Noether Invariants and Complete Lie-Point Symmetries for Equations of the Hill Type

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We carry out a systematic analysis of second-order differential equations of the Hill type in the framework of the Lie group theory of point transformations. Both the homogeneous and the inhomogeneous cases are treated. We find the complete Lie-point symmetry group, to be associated with these equations. This group contains, as subgroups, $SO(2,1)$ and $E_2$, which are important in evaluating the energy spectrum, as well as the degeneracy of levels of quantum mechanical systems related to Hill equations. A set of Noether invariants which come from a symmetry subgroup endowed with five linearly independent generators is also determined.

§ 1. Introduction

Exact invariants (first integrals) are relevant in the study of both classical and quantum dynamical systems generally endowed with variable coefficients. The existence of a complete set of invariants of a given ordinary differential equation (ODE) enables one to solve the equation itself. Moreover, from the knowledge of at least one invariant one can reduce the order of the ODE, and may investigate the behavior of the underlying dynamical system by a qualitative point of view. The invariants can also be exploited to obtain the coherent states of oscillatory systems.

The construction of invariants of an ODE can be carried out by resorting to different techniques. However, among these a privileged role is surely played by the Lie group approach, by means of which one can determine the symmetry group of the dynamical system under consideration. In some cases of a physical significance, the (full) symmetry group contains a subgroup generated by infinitesimal operators related to invariants of the Noether type, together with a set of additional generators which do not lead to Noether invariants.

Although the Lie group methodology sometimes requires calculations less immediate than other procedures, it permits, on the other hand, not only to derive Noether invariants in an algorithmic way, but gives rise to additional generators (besides the Noether ones) providing further information on a given ODE. We can also point out that symmetry groups are essential in setting up programs of geometric quantization of the dynamical systems.

Keeping in mind the above considerations, in this work we perform a group analysis of both the homogeneous and inhomogeneous Hill equations, which appear in many branches of physics and describe, in particular, important dynamical systems such as the harmonic, the damped and the driven oscillators. In doing so, we have tried to minimize the mathematical difficulties inherent to the Lie group strategy,
adopting a terse formalism quite suitable for applications. The paper is organized as follows. In § 2 we outline some mathematical preliminaries characterizing the Lie group approach to the second order ODEs.

In § 3 we investigate the homogeneous Hill equation within the Lie group scheme. We obtain the (Lie-point) complete symmetry group and a set of five Noether invariants, where only two of them are functionally independent. Additional generators are also examined and examples of alternative Lagrangians related to the equation under consideration are displayed.

In § 4 an analysis similar to that carried out in § 3 is presented for the inhomogeneous Hill equation.

Finally, some concluding remarks are made in § 5, while Appendixes A and B contain details of the calculations.

§ 2. The Lie group approach

a. Divergence symmetries

Let us deal with the second order ordinary differential equation

\[ u_{xx} = F(x, u, u_x), \tag{2.1} \]

where \( u = u(x) \), \( u_x = du/dx \) and \( F \) is a given function.

Let us assume that (2.1) can be derived by the variational integral (action)

\[ S = \int_{x_1}^{x_2} \mathcal{L}(x, u, u_x) dx , \tag{2.2} \]

where \( \mathcal{L} \) is the (density) Lagrangian. In this case (2.1) is nothing but the Euler-Lagrange equation

\[ \frac{\delta S}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - D_x \frac{\partial \mathcal{L}}{\partial u_x} = 0 , \tag{2.3} \]

being \( D_x \) the total derivative with respect to \( x \).

Now consider a Lie group \( G \) of local point transformations, depending on one parameter \( \varepsilon \) and with non-zero Jacobian, acting on \( (x, u) \), namely

\[ x' = M(x, u; \varepsilon) , \tag{2.4a} \]

\[ u' = N(x, u; \varepsilon) , \tag{2.4b} \]

where the functions \( M \) and \( N \) are differentiable with respect to \( x \) and the value \( \varepsilon = 0 \) corresponds to the identity transformation \( x = M(x, u; 0) \), \( u = N(x, u; 0) \).\(^9\)

Equation (2.4) is generated by the infinitesimal operator (vector field)

\[ V = \xi(x, u) \partial_x + \phi(x, u) \partial_u , \tag{2.5} \]

where \( \partial_x = \partial/\partial x \), \( \partial_u = \partial/\partial u \) and

\[ \xi(x, u) = \frac{\partial M(x, u; \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon = 0} , \quad \phi(x, u) = \frac{\partial N(x, u; \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon = 0} . \tag{2.6} \]
If \( \varepsilon \) is a small quantity, Eq. (2.4) provides the infinitesimal transformations

\[
x' = x + \varepsilon \xi(x, u), \quad u' = u + \varepsilon \phi(x, u),
\]

(2.7)

at the first order in \( \varepsilon \).

The variation of \( u_x \) under (2.7) is given by

\[
\begin{align*}
u'_x &= u_x + \varepsilon \phi^x, \\
\end{align*}
\]

(2.8)

where \( u'_x = du'/dx' \), and

\[
\phi^x = \phi_x + [\phi_u - (\xi_x + \xi_u u_x)] u_x.
\]

(2.9)

Here, and in the following, subscripts affecting the generating functions \( \xi = \xi(x, u) \) and \( \phi = \phi(x, u) \) denote partial derivatives.

The group \( G \) can be extended so as to comprise the transformation of \( u_x \). Thus the finite transformations corresponding to (2.7) and (2.8) read

\[
\begin{align*}
x' &= \exp(\varepsilon V) x, \\
u' &= \exp(\varepsilon V) u, \\
u'_x &= \exp(\varepsilon pr^{(1)} V) u_x,
\end{align*}
\]

(2.10)

where

\[
pr^{(1)} V = V + \phi^x \partial_{ux},
\]

(2.11)

with \( \partial_{ux} = \partial/\partial u_x \), is the first prolongation of the infinitesimal generator \( V \).

The action of (2.10) on the Lagrangian \( \mathcal{L}(x, u, u_x) \) can be obtained keeping in mind that

\[
\mathcal{L}(x, u, u_x) = \{\exp[-\varepsilon pr^{(1)} V(x', u', u'_x)]\} \mathcal{L}(x', u', u'_x).
\]

(2.12)

Substitution from (2.12) into (2.2) yields

\[
S = \int_{x_1}^{x_2} \tilde{\mathcal{L}}(x', u', u'_x; \varepsilon) dx',
\]

(2.13)

where \( \tilde{\mathcal{L}} \) is defined by

\[
\tilde{\mathcal{L}}(x', u', u'_x; \varepsilon) = \{\exp[-\varepsilon pr^{(1)} V(x', u', u'_x)]\} \mathcal{L}(x', u', u'_x)(\partial_x + u'_x \partial_u) x.
\]

(2.14)

If it occurs that

\[
\tilde{\mathcal{L}}(x', u', u'_x; \varepsilon) = \mathcal{L}(x', u', u'_x),
\]

(2.15)

then the functional (2.2) is said to be invariant under the group of transformations (2.10). In this case the group \( G \) transforms solutions of the Euler-Lagrange equation (2.3) to solutions of the same equation. Here we can observe that this property is maintained by replacing (2.15) with the condition

\[
\tilde{\mathcal{L}}(x', u', u'_x; \varepsilon) = \mathcal{L}(x', u', u'_x) + D_x \varphi(x', u', \varepsilon),
\]

(2.16)

where \( \varphi \) is a differentiable function.

