's local frame), and

\[
(u^t)^2 = \frac{y_\pm^2}{\omega_\pm^2} \left[ 1 - 3M/r + 2a y_\pm \right] = \frac{M}{r} \frac{1}{\omega_\pm^2} \frac{1}{\Delta - (\sqrt{Mr} + a)^2} \tag{7}
\]

follows from (4) and (2). The observed total width of the line arises from the different frequency shifts \( g \) carried by photons which reach the observer at infinity. Since \( g \) depends on the direction cosine \( \varepsilon_\phi \), the maximum width of the line \( \delta g \) (which is a result of integration over one entire circle) corresponds to the range of \( \varepsilon_\phi \) consistent with the escape to infinity, \( \delta \varepsilon_\phi \). We have from equation (6)

\[
\delta g = (\delta \varepsilon_\phi) u^t \omega_\pm \sqrt{\Delta}, \tag{8}
\]

and thus

\[
\delta = \frac{\delta g}{(\delta \varepsilon_\phi)^2} = \frac{1 - 2M/r + a^2/r^2}{r} = \frac{M}{r} \frac{1}{\Delta - (\sqrt{Mr} + a)^2}. \tag{9}
\]

From this equation it appears convenient to include the parameter \( \delta \) in our consideration because it can be written in a form independent of the inclination angle. To determine \( \delta \) in terms of \( \delta g \), one needs to fix the range of \( \delta \varepsilon_\phi \). This can be easily estimated for \( r \gg M \) (cf. Appendix). In general, \( \delta \varepsilon_\phi \) is at most 2 (which is acquired for \( \theta_\phi = \pi/2 \), i.e., edge-on view of the disk), and decreases with the inclination angle \( \theta_\phi \) of the black hole-disk system relative to the line of sight (it also depends, quite weakly, on the rotational parameter \( a \) and the radius of emission \( r \)). It appears unpractical to deal with a lengthy analytic expression for \( \delta \varepsilon_\phi \). Instead, given \( a/M, r/M \), and \( \theta_\phi \), the actual value of \( \delta \varepsilon_\phi \) can be found by numerical ray tracing (Fanton et al. 1997).

We simulated a time-variable line profile to illustrate the main features expected in the observed signal (we used a modification of the code described by Karas, Vokrouhlický 1994). Figure 1 shows the measured count rate in the line (background subtracted) as a function of energy and orbital phase in the equatorial plane of an extreme \( (a = M) \) Kerr black hole. In this example, only that contribution to the total light is taken into account, which originates at \( 10M \lesssim r \lesssim 14M \), where an orbiter affects the disk. The energy is normalized to the emission energy of the line in the local frame of the disk material, as usual (energy axis thus indicates the redshift factor \( g \)). The phase axis is normalized with respect to the Keplerian orbital motion at the central radius, \( r = 12M \); the interval of the two periods is shown for clarity. One can observe:

(i) Modulation of the count rate in the line by the orbiter crossing the disk upwards at radius \( r \) with period
574 O. Semerák, V. Karas, and F. de Felice [Vol. 51, 

Energy

Fig. 1. Simulation of the time-variable line profile from a hot spot orbiting with the disk. A flare occurs at the moment of the star-disk collision (the sharp increase in flux), giving rise to a spot on the disc surface. The observed count rate is then modified by the orbital motion of the spot (period $T$) and by relativistic lensing (with a corresponding change of energy in the range $\delta g$). The spot decays gently with time, until another one is generated in a subsequent collision (i.e., after period $T$). This graph has been obtained by ray tracing in the Kerr metric. Decay time of the spot is taken as a free parameter. See the text for details.

take into account a proper method of background continuum subtraction (the underlying continuum has been ignored here). The mutual interplay between spectral features (around 6.4 keV) and continuum has been subject of recent studies (Young et al. 1998; Zycki et al. 1998; Martocchia et al. 1999), which show substantial smoothing of the profile in some situations.

The uncertainties connected with the badly known $\delta g$, can be overcome by introducing another independent observable. A promising possibility arises from a more realistic view of a star-disk collision. As mentioned above, these collisions disturb the disk periodically with latitudinal frequency $f$. If the orbiter passes across the disk at small $r$, it may induce waves in the disk matter which remain trapped in the range of several gravitational radii and are governed by the epicyclic frequency $\kappa$ (Okazaki et al. 1987; Nowak, Wagoner 1992):

$$\kappa^2 = \omega^2_r (1 - 6M/r + 8ay/3a^2/r^2)$$

$$= \omega^2_r [\Delta - 4(\sqrt{Mr} + a)^2].$$

(10)

These modes can be excited and trapped in a rather narrow range of radius, and one can thus expect that they will reflect just the orbital radius of the satellite. However, the problem of distinguishing between different possible modes is a subject of much debate and has not yet been settled.