In many variational problems of physical interest, tackled in the context of the
Lie group theory, one needs to introduce the concept of a symmetry group for a functional of the following type\textsuperscript{11)

\[ S = \int_{x_1}^{x_2} \mathcal{L}(x, u, u_x, u_{xx}, \ldots) \, dx. \] (2.17)

For our purpose we limit ourselves in dealing with the action integral (2.2), where we recall that a vector field \( V \) of the form (2.5) is an infinitesimal divergence symmetry of \( S \) if there exists a function \( B(x, u) \) so that

\[ (pr^{(1)} V) \mathcal{L} + \mathcal{L} D_x \xi = \mathcal{L} B, \] (2.18)

for all \( x, u \). Each infinitesimal divergence symmetry generates a one-parameter group \( G_{\varepsilon} = \exp(\varepsilon V) \) of transformations on \( (x, u) \), which is called a symmetry group of the associated Euler-Lagrange equation. Finally, we remind the reader that if the Euler-Lagrange equation (2.3) admits a symmetry group \( G \), then the conservation equation

\[ D_x I = 0, \] (2.19)

where

\[ I = (\xi u_x - \phi) \partial_{ux} \mathcal{L} - \xi \mathcal{L} + B, \] (2.20)

holds, if and only if the action integral (2.2) is invariant with respect to the group \( G \). This is a generalization of the original Noether theorem.\textsuperscript{6,10,11)

As we shall see later, we point out that Eq. (2.18) can be used to derive the infinitesimal divergence symmetries of an ordinary differential equation of the form (2.1). In our applications, in order to distinguish between different kinds of symmetries possessed by (2.1), we shall name the Lie symmetry algebra and the related Lie symmetry group coming from (2.18), Noether symmetry algebra and Noether symmetry group, respectively. The conserved quantity (2.20) will be called Noether invariant.

b) **Complete Lie-point symmetries**

The relation (2.18) can be considered as the starting point to determine the Noether symmetries of Eq. (2.1). However, these are not all the symmetries exhibited by such an equation. Really, it is possible to look for additional symmetries which form, together with the Noether symmetries, the so-called (Lie-point) complete symmetry group of Eq. (2.1). This group, which contains as a subgroup the Noether symmetry group, is generated by infinitesimal operators (vector fields) of the form (2.5).

The complete symmetry group, which transforms solutions of Eq. (2.1) to other solutions,\textsuperscript{11} is defined as follows. Consider the group \( G \) of local point transformations (2.4). Then \( G \) is called a complete symmetry group of (2.1) if \( u'(x') = g \circ u(x) \) is a solution of (2.1) for \( g \in G \) so that \( g \circ u \) is defined whenever \( u(x) \) satisfies (2.1). (The symbol \( \circ \) denotes composition of functions.)

We remark that in our context, it is convenient to distinguish between Noether symmetries and additional symmetries. Both are Lie-point symmetries, in the sense
that the functions $\xi$ and $\phi$ appearing in the expression of the vector field (2.5) do not depend on derivatives of $u$. Anyway, as we shall see later, the Noether symmetries have the property of generating Noether invariants, while additional symmetries do not enjoy this feature.

The complete symmetry group $G$ for Eq. (2.1) can be obtained via an algorithmic procedure. This provides the Lie algebra of vector fields as (2.5), corresponding to the Lie group $G$. The coefficients $\xi$ and $\phi$ can be determined by the condition (see Ref. 11, p. 165)

$$\mathcal{P}^{(2)} V[u_{xx} - F(x, u, u_x)] = 0, \quad (2.21)$$

whenever $u_{xx} - F(x, u, u_x) = 0$, for every infinitesimal generator $V$ of $G$. $\mathcal{P}^{(2)} V$ is the second prolongation of $V$, defined by

$$\mathcal{P}^{(2)} V = V + \phi^x \partial_{u_x} + \phi^{xx} \partial_{u_{xx}}, \quad (2.22)$$

where $\phi^x$ is given by (2.9) and

$$\phi^{xx} = D_x^2(\phi - \xi u_x) + \xi u_{xxx}.$$ 

Equation (2.21), written explicitly, reads

$$\phi_x - 2\xi_x - 3\xi u_x)F_x - \xi F_x - \phi F_x - [\phi_x + (\phi_u - \xi)u_x - \xi u u_x^2]F_{ux}$$

$$+ \phi_{xx} + (2\phi_{ux} - \xi u_x)u_x + (\phi_{uu} - 2\xi u_x)u_x^2 - \xi uu_x^3 = 0. \quad (2.23)$$

This relation enables one, in theory, to yield all the Lie-point symmetries (including those of the Noether type) for a given differential equation of the form (2.1). Finally, we notice that the condition (2.23) arises from the requirement that Eq. (2.1) be form invariant under the Lie-point transformations (2.4) (see Appendix A).

### § 3. The homogeneous Hill equation

Many second-order ordinary differential equations which occur in various fields such as, for example, electromagnetism, quantum theory\(^{(12)}\) and accelerator physics\(^{(13)}\) can be cast in the standard form of a homogeneous Hill equation, namely

$$u_{xx} + \omega^2(x) u = 0, \quad (3.1)$$

where $\omega(x)$ is a known differentiable function. Therefore, the search for Noether invariants and the (Lie-point) complete symmetry algebra for Eq. (3.1) deserves a special attention.

In doing so, we need to exploit (2.18). Upon substitution of the Lagrangian

$$\mathcal{L} = (1/2)(u_x^2 - \omega^2(x)u^2) \quad (3.2)$$

related to Eq. (3.1), the explicit form of (2.18) turns out to be

$$-\xi \omega x u^2 - \phi \omega^2 u + [\phi_x + \phi u u_x - (\xi x + \xi u u_x)u_x]u_x$$

$$+ \frac{1}{2}(u_x^2 - \omega^2 u^2)(\xi x + \xi u u_x) = B_x + B u_x, \quad (3.3)$$
where $\omega_x = d\omega/dx$.

Equating the coefficients of powers of $u_x$ to zero, from (3·3) we are led to the expressions

$$\phi = \frac{1}{2} \xi u_x + \phi,$$
$$B = \frac{1}{4} \xi_x u_x^2 + \phi u_x,$$

where the functions $\xi$ and $\phi = \phi(x)$ obey the equations

$$\xi_{xxx} + 4 \omega \omega_x \xi_x + 4 \omega^2 \xi = 0,$$
$$\phi_{xx} + \omega^2(x)\phi = 0.$$

Now let us attempt to look for a solution of (3·6) in the form

$$\xi = \rho \left( c_1 \cos \theta + c_2 \sin \theta + c_3 \right),$$

where $\rho = \rho(x)$ and $\theta = \theta(x)$ are functions to be determined, and $c_1, c_2, c_3$ are arbitrary constants.

Inserting (3·8) into (3·6), after some manipulations we have that (3·8) fulfills (3·6), if

$$\sigma_{xx} + \omega^2(x)\sigma = \frac{c}{\delta^2},$$

where $c$ is a constant of integration, and

$$\rho = \delta^2, \quad \theta = \int \frac{dx}{\delta \xi}.$$

In a similar way we observe that Eq. (3·7) affords the solution

$$\phi = \eta \left( c_4 \cos a + c_5 \sin a \right),$$

where $c_4, c_5$ are arbitrary constants, and $\eta = \eta(x), a = a(x)$ are such that

$$\eta_{xx} + \omega^2(x)\eta = \frac{1}{\eta^2},$$
$$a = \int \frac{dx}{\eta^2}.$$

If we choose $c = 1/4$ in (3·9), then the functions $a, \theta$, and $\eta, \sigma$, are respectively mutually dependent, i.e.,

$$a = \frac{\theta}{2}, \quad \eta = \sqrt{2}\sigma.$$

Now we are ready to write down the general expression for the generator of the Noether symmetries. Bearing in mind (3·4), (3·8), (3·11) and (3·14), from (2·5) we get
\[ V = \sigma^2 (c_1 \cos \theta + c_2 \sin \theta + c_3) \partial_x + \left\{ \frac{1}{2} u \left[ c_1 (2 \sigma \sigma_x \cos \theta - \sin \theta) + c_2 (2 \sigma \sigma_x \sin \theta + \cos \theta) + 2 c_3 \sigma_x \right] + \sqrt{2} \sigma^2 \left( c_4 \cos \frac{\theta}{2} + c_5 \sin \frac{\theta}{2} \right) \right\} \partial_u, \quad (3.15) \]

where \( c_1, \ldots, c_5 \) are arbitrary constants. Thus the Noether symmetry algebra for the Hill equation (3.1) is spanned by the five vector fields

\[ V_1 = \sigma^2 \cos \theta \partial_x + \left( \sigma \sigma_x \cos \theta - \frac{1}{2} \sin \theta \right) \partial_u, \quad (3.16a) \]
\[ V_2 = \sigma^2 \sin \theta \partial_x + \left( \sigma \sigma_x \sin \theta + \frac{1}{2} \cos \theta \right) \partial_u, \quad (3.16b) \]
\[ V_3 = \sigma^2 \partial_x + \sigma \sigma_x u \partial_u, \quad (3.16c) \]
\[ V_4 = \sqrt{2} \sigma \cos \frac{\theta}{2} \partial_u, \quad (3.16d) \]
\[ V_5 = \sqrt{2} \sigma \sin \frac{\theta}{2} \partial_u, \quad (3.16e) \]

which come from (3.15) by setting \( c_1 = 1, c_i = 0 \ (i \neq 1), c_2 = 1, c_i = 0 \ (i \neq 2) \), and so on.

At this stage we can resort to Eq. (2.23) to find additional Lie-point symmetries for Eq. (3.1). To this end, let us put \( F = -\phi(x) u \). Then Eq. (2.23) becomes

\[ -\omega^2 u (\phi_u - 2 \xi_x - 3 \xi_u u_x) + 2 \omega \phi_x u^2 + \omega^2 \phi + (2 \phi_{ux} - \xi_{xx}) u_x + \phi_{xx} + (\phi_{uu} - 2 \xi_{ux}) u_x^2 - \xi_{uux} u_x^3 = 0, \quad (3.17) \]

that implies

\[ \xi = a_1 u + a_2, \quad (3.18) \]
\[ \phi = a_{1x} u^2 + b_1 u + b_2, \quad (3.19) \]

and

\[ a_{1xx} + \omega^2 a_1 = 0, \quad (3.20a) \]
\[ a_{2xx} - 2 b_{1x} = 0, \quad (3.20b) \]
\[ a_{2xx} + 4 \omega^2 a_{2x} + 4 \omega \phi_{x} a_2 = 0, \quad (3.20c) \]
\[ b_{2xx} + \omega^2 b_2 = 0, \quad (3.20d) \]

where \( a_1, a_2, b_1 \) and \( b_2 \) are functions of integration depending on the variable \( x \). Equation (3.20) can be solved to get exact analytical expressions for \( a_1, a_2, b_1 \) and \( b_2 \), provided that constraints like (3.9) and (3.12) are fulfilled. Consequently, (3.18) and (3.19) take the form

\[ \xi = \sqrt{2} \sigma \left( c_4 \cos \frac{\theta}{2} + c_5 \sin \frac{\theta}{2} \right) u + \sigma^2 (c_1 \cos \theta + c_2 \sin \theta + c_3), \quad (3.21) \]
\[ \phi = u^2 \left[ c_6 \left( \sqrt{2} \sigma \cos \theta \frac{\theta}{2} - \frac{1}{\sqrt{2}} \sigma \sin \frac{\theta}{2} \right) + c_7 \left( \sqrt{2} \sigma \sin \theta \frac{\theta}{2} + \frac{1}{\sqrt{2}} \sigma \cos \frac{\theta}{2} \right) \right] \]

\[ + u \left[ c_1 \left( \sigma \sigma_x \cos \theta - \frac{1}{2} \sigma \sin \theta \right) + c_5 \left( \sigma \sigma_x \sin \theta + \frac{1}{2} \sigma \cos \theta \right) + c_6 \sigma \sigma_x + c_8 \right] \]

\[ + \sqrt{2} \sigma \left( c_4 \cos \frac{\theta}{2} + c_8 \sin \frac{\theta}{2} \right), \quad (3 \cdot 22) \]

where the function \( \sigma \) and \( \theta \) are defined by (3 \cdot 10), and \( c_1, \ldots, c_8 \) are arbitrary constants.

The substitution of (3 \cdot 21) and (3 \cdot 22) in (2 \cdot 5) yields the generator of the complete Lie-point symmetries related to Eq. (3 \cdot 1). We deduce, by inspection, that this vector field includes the Noether vector field (3 \cdot 5), that is constituted by the terms involving the constants \( c_1, c_2, c_3, c_4, c_5 \), and the additional vector field, formed by the terms containing the constants \( c_6, c_7, c_8 \). Hence, the explicit expressions of the additional vector fields associated with Eq. (3 \cdot 1) are

\[ V_6 = \frac{\sqrt{2}}{\sigma} \left[ \sigma^2 \cos \frac{\theta}{2} u \partial_x + \left( \sigma \sigma_x \cos \frac{\theta}{2} - \frac{1}{2} \sigma \sin \frac{\theta}{2} \right) u^2 \partial_u \right], \quad (3 \cdot 23a) \]

\[ V_7 = \frac{\sqrt{2}}{\sigma} \left[ \sigma^2 \sin \frac{\theta}{2} u \partial_x + \left( \sigma \sigma_x \sin \frac{\theta}{2} + \frac{1}{2} \sigma \cos \frac{\theta}{2} \right) u^2 \partial_u \right], \quad (3 \cdot 23b) \]

\[ V_8 = u \partial_u. \quad (3 \cdot 23c) \]