4. Determining $M$, $a$, and $r$

The formulas (2), (5), (9), and/or (10) provide a closed system of ordinary equations for $M$, $a$, and $r$ (and possibly $\delta g$, if one succeeds in measuring all four observables) as functions of the quantities $\omega_r$, $|\Omega_\infty|$, $|\delta|$, and/or $|\kappa|$. The equations can be tackled in different manners depending on which of the quantities is determined with the best confidence. We now discuss the derivation of the most desired combinations of parameters, $r/M$ and $a/M$, from different starting relations:

(i) With knowledge of $|\Omega_\infty|$ and $|\kappa|$ one can start from equations (5) and (10) and find

$$\Delta = \frac{2\Omega^2_\infty + \kappa^2}{3\omega^2_r},$$

$$\frac{(\sqrt{Mr} + a)^2}{r^2} = \frac{\Omega^2_\infty - \kappa^2}{6\omega^2_r},$$

(11)

(12)

which can be solved with respect to $M/r$ and $a/r$:

$$M = r = 1 - \frac{\Omega^2_\infty + 5\kappa^2}{6\omega^2_r}$$

$$- 2\left(\frac{\Omega^2_\infty - \kappa^2}{6\omega^2_r}\right)^{1/2} \left(1 - \frac{\Omega^2_\infty + 2\kappa^2}{3\omega^2_r}\right)^{1/2},$$

(13)

$$a^2 = \frac{2\Omega^2_\infty + \kappa^2}{3\omega^2_r} + \frac{2M}{r} - 1.$$
The values of $r/M$ and $a/M$ are then reached by obvious manipulations. One more relation is needed to obtain the absolute values $M$, $a$, and $r$. Using equation (2), we fix
\[
\frac{1}{r} = \left( \frac{a}{r} \pm \sqrt{\frac{r}{M}} \right) \omega_{\pm},
\]
(15)

and thus the value of $M/r$ follows immediately. This can be compared with the result (13) to fix (or verify) the value of $\delta e$. (ii) Starting from equation (9), a convenient possibility is to combine it with equations (5) and (10). We find
\[
\delta^2 = \frac{2M}{3r} \left( \Omega_\infty^2 + \kappa^2 \right),
\]
(16)

and thus the value of $M/r$ follows immediately. This can be compared with the result (13) to fix (or verify) the value of $\delta e$. (iii) Suppose we know $\omega_{\pm}$, $|\Omega_\infty|$, and $|\delta|$. Introducing
\[
a = \omega_+^{-1} - y_{\pm}^{-1}
\]
(17)
fraction from equation (2) to equations (5) and (9), we obtain
\[
r^2 = \frac{3}{4 \omega_+ y_+ + \Omega_\infty^2 - 5 \omega_+^2},
\]
(18)
y_{\pm} is then given by quartic equation. This is rather cumbersome to be handled analytically and it is more suitable to find the solution numerically for each given set of data. However, a simple explicit solution can be found analytically if $r^2 > \kappa^2$; see the Appendix. (iv) The case when one measures $\omega_{\pm}$, $|\Omega_\infty|$, and $|\delta|$, and thus turns to equations (2), (10), and (9), also leads to a quartic equation.

In the Schwarzschild case, $a = 0$, equations (2) and (5) are reduced to
\[
\omega_{\pm} = y_{\pm} = \pm |\Omega_\infty| = \pm \sqrt{M/r^3},
\]
(19)
equation (9) reads
\[
\delta^2 = r^2 \omega_+^{-1} \left( \frac{1 - 2r^2 \omega_+^2}{1 - 3r^2 \omega_+^2} \right),
\]
(20)
and equation (10)
\[
\kappa^2 = \omega_+^2 (1 - 6M/r).
\]
(21)
The physical solution of equation (20) is
\[
r^2 = (4 \omega_+^2)^{-1} \left[ 3 \delta^2 + 1 - \sqrt{(2 \delta^2 + 1)^2 - 8 \delta^2} \right],
\]
(22)
while that of equation (21)
\[
r^2 = (6 \omega_+^2)^{-1} \left( 1 - \kappa^2 / \omega_+^2 \right);
\]
(23)
$M$ follows then from equation (19).

Note that the above formulas are written in geometrized units. Corresponding quantities in physical units are obtained by the following conversions:
\[
\frac{M^\text{phys}}{M_\odot} = \frac{M}{1.477 \times 10^8 \text{cm}^3},
\]
(24)
\[
a^\text{phys} = a, \quad r^\text{phys} = r;
\]
(25)
also,
\[
\frac{a}{M} = \frac{GM^\text{phys}}{c^2}, \quad \frac{r}{M} = \frac{GM^\text{phys}}{c^2}.
\]
(26)

To obtain the frequency in physical units [Hz] (either from $\omega$ or $\Omega$), one uses the relation
\[
f^\text{phys} = \frac{\omega^\text{phys}}{2\pi} = \frac{c}{2\pi} = (4.771 \times 10^9 \text{cm s}^{-1}) \omega.
\]
(27)

In the usual notation of relativistic astrophysics the (geometrized) frequencies are scaled by $M^{-1}$ (as in table 1). Their numerical values must be multiplied by the factor
\[
\frac{c}{2\pi M} = (3.231 \times 10^4) \left( \frac{M}{M_\odot} \right)^{-1} \text{[Hz]}.
\]
(28)

5. Discussion

We examined an interesting possibility of determining the intrinsic parameters of a black hole-accretion disk system in sources with periodic modulation of variability which is caused by an orbiting satellite. The method combines different pieces of information contained in a time-variable spectral and photometric signal, which have been discussed (widely but separately) in the recent literature. It can provide all the relevant quantities in physical units, but of course it can also be combined with another independent determination of some of the parameters. We assumed that the orbiting secondary is only weakly affected by the disk, which allowed us to approximate its orbit by a nearly equatorial spherical geodesic. This restriction must be abandoned if the secondary is not sufficiently compact.

The proposed approach can be applied to galactic nuclei as well as to stellar-mass objects in the Galaxy, but the identity and physical properties of the secondary depend on the mass of the centre around which it revolves. In the latter case the lifetime of the system is more restricted by tidal interactions and the secondary appears unlikely to survive sufficiently long. The effects of the tidal distortion of a stellar body passing near a black hole have been discussed by several authors, who show that the star becomes squeezed and heated (Rees 1988; Luminet, Marck 1985; Evans, Kochanek 1989). Tidal disruption occurs in the case of a close encounter, when the satellite plunges below the tidal radius.
\[ R_t \sim 10^{11} \left( \frac{M/M_*}{R_*/R_{g}} \right)^{1/3} \text{ cm}, \]

where \( M_* \) and \( R_* \) denote the mass and the radius of the satellite. On the other hand, the distortions of a compact satellite near to a supermassive black hole are negligible. The duration of a spot or a vortex in an accretion disk has also been discussed by several authors, but still remains uncertain (Adams, Watkins 1995; Bracco et al. 1998). We assumed that the lifetime exceeds the corresponding orbital period.

Several possible targets have already been discussed. For example, optical outbursts in the blazar OJ 287 have recently been modelled in terms of a black-hole binary system by Villata et al. (1998). The source exhibits several time-scales: a feature-less short-term variability, 12-yr cycle, and, possibly, a 60-yr cycle. These authors, however, presume both components to be of comparable masses, while in our calculation the frequencies are determined under the assumption that the secondary is much less massive than the primary (cf. also Sundelius et al. 1997). There are more BL Lac objects which show a periodic component of optical variability, but the time-scales are always \( \gtrsim 10 \) yr in the optical band, and statistical compilation of data is therefore very difficult (Liu et al. 1997). As a better example, a 16-hr periodicity was reported in the X-ray signal from the Seyfert galaxy IRAS 18325-5926 (Iwasawa et al. 1998).

We conclude with M. J. Rees (1998): "There is a real chance that someday observers will find evidence that an AGN is being modulated by an orbiting star, which would act as a test particle whose orbital precession would prove the metric in the domain where the distinctive features of the Kerr geometry should show up clearly." Indeed, to be able to resolve the effects of general relativity in AGN, the satellite would have to orbit near the centre and produce periodicity on a few-hours time-scale.

O.S. thanks the University of Padova and the International Centre for Theoretical Physics in Trieste for hospitality, and he acknowledges support from the grant GACR 202/99/0261 in Prague. F.de F. thanks for support from Agenzia Spaziale Italiana, Gruppo Nazionale per la Fisica Matematica del C.N.R., and Ministero della Ricerca Scientifica e Tecnologica of Italy. V. K. thanks for hospitality of the International School for Advanced Studies in Trieste; support from the grants GACR 205/97/1165 and 202/98/0522 is acknowledged.

Appendix. A Simple Explicit Solution of equations (2), (5), and (9) for \( r^2 \gg a^2 \)

It is relevant to suppose that the secondary orbits at \( r \gtrsim 10M \), and typically at \( r \sim 30M \). In this region a significant part of the disk radiation arises. Also, the secondary would be tidally disrupted if it plunges too close to the black hole horizon (to \( r \sim M \)), while at too large an orbital radius the relativistic dragging effect vanishes (with \( \propto r^{-3} \)) and the frequencies \( \omega_{\pm}, \Omega_{\infty} \) would become indistinguishable.