Each infinitesimal operator \( V_1, \ldots, V_8 \) generates a one-parameter subgroup of Lie-point symmetries for the homogeneous Hill equation (3 \cdot 1). We find that the generators \( V_1, \ldots, V_8 \) satisfy the following commutation relations:

\[ [V_1, V_2] = V_3, \quad [V_1, V_3] = V_2, \quad [V_1, V_4] = \frac{1}{2} V_5, \]

\[ [V_1, V_5] = \frac{1}{2} V_4, \quad [V_2, V_3] = -V_1, \quad [V_2, V_4] = -\frac{1}{2} V_5, \]

\[ [V_2, V_5] = \frac{1}{2} V_4, \quad [V_3, V_4] = -\frac{1}{2} V_5, \quad [V_3, V_5] = \frac{1}{2} V_4, \]

\[ [V_4, V_5] = 0, \quad (3 \cdot 24) \]

together with

\[ [V_1, V_6] = \frac{1}{2} V_7, \quad [V_1, V_7] = \frac{1}{2} V_6, \quad [V_1, V_8] = 0, \]

\[ [V_2, V_6] = -\frac{1}{2} V_5, \quad [V_2, V_7] = \frac{1}{2} V_5, \quad [V_2, V_8] = 0, \]

\[ [V_3, V_6] = -\frac{1}{2} V_7, \quad [V_3, V_7] = \frac{1}{2} V_6, \quad [V_3, V_8] = 0, \]

\[ [V_4, V_6] = V_1 + V_3, \quad [V_4, V_7] = V_2 + \frac{3}{2} V_8, \quad [V_4, V_8] = V_4, \]
\[ [V_6, V_6] = V_2 - \frac{3}{2} V_8, \quad [V_6, V_7] = V_3 - V_1, \quad [V_6, V_8] = V_5, \]
\[ [V_6, V_7] = 0, \quad [V_6, V_8] = -V_6, \quad [V_7, V_8] = -V_7. \]  
(3·25)

The commutation relations (3·24) and (3·25) constitute the Lie-point complete symmetry algebra endowing with Eq. (3·1). The Noether symmetry algebra is given by (3·24). It is noteworthy to observe that the latter contains the noncompact subalgebra \( so(2, 1) \), defined by
\[ [V_i, V_z] = V_z, \quad [V_i, V_10] = V_i, \quad [V_z, V_10] = -V_i, \]
(3·26)

that is semisimple and whose Casimir invariant \( C \) reads
\[ C = V_z^2 - V_{10}^2. \]  
(3·27)

The symmetry group \( SO(2, 1) \), corresponding to \( so(2, 1) \), is just the dynamical group for Eq. (3·1) (see Ref. 14). This noncompact group is important in evaluating the energy spectrum as well as the degeneracy of the levels of quantum mechanical systems related to ordinary differential equations of the Hill type.\(^{12}\)

Another interesting property emerging from the Noether symmetry algebra is the following. The subalgebra defined by the commutation relations
\[ [V_3, V_4] = -V_2 V_5, \quad [V_3, V_5] = V_4, \quad [V_4, V_5] = 0, \]  
(3·28)

coming from (3·24), corresponds to the Euclidean group in the plane, \( E_2 \), which is not semisimple and is such that
\[ K = V_2^2 + V_5^2 \]  
(3·29)

commutes with all the generators \( V_3, V_4, V_5 \). We remark that \( E_2 \) arises as the contraction group of \( SO(2, 1) \) in the context when studying \( SO(2, 1) \)-type spectrum generating algebras.\(^{12,14}\)

Furthermore, we notice also that the additional vector fields \( V_6 \) and \( V_7 \) can be combined with the Noether generators \( V_3, V_4 \) and \( V_5 \) to provide the subalgebra underly ing the rotation group \( SO(3) \). Indeed, we have
\[ [Y_1, Y_2] = Y_5, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2, \]  
(3·30)

where
\[ Y_1 = V_5 - V_6, \quad Y_2 = V_4 + V_7, \quad Y_3 = 2 V_5. \]  
(3·31)

The Casimir invariant for \( so(3) \) (that is a semisimple Lie algebra) can be written as
\[ C = Y_1^2 + Y_2^2 + Y_3^2. \]  
(3·32)

The operators (3·27), (3·29) and (3·32) play a basic role in the study of spectrum generating algebras.\(^{12,14}\)

The Noether generators (3·16) allow us to write down a set of \( x \)-dependent constants of the motion for the Hill equation (3·1). In doing so, we need to use the expression (2·20), where the quantities \( \xi, \phi, \mathcal{L} \) and \( B \) are given by (3·8), (3·4), (3·2)
and (3.5), respectively. We get

\[ I_1 = \left[ \sigma^2 u_x^2 + \left( \sigma_x^2 - \frac{1}{4\sigma^2} \right) u^2 - 2\sigma x u u_x \right] \cos \theta + \left( u u_x - \frac{\sigma_x}{\sigma} u^2 \right) \sin \theta, \]
\[ I_2 = \left[ \sigma^2 u_x^2 + \left( \sigma_x^2 - \frac{1}{4\sigma^2} \right) u^2 - 2\sigma x u u_x \right] \sin \theta - \left( u u_x - \frac{\sigma_x}{\sigma} u^2 \right) \cos \theta, \]
\[ I_3 = \frac{u^2}{4\sigma^2} + (\sigma u_x - \sigma x u)^2, \]
\[ I_4 = \sqrt{2}(\sigma u - \sigma x u) \cos \frac{\theta}{2} - \frac{u}{\sqrt{2\sigma}} \sin \frac{\theta}{2}, \]
\[ I_5 = \sqrt{2}(\sigma u - \sigma x u) \sin \frac{\theta}{2} + \frac{u}{\sqrt{2\sigma}} \cos \frac{\theta}{2}. \]

Only two of these Noether invariants are functionally independent, i.e., \( I_4 \) and \( I_5 \). In fact, it turns out that

\[ I_1 = I_2 I_3, \quad I_2 = \frac{1}{2}(I_4^2 - I_5^2), \quad I_3 = \frac{1}{2}(I_4^2 + I_5^2). \]

The constant of the motion \( I_3 \) coincides with that already found by Ermakov \( ^{15-17} \) and, much later, by other authors. \( ^{4,19} \) We notice also that from (3.33d) and (3.33e) we can reproduce the general solution of Eq. (3.1), namely

\[ u = \sqrt{2\sigma} \left( I_5 \cos \frac{\theta}{2} - I_4 \sin \frac{\theta}{2} \right). \]

We point out that the role played by the Noether vector fields (3.16) is a well-established one, i.e., these operators generate a subgroup of the (Lie-point) symmetry group associated with the homogeneous Hill equation (3.1), leaving the action integral (2.17) invariant with \( \mathcal{L} \) given by (3.2).