At large enough radii \( (r \sim 30M) \), several terms in equations (2), (5), and (9) become negligible when compared with the rest. Ignoring them, one arrives at a simplified set of relations which keeps an acceptable precision of the resulting parameters without the need to solve the fourth-order equation. (Exact equations can be used to improve the estimates numerically.) One chooses some particular approximation according to what precision is desired at each of the quantities \( M, a, r \), and also according to precision of each of the input data \( |\omega_{\pm}|, |\Omega_{\infty}|, |\delta| \). For instance, since \( ay_{\pm} \lesssim 1/30 \), we can take

\[ \omega_{\pm} \approx y_{\pm}(1 - ay_{\pm}), \quad y_{\pm} \approx \omega_{\pm}(1 + ay_{\pm}) \]  

(A1)

(for the determination of \( M \) from the known \( r \), or vice versa, it is even sufficient to neglect \( ay_{\pm} \) altogether and use just the Schwarzschild form of equations). Other two relevant bounds are:

\[ \frac{a^2}{r^2} < 1 \quad \frac{a^2}{r^2} \frac{a}{ay_{\pm}} = \frac{a}{M/r} \frac{a}{\sqrt{M/r}} < \frac{1}{3} \]  

(A2)

An example of a reasonable estimate is provided as follows (we suppose a prograde orbit of the secondary, and consequently we take plus sign in the relations): Re-
stricting to the Schwarzschild limit of equation (9),

$$\delta^2 = \frac{M}{r} - \frac{2M}{r} - \frac{3M}{r^2},$$  \hspace{1cm} (A3)

one comes to an approximative value for $M/r$ (the approximative values will be denoted by a tilde below),

$$4 \tilde{M}/r = 1 + 3 \delta^2 - \sqrt{(1 + 3 \delta^2)^2 - 8 \delta^2}. $$  \hspace{1cm} (A4)

An approximative expression for $a/r$ follows then from equation (5):

$$\frac{3 \tilde{a}}{r} = 2 \sqrt{\frac{M}{r} - \frac{4 \tilde{M}}{r} - 3 \left(1 - \frac{\Omega_{\infty}^2}{\omega_+^2}\right)}. $$  \hspace{1cm} (A5)

Finally, equation (2) yields $1/\tilde{r}$ according to the relation (15). The unknowns $\tilde{M}$, $\tilde{a}/\tilde{M}$, and $\tilde{r}/\tilde{M}$ are reached by obvious combinations.

The results of this approximation are illustrated in table 1 for several typical $\alpha$ and $r$, together with the respective values of the observable quantities. Here, $\alpha$ and $r$ are assumed to be given (in units of $M$; see the text above for conversions to physical units). For a prograde orbit of the secondary, $\omega_+$ and $\Omega_{\infty}(\omega_+)$ are the corresponding azimuthal and latitudinal angular frequencies, $\delta = \delta(\omega_+)$ is the dimensionless parameter defined by equation (9). From these quantities one computes the estimates $\tilde{a}$ and $\tilde{r}$ (in units of $M$) using our approximate equations (A4) and (A5). Notice that acceptable results are reached for $r \geq 10 M$ where approximative values are close to the simulated exact ones. The accuracy decreases if the secondary is orbiting too close. On the other hand, too distant orbits are also unsuitable: dragging effects quickly weaken with the distance from the centre and the values of $\omega_+$ and $\Omega_{\infty}$ become too close to each other.

In other words, the minimum sampling rate (inverse of the Nyquist critical frequency; Press et al. 1992) is restricted by the difference $\tau = \Omega_{\infty}^{-1} - \omega_+^{-1}$, requiring that both frequencies can be safely distinguished in the observed signal. For example, for $M = 10^8 M_{\odot}$, maximum rotation ($\alpha = 1$), and $r = 20$ gravitational radii, one can find (from table 1) $\tau = 1.8$, which in figure 1 corresponds to time interval $T - t = 2 \pi M c^{-1} \tau = 1.5$ hr.

References

Charles P.A. 1997, in Proc. 18th Texas Conference (World Scientific, Singapore) in press
Kato S., Fukue J., Mineshige S. 1998, Black-hole Accretion Disks (Kyoto University Press, Kyoto)
Miser C.W., Thorne K.S., Wheeler J.A. 1973, Gravitation (Freeman, San Francisco) ch3
Rees M.J. 1988, Nature 333, 523
Semenâk O., de Felice F. 1997, Class. Quantum Grav. 14, 2381
Stella L. 1990, Nature 344, 747
Appendix

© Astronomical Society of Japan • Provided by the NASA Astrophysics Data System