Now we shall analyse the additional vector fields (3.23), which form a Lie subalgebra of the complete symmetry algebra (see (3.25)). The corresponding subgroup of this additional subalgebra does not preserve the action integral. However, the additional generators (3.23) lead to alternative Lagrangians which may be \( x \)-dependent and are equivalent to the Lagrangian (3.2), in the sense that these new Lagrangians give rise to the same Euler-Lagrange equation as (3.2). To show this, let us expand the transformed Lagrangian (2.14) to first order in the parameter \( \epsilon \). We obtain

\[ \mathcal{L}' = \mathcal{L} - \epsilon \mathcal{L}_1, \]

where

\[ \mathcal{L}_1 = (\rho \gamma^{(1)} V) \mathcal{L} + \mathcal{L} D_{\xi} \xi. \]

Then, if we introduce in (3.36) the vector field \( V_6 \) (see (3.23a)), we have

\[ \mathcal{L}_1 = \frac{1}{\sqrt{2}} \left[ 3\sigma x u (u_x^2 - u^2 u_x^2) - \sigma (u_x^3 + 3\sigma u^2 u_x + 2\omega u x u^3) \right] \cos \frac{\theta}{2}. \]
\[-\frac{3}{2\sqrt{2}} \frac{u}{\sigma}(u_x^2 - \omega^2 u^2) \text{sin}\frac{\theta}{2}, \quad (3.37)\]

and

\[\frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial \mathcal{L}}{\partial u} = 3I_4(u_{xx} + \omega^2 u), \quad (3.38)\]

where \( I_4 \) is the constant of the motion expressed by (3.33d).

Putting the right-hand side of (3.38) equal to zero, we deduce

\[I_4 = 0 \quad (3.39)\]

and

\[u_{xx} + \omega^2(x)u = 0 . \quad (3.40)\]

By virtue of (3.34), (3.39) implies

\[u = \sqrt{2I_4} \cos\frac{\theta}{2}, \quad (3.41)\]

which satisfies Eq. (3.1). On the other hand, Eq. (3.40) is just the Hill equation (3.1).

In other words, we may conclude that the transformed \( x \)-dependent Lagrangian (3.37) enables us to achieve the same solutions as the Lagrangian (3.2).

The analysis of the vector field \( V_7 \) produces a similar result.

When dealing with the operator \( V_8 \), a simple calculation provides (see (2.10) and (2.14)):

\[
\mathcal{L}' = [\exp(-2\varepsilon)]\mathcal{L}. \quad (3.42)
\]

The Lagrangians like (3.37) and (3.42), derivable by the additional vector fields \( V_6 \), \( V_7 \) and \( V_8 \), violate the condition (2.16) of invariance under the additional symmetry subgroups of the action integral related to Eq. (3.1). Consequently, no Noether invariant can be associated with \( V_6 \), \( V_7 \) and \( V_8 \). Anyway, the latter vectors preserve the property of transforming solutions of the homogeneous Hill equation (3.1) to solutions.

§ 4. The inhomogeneous Hill equation

The second-order ordinary differential equation

\[u_{xx} + g(x)u_x + \omega^2(x)u = f(x), \quad (4.1)\]

where \( g(x) \), \( \omega(x) \) and \( f(x) \) are given differentiable functions, has important applications in many areas of classical and quantum physics. Therefore, the search for its Noether invariants and its (Lie-point) symmetry group is of particular interest.

Equation (4.1) can be obtained via a variational problem by the Lagrangian

\[\mathcal{L} = \frac{1}{2}[\exp\mu](u_x^2 - \omega^2 u^2 + 2fu), \quad (4.2)\]
where \( \mu = \mu(x) \) is defined by
\[
\mu = \int g(x) \, dx . \tag{4.3}
\]

For our purposes, it is convenient to write Eq. (4.1) in the form of the inhomogeneous Hill equation
\[
v_{xx} + k^2(x) v = h(x) . \tag{4.4}
\]
This can be done through the transformation
\[
u = \left[ \exp \left( -\frac{\mu}{2} \right) \right] v , \tag{4.5}
\]
where the functions \( k = k(x) \) and \( h = h(x) \) are expressed by
\[
4k^2 = 4\omega^2 - g^2 - 2g_x \tag{4.6}
\]
and
\[
h = \left[ \exp \frac{\mu}{2} \right] f . \tag{4.7}
\]

With the help of (4.5) and (4.7), the Lagrangian (4.2) becomes
\[
\mathcal{L} = \frac{1}{2} \left[ v_x^2 - k^2 v^2 - \frac{1}{2} (v^2 g)_x + 2hv \right] . \tag{4.8}
\]

The Noether symmetries can be obtained inserting (4.8) into (2.18). Carrying out the calculations, we find
\[
\xi \left[ -\left( kk_x + \frac{1}{4} g_{xx} \right) v^2 - \frac{1}{2} g_x v v_x + h_x v \right] + \phi \left[ -\left( k^2 + \frac{1}{2} g_x \right) v - \frac{1}{2} g v_x + h \right] \nonumber
\]
\[
+ [\phi_x + (\phi_x - \xi_x) v x - \xi_x v x^2] \left( v_x - \frac{1}{2} g v \right) \nonumber
\]
\[
+ \frac{1}{2} (\xi_x + \xi_x v_x) \left[ v_x^2 - \left( k^2 + \frac{1}{2} g_x \right) v^2 - g v v_x + 2hv \right] = B_x + B_v v_x . \tag{4.9}
\]
Following a procedure similar to the one outlined for the homogeneous Hill equation, Eq. (4.9) provides the relations
\[
\xi_{xxx} + 4kk_x \xi + 4k^2 \xi_x = 0 , \tag{4.10}
\]
\[
\phi = \frac{1}{2} \xi_x v + \phi , \tag{4.11}
\]
\[
B = \frac{1}{4} \left[ \xi_x - (g \xi)_x \right] v^2 + \left( \phi_x - \frac{1}{2} g \phi \right) v + \beta , \tag{4.12}
\]
where the functions \( \phi = \phi(x) \) and \( \beta = \beta(x) \) are defined by
\[
\phi_{xx} + k^2 \phi = Q , \tag{4.13}
\]
The general solution of Eq. (4.10) is
\[
\xi = \gamma^2 (c_1 \cos \lambda + c_2 \sin \lambda + c_3),
\]
where \(c_1, c_2, c_3\) are arbitrary constants and the functions \(\gamma = \gamma(x)\) and \(\lambda = \lambda(x)\) are such that
\[
\gamma_{xx} + k^2 \gamma = \frac{1}{4} \gamma^2,
\]
\[
\lambda = \int \frac{dx}{\gamma^2}.
\]
Substituting (4.16) in (4.14) and solving Eq. (4.13) yields
\[
\psi = \sqrt{2} \gamma \left( c_4 \cos \frac{\lambda}{2} + c_5 \sin \frac{\lambda}{2} - \sqrt{2} \cos \frac{\lambda}{2} \int Q \gamma \sin \frac{\lambda}{2} dx + \sqrt{2} \sin \frac{\lambda}{2} \int Q \gamma \cos \frac{\lambda}{2} dx \right),
\]
where \(c_4\) and \(c_5\) are arbitrary constants, and the explicit expression of \(Q\) is (see (4.14)):
\[
Q = c_1 \left[ (\gamma^2 h_x + 3 \gamma \gamma_x) \cos \lambda - \frac{3}{2} h \sin \lambda \right]
+ c_2 \left[ (\gamma^2 h_x + 3 \gamma \gamma_x) \sin \lambda + \frac{3}{2} h \cos \lambda \right] + c_3 (\gamma^2 h_x + 3 \gamma \gamma_x h).
\]
By virtue of (4.11), (4.16), (4.19) and keeping in mind (2.5), we are led to the following set of Noether vector fields for equation (4.4):
\[
V_1 = \gamma^2 \cos \lambda \partial_x + \left[ v \left( \gamma \gamma_x \cos \lambda - \frac{1}{2} \sin \lambda \right) + \phi_1 \right] \partial_v,
\]
\[
V_2 = \gamma^2 \sin \lambda \partial_x + \left[ v \left( \gamma \gamma_x \sin \lambda + \frac{1}{2} \cos \lambda \right) + \phi_2 \right] \partial_v,
\]
\[
V_3 = \gamma^2 \partial_x + (v \gamma \gamma_x + \phi_3) \partial_v,
\]
\[
V_4 = \phi_4 \partial_v,
\]
\[
V_5 = \phi_5 \partial_v,
\]
where
\[
\phi_j = 2 \gamma \left( -\cos \frac{\lambda}{2} \int Q \gamma \sin \frac{\lambda}{2} dx + \sin \frac{\lambda}{2} \int Q \gamma \cos \frac{\lambda}{2} dx \right), \quad j = 1, 2, 3,
\]
\[
\phi_4 = \sqrt{2} \gamma \cos \frac{\lambda}{2}, \quad \phi_5 = \sqrt{2} \gamma \sin \frac{\lambda}{2}.
\]
\[ Q_1 = Q_0 \cos \lambda - \frac{3}{2} \gamma \sin \lambda, \quad (4.23a) \]
\[ Q_2 = Q_0 \sin \lambda + \frac{3}{2} \gamma \cos \lambda, \quad (4.23b) \]
\[ Q_3 = \gamma^2 h_z + 3 \gamma \gamma_h. \quad (4.23c) \]

To find additional symmetry generators (besides the Noether ones) for Eq. (4.4), we have to exploit Eq. (2.23), where now \( F = -k^2 v + h \). After some manipulations, we arrive at the quantities
\[ \xi = a_1 v + a_2, \quad (4.24) \]
\[ \phi = a_1 v^2 + b_1 v + b_2, \quad (4.25) \]
for the generating functions involved in the infinitesimal operator (2.5), where \( a_1, a_2, b_1 \) and \( b_2 \) satisfy the differential equations
\[ a_{1xx} + k^2 a_1 = 0, \quad (4.26a) \]
\[ 2b_{1x} = a_{2xx} + 3a_1 h, \quad (4.26b) \]
\[ a_{2xxx} + 4kk_x + 4k^2 a_{2x} = -(3a_1 h + a_1 h_x), \quad (4.26c) \]
\[ b_{2xx} + k^2 b_2 = a_2 h_z + (2a_{2x} - b_1) h. \quad (4.26d) \]

These allow the solutions
\[ a_1 = \sqrt{2} \gamma \left( c_0 \cos \frac{\lambda}{2} + c_1 \sin \frac{\lambda}{2} \right), \quad (4.27) \]
\[ a_2 = \gamma^2 \left[ c_1 \cos \lambda + c_2 \sin \lambda + c_3 - \cos \lambda \int R \sin \lambda dx + \sin \lambda \int R \cos \lambda dx \right], \quad (4.28) \]
\[ b_1 = \frac{1}{2} a_{2x} + \frac{3}{2} \int a_1 h dx + c_8, \quad (4.29) \]
\[ b_2 = \sqrt{2} \gamma \left[ c_4 \cos \frac{\lambda}{2} + c_6 \sin \frac{\lambda}{2} - \sqrt{2} \cos \frac{\lambda}{2} \int S \gamma \sin \frac{\lambda}{2} dx + \sqrt{2} \sin \frac{\lambda}{2} \int S \gamma \cos \frac{\lambda}{2} dx \right], \quad (4.30) \]

where \( c_1, \ldots, c_8 \) are arbitrary constants, and
\[ R = \frac{1}{\gamma^2} \int P \gamma^2 dx, \quad (4.31) \]
\[ P = -c_6 \left[ \sqrt{2} (\gamma h_z + 3 \gamma \gamma h) \cos \frac{\lambda}{2} - \frac{1}{\sqrt{2} \gamma} \sin \frac{\lambda}{2} \right] \]
\[ + c_1 \left[ -\sqrt{2} (\gamma h_z + 3 \gamma \gamma h) \sin \frac{\lambda}{2} + \frac{1}{\sqrt{2} \gamma} \cos \frac{\lambda}{2} \right], \quad (4.32) \]
\[ S = a_2 h_z + (2a_{2x} - b_1) h. \quad (4.33) \]

The generating functions (4.24) and (4.25) related to the additional vector fields \( V_6 \),
$V_6$ and $V_8$ for Eq. (4.4), can be derived by choosing, respectively, $c_6=1$, $c_j=0$ ($j\neq 6$); $c_7=1$, $c_j=0$ ($j\neq 7$), and $c_8=1$, $c_j=0$ ($j\neq 8$) in the expressions (4.27), (4.28), (4.29) and (4.30). Thus from (2.5) we get

$$V_6 = \left[ \sqrt{2}\gamma \cos \frac{A}{2} + \gamma^2 \left( -\cos \lambda \int R_1 \sin \lambda dx + \sin \lambda \int R_1 \cos \lambda dx \right) \right] \partial_x$$

$$+ \left\{ \left( \sqrt{2} \gamma \cos \frac{A}{2} - \frac{1}{\sqrt{2} \gamma} \sin \frac{A}{2} \right) v^2 + \frac{v^2}{2} \left( -2 \gamma \gamma \cos \lambda + \sin \lambda \right) \int R_1 \sin \lambda dx \right\}$$

$$+ (2 \gamma \gamma \sin \lambda + \cos \lambda) \int R_1 \cos \lambda dx + 3 \sqrt{2} \int h \gamma \cos \frac{A}{2} dx \right\}$$

$$+ 2 \gamma \left( -\cos \frac{A}{2} \int S_1 \gamma \sin \frac{A}{2} dx + \sin \frac{A}{2} \int S_1 \gamma \cos \frac{A}{2} dx \right) \partial_x, \quad (4.34a)$$

$$V_7 = \left[ \sqrt{2} \gamma \sin \frac{A}{2} + \gamma^2 \left( -\cos \lambda \int R_2 \sin \lambda dx + \sin \lambda \int R_2 \cos \lambda dx \right) \right] \partial_x$$

$$+ \left\{ \left( \sqrt{2} \gamma \sin \frac{A}{2} + \frac{1}{\sqrt{2} \gamma} \cos \frac{A}{2} \right) v^2 + \frac{v^2}{2} \left( -2 \gamma \gamma \cos \lambda + \sin \lambda \right) \int R_2 \sin \lambda dx \right\}$$

$$+ (2 \gamma \gamma \sin \lambda + \cos \lambda) \int R_2 \cos \lambda dx + 3 \sqrt{2} \int h \gamma \sin \frac{A}{2} dx \right\}$$

$$+ 2 \gamma \left( -\cos \frac{A}{2} \int S_2 \gamma \sin \frac{A}{2} dx + \sin \frac{A}{2} \int S_2 \gamma \cos \frac{A}{2} dx \right) \partial_x, \quad (4.34b)$$

$$V_8 = \left[ \gamma + 2 \left( \cos \frac{A}{2} \int h \gamma \sin \frac{A}{2} dx + \sin \frac{A}{2} \int h \gamma \cos \frac{A}{2} dx \right) \right] \partial_x, \quad (4.34c)$$

where $R_1$, $S_1$ and $R_2$, $S_2$ are given by (4.31) and (4.33) by putting $c_6=1$, $c_j=0$ ($j\neq 6$), and $c_7=1$, $c_j=0$ ($j\neq 7$), respectively.

The constants $c_1$, $c_4$, $c_2$, $c_3$ and $c_5$ refer to the Noether vector fields (4.21) previously found by means of (2.18).

The vector fields (4.21) and (4.34) constitute the (Lie-point) complete symmetry algebra for the inhomogeneous Hill equation (4.4). It can be shown straightforwardly that such vector fields satisfy the same commutation relations (3.24) and (3.25) which hold for the case of the homogeneous Hill equation (3.1).

Proceeding in a manner similar to that adopted in the study of Eq. (3.1), the generators (4.21) enable us to derive a set of constants of the motion for Eq. (4.4). Omitting any calculation for the sake of brevity, these read (see (2.20)):

$$I_1 = \frac{1}{2} \cos \lambda \left[ (\gamma v_x - \gamma_x v)^2 - \frac{1}{4 \gamma_x^2} v^2 - 2 \gamma^2 h v \right]$$

$$+ \frac{1}{2} \sin \lambda \left( v v_x - \frac{\gamma_x}{\gamma} v^2 \right) + \phi_1 v - \phi_1 v_x + \beta_1, \quad (4.35a)$$

$$I_2 = \frac{1}{2} \sin \lambda \left[ (\gamma v_x - \gamma_x v)^2 - \frac{1}{4 \gamma_x^2} v^2 - 2 \gamma^2 h v \right]$$
Noether Invariants and Complete Lie-Point Symmetries

\[ + \frac{1}{2} \cos \lambda \left( -uvx + \frac{\gamma^2}{\gamma} \right) + \phi_2 v - \phi_2 v_2 + \beta_2, \quad (4.35b) \]

\[ I_3 = \frac{1}{2} (\gamma v_x - \gamma v_v)^2 + \frac{1}{8} \gamma^2 v^2 - \gamma^2 hv + \phi_3 v - \phi_3 v_3 + \beta_3, \quad (4.35c) \]

\[ I_4 = \phi_4 v - \phi_4 v_4 + \beta_4, \quad (4.35d) \]

\[ I_5 = \phi_5 v - \phi_5 v_5 + \beta_5, \quad (4.35e) \]

where \( \phi_1, \ldots, \phi_5 \) are given by (4.22), and

\[ \beta_j = \int \hbar \phi_j dx. \quad (j=1, \ldots, 5) \quad (4.36) \]

Only two of the Noether invariants (4.35) are functionally independent. In fact, we can see that

\[ I_1 = \frac{1}{4} (I_4^2 - I_5^2), \quad I_2 = \frac{1}{2} I_4 I_5, \quad I_3 = \frac{1}{4} (I_4^2 + I_5^2) \quad (4.37) \]

(see Appendix B).

From (4.35d) and (4.35e) we deduce the general solution of Eq. (4.4), that is

\[ v = \sqrt{2} \gamma \left( I_5 \cos \frac{\lambda}{2} - I_4 \sin \frac{\lambda}{2} \right) + 2 \gamma \left( \sin \frac{\lambda}{2} \int h \gamma \cos \frac{\lambda}{2} dx - \cos \frac{\lambda}{2} \int h \gamma \sin \frac{\lambda}{2} dx \right), \quad (4.38) \]

where \( \gamma \) and \( \lambda \) are defined by (4.17) and (4.18).

Finally, we observe that no Noether invariant corresponds to each additional vector field (4.34). However, as it happens for the homogeneous Hill equation (3.1), the operators (4.34) lead to Lagrangians which transform solutions of the inhomogeneous Hill equation (4.4) to solutions.

§ 5. Conclusion

We have examined the homogeneous and the inhomogeneous Hill equations in the framework of the Lie group theory of point transformations. The Lie group approach is particularly effective, since it furnishes both a set of \( x \)-dependent Noether invariants and the complete Lie-point symmetry group associated with the predicted equations. The underlying complete symmetry algebra turns out to be of the SL(3, \( R \)) type.\(^{18,19} \) It contains the noncompact subalgebra so(2, 1), corresponding to the dynamical group SO(2, 1), and the subalgebra related to \( E_2 \), the Euclidean group in the plane. Furthermore, combining together certain generators of the complete symmetry group, the three-dimensional rotation group is obtained. The behavior of the additional generators (besides the Noether ones) has also been investigated. The former does not generate Noether invariants, but allows us to find new alternative Lagrangians equivalent to the original ones, which transform solutions of the Hill equations to solutions.

Noether invariants yield the general solutions of the Hill equations (3.1) and (4.4) (see (3.34) and (4.38)). Formulae (3.34) and (4.38) can be easily applied to reproduce
some cases of special physical interest, as for example the harmonic, the damped and the driven oscillators. In such cases equations (3·9) and (4·17) can be trivially solved.

Considering equations (3·9) and (4·17), we point out that they can be regarded as Bohl differential equations\(^\text{20}\) corresponding to equations of the form

\[
X'' + m(x)X = 0, \tag{5·1}
\]

where \(X = X(x)\), and \(m(x)\) is a given function. In other words, let us suppose that \(X_1(x)\) and \(X_2(x)\) are two independent solutions of Eq. (5·1). Then the transformation

\[
Y = (X_1^2 + X_2^2)^{\frac{1}{2}},
\]

which is called the Bohl transformation of the pair \(X_1\) and \(X_2\), relates the linear differential equation (5·1) to the nonlinear one

\[
Y'' + m(x)Y = \frac{1}{Y}. \tag{5·2}
\]

The Bohl transformation is useful in establishing oscillation criteria for differential systems like (5·1).\(^\text{19}\)

To conclude, we must remember an important question which is open and lies within the context of the present work. The question concerns the existence of a possible connection between the Lie classification of second order ordinary differential equations of the type (2·1) and the Painlevé classification.\(^\text{21}\) Studies of this problem are still in their infancy,\(^\text{20}\) and deserve further investigation.

\section*{Appendix A}

Let us assume that the Lie group \(G\) of transformations (2·4) leaves Eq. (2·1) form invariant. (In this case one says that Eq. (2·1) admits \(G\) as a symmetry group.) The generator of \(G\), given by (2·5), is determined by the functions \(\xi\) and \(\phi\). To find these, we start from the infinitesimal transformations (2·7), from which

\[
dx' = dx + \varepsilon(\xi dx + \xi_u du), \quad du' = du + \varepsilon(\phi dx + \phi_u du). \tag{A·1}
\]

Consequently, we obtain up to the first order in the parameter \(\varepsilon\)

\[
u_1' = \nu_1 + \varepsilon[\phi_1 + (\phi_1 - \xi_1)\nu_1 - \xi_1\nu_1^2], \tag{A·2}
\]

\[
u_2' = \nu_2 + \varepsilon[(\phi_2 - 2\xi_2 - 3\xi_2\nu_1)\nu_2 + \phi_2 + (2\phi_2 - \xi_2)\nu_2
\]

\[+ (\phi_2 - 2\xi_2)\nu_2^2 - \xi_2\nu_1\nu_2^2]. \tag{A·3}
\]

Now the requirement that Eq. (2·1) be invariant under the transformations (2·4) implies

\[
u_1'' = F(x', u', u_2'). \tag{A·4}
\]

Taking into account the infinitesimal transformations (2·7) and expanding in series the right-hand side of (A·4) up to the first order in the parameter \(\varepsilon\), we get

\[
u_1'' = \nu_1 + \varepsilon[\xi F + \phi F_u + [\phi + (\phi - \xi)\nu_1 - \xi u_2^2] F_u], \tag{A·5}
\]

from which Eq. (2·23) arises with the help of (A·3).
Appendix B

In order to demonstrate the relations (4·37), it is convenient to write explicitly the constants of the motion \( I_4 \) and \( I_5 \) (see (4·35)), namely

\[
I_4 = \sqrt{2} \left[ (\gamma \cos \frac{\lambda}{2} - \frac{1}{2\gamma} \sin \frac{\lambda}{2}) v - \nu \cos \frac{\lambda}{2} + \int h \gamma \cos \frac{\lambda}{2} dx \right],
\]

\[
I_5 = \sqrt{2} \left[ (\gamma \sin \frac{\lambda}{2} + \frac{1}{2\gamma} \cos \frac{\lambda}{2}) v - \nu \sin \frac{\lambda}{2} + \int h \gamma \sin \frac{\lambda}{2} dx \right].
\]

Furthermore, we observe that

\[
\int \gamma Q_3 \sin \frac{\lambda}{2} dx = r_3 \sin \frac{\lambda}{2},
\]

\[
\int \gamma Q_3 \cos \frac{\lambda}{2} dx = r_3 \cos \frac{\lambda}{2},
\]

\[
\int \gamma (Q_3 \cos \frac{3\lambda}{2} - \frac{3}{2} r_3 \sin \frac{3\lambda}{2}) dx = r_3 \cos \frac{3\lambda}{2},
\]

\[
\int \gamma (Q_3 \sin \frac{3\lambda}{2} + \frac{3}{2} r_3 \cos \frac{3\lambda}{2}) dx = r_3 \sin \frac{3\lambda}{2},
\]

where \( Q_3 \) is expressed by (4·23c).

Equations (B·3) can be exploited to put \( \phi_1 \), \( \phi_2 \) and \( \phi_3 \) (see (4·22a)) in the form

\[
\phi_1 = \gamma \left[ \cos \frac{\lambda}{2} \int h \gamma \cos \frac{\lambda}{2} dx - \sin \frac{\lambda}{2} \int h \gamma \sin \frac{\lambda}{2} dx \right],
\]

\[
\phi_2 = \gamma \left[ \cos \frac{\lambda}{2} \int h \gamma \sin \frac{\lambda}{2} dx + \sin \frac{\lambda}{2} \int h \gamma \cos \frac{\lambda}{2} dx \right],
\]

\[
\phi_3 = \gamma \left[ \cos \frac{\lambda}{2} \int h \gamma \cos \frac{\lambda}{2} dx + \sin \frac{\lambda}{2} \int h \gamma \sin \frac{\lambda}{2} dx \right].
\]

Indeed, consider for instance \( \phi_1 \), defined by (4·29a) (for \( j=1 \)), i.e.,

\[
\phi_1 = 2\gamma \left( -\cos \frac{\lambda}{2} \int Q_3 \gamma \sin \frac{\lambda}{2} dx + \sin \frac{\lambda}{2} \int Q_3 \gamma \cos \frac{\lambda}{2} dx \right).
\]

From (4·23a) and with the help of the identities

\[
\sin^3 \frac{\lambda}{2} = 3\sin \frac{\lambda}{2} - 4\sin^3 \frac{\lambda}{2}, \quad \cos^3 \frac{\lambda}{2} = -3\cos \frac{\lambda}{2} + 4\cos^3 \frac{\lambda}{2},
\]

we have

\[
2Q_3 \sin \frac{\lambda}{2} = -\left( Q_3 \sin \frac{\lambda}{2} + \frac{3}{2} h \cos \frac{\lambda}{2} \right) + Q_3 \left( \sin \frac{3\lambda}{2} + \frac{3}{2} h \cos \frac{3\lambda}{2} \right),
\]

\[
2Q_3 \cos \frac{\lambda}{2} = Q_3 \cos \frac{\lambda}{2} - \frac{3}{2} h \sin \frac{\lambda}{2} + Q_3 \left( \cos \frac{3\lambda}{2} - \frac{3}{2} h \sin \frac{3\lambda}{2} \right).
\]
Inserting (B·8) and (B·9) in (B·7) and taking account of (B·3), we easily find (B·4). Analogous considerations lead to (B·5) and (B·6).

Now, due to formulae (B·4), (B·5) and (B·6) we get

\[ \beta_1 = \int \! \beta \phi_1 \, dx = \frac{1}{2} \left( \int \! \beta \gamma \cos \frac{\lambda}{2} \, dx \right)^2 - \frac{1}{2} \left( \int \! \beta \gamma \sin \frac{\lambda}{2} \, dx \right)^2, \]

(B·10)

\[ \beta_2 = \int \! \beta \phi_2 \, dx = \left( \int \! \beta \gamma \cos \frac{\lambda}{2} \, dx \right) \left( \int \! \beta \gamma \sin \frac{\lambda}{2} \, dx \right), \]

(B·11)

\[ \beta_3 = \int \! \beta \phi_3 \, dx = \frac{1}{2} \left( \int \! \beta \gamma \cos \frac{\lambda}{2} \, dx \right)^2 + \frac{1}{2} \left( \int \! \beta \gamma \sin \frac{\lambda}{2} \, dx \right)^2. \]

(B·12)

Then the result (4·37) can be achieved directly from (B·1) and (B·2) by the use of (B·4)∼(B·6) and (B·10)∼(B·12).

References

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