Clebsch-Gordan Formulas of the SU(1, 1) Group

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The Clebsch-Gordan formulas for d-function are studied for both nonunitary finite dimensional representation and unitary infinite dimensional representation of the SU(1, 1) group. It is shown that, in addition to the usual Clebsch-Gordan formulas for the above-mentioned two representations, there exists an additional type of Clebsch-Gordan formula connecting the nonunitary and unitary d-functions. The new kind of Clebsch-Gordan coefficient of the SU(1, 1) obtained in our previous paper arises naturally in this formula. The three Clebsch-Gordan formulas are proved to be interrelated with each other by analytic continuation. It is, further, pointed out that all the recurrence formulas of the unitary d-functions—obtainable from the known recurrence relations of Jacobi polynomials—belong actually to this new type of Clebsch-Gordan formula.

Explicit algebraic formulas of d-functions are derived by employing an elementary algebraic method analogous to the standard treatment of angular momentum.

§ 1. Introduction

The coupling rule of d-function of the SU(2) group is well known as Clebsch-Gordan formula: it reads in obvious notations as

\[
d^{(2)}_{m_1 m_2} (\beta) = \sum_{m_1' m_2'} (j_1 j_2 m_1 m_2 | jm) (j_1 j_2 m_1' m_2' | jm') d^{(2)}_{m_1' m_2'} (\beta) \cdot d^{(2)}_{m_1 m_2} (\beta).
\]

Besides its fundamental importance from the standpoint of group theory, this formula can be conveniently used in obtaining explicit algebraic forms of the d-functions. Indeed, all the recurrence relations of d-functions can be regarded as special cases of the above formula.

For the SU(1, 1) group, it is clear that such a formula should also exist for a given type of representation such as, for example, for the finite dimensional nonunitary representation and for the unitary—necessarily infinite dimensional—representation.

It is one of the purposes of present paper to show that there exists an additional type of Clebsch-Gordan formula in the SU(1, 1) which connects the nonunitary and unitary d-functions. Any recurrence formulas—derivable directly from the known properties of Jacobi polynomials—of the d-functions of unitary representation of the SU(1, 1) belong actually to this new type of Clebsch-Gordan formula. The new kind of Clebsch-Gordan coefficient obtained in our SU(1, 1) Quasi-Spin Formalism of Many-Boson System arises naturally in this formula.

In the next section, a brief introduction to the SU(1, 1) group is presented. The explicit algebraic formulas of d-functions are derived in § 3 for both non-
unitary finite dimensional and unitary infinite dimensional representations. Although a part of the results given there has been known, we present these for the sake of methodological interest. That is, we derive all the results by employing an elementary algebraic method in a manner just analogous to the standard treatment of angular momentum. The symmetry properties of $d$-functions are discussed in detail. The Clebsch-Gordan formulas are obtained in §4. It is pointed out in §5 that the three types of the Clebsch-Gordan formulas derived in §4 are interrelated with each other by simple analytic continuations.

Of the two classes of the unitary representation—the discrete and continuous—of the $SU(1, 1)$, the present paper concerns discrete series only. It is the discrete series that plays an essential role in many-boson problem.\(^{43}\)

A remark is presented in §4C on the importance of the finite dimensional nonunitary representation in physics.

§ 2. The $SU(1, 1)$ group

The $SU(1, 1)$ is the twice-covering group of the well-known three-dimensional Lorentz group $O(2, 1)$: its Lie algebra is defined by

$$[K_3, K_+] = K_+$$

and

$$[K_3, K_-] = -2K_3,$$  \hspace{1cm} (2.1)

and Casimir operator is given by

$$I_5 = -K_x^2 - K_y^2 + K_z^2,$$  \hspace{1cm} (2.2)

where $K_3 = K_x \pm iK_y$.

Throughout the present paper, $K_3$ will always be taken to be hermitian, so that its eigenvalues are real. Let us discuss briefly the representations of (2.1) where $K_3$ is diagonal.

2A. Finite dimensional representations

Since the Lie algebra (2.1) is a complex extension of that of the $SU(2)$, the finite dimensional nonunitary representations of the $SU(1, 1)$ can be obtained from the corresponding unitary representations of the $SU(2)$ by taking $J_+$ and $J_-$ as mutually anti-hermitian. Therefore, the matrix elements of the generators in the $(2J + 1)$ dimensional representation can be taken to be

$$K_3 |JM\rangle = M |JM\rangle$$  \hspace{1cm} (2.3a)

and

$$K_\pm |JM\rangle = \mp \sqrt{(J \mp M)(J \pm M + 1)} |JM \pm 1\rangle,$$  \hspace{1cm} (2.3b)

\(^{43}\) Our approach could equally be applied to the continuous series. However, as will be clear later, we first need a detailed knowledge of the relevant Clebsch-Gordan coefficients. Since we have not yet generalized our new kind of Clebsch-Gordan coefficient to include the continuous series, we shall at present leave the continuous series untouched. It is to be noted that the continuous series, mainly the principal series, play a predominant role in the so-called harmonic analysis of $S$-matrix.\(^{9}\) In this case, however, we usually need not a detailed algebraic form of the $d$-function.
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from which we fix our phase. With this choice of phase, the 2×2 matrix representation (d-function) of \(\exp(-i\beta K_\theta)\) will be

\[
d^{(\frac{1}{2})} (\beta) = \begin{pmatrix} \cosh \frac{\beta}{2} & \sinh \frac{\beta}{2} \\ \sinh \frac{\beta}{2} & \cosh \frac{\beta}{2} \end{pmatrix}.
\]

(2.4)

It is to be noted that our phase convention corresponds just to the replacements

\[J_z \leftrightarrow K_3, \quad K_\theta \leftrightarrow -iJ_y, \quad K_\phi \leftrightarrow iJ_z\]

in the SU(2) Lie algebra,

\[[J_\pm, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_z.
\]

Therefore, (2.4) may be written simply as

\[d^{(\frac{1}{2})} (\beta) = e^{(\beta/2)\sigma_\theta}.
\]

2B. Unitary representations

Since the SU(1, 1) is noncompact, its unitary representations are always infinite dimensional except for the trivial identity representation. The unitary representations have been investigated in detail first by V. Bargmann\(^4\) and later by many authors. We refer here to only one article by Biedenharn and his coworkers\(^5\) who employed an algebraic approach to this problem. Our treatment is along the same spirit as theirs.

The unitary irreducible representation of the SU(1, 1) is usually classified into two classes, the discrete and continuous series. The former is further divided into two classes, the positive discrete series and the negative discrete series.

The positive discrete series consist of the infinite chain of states; \(|\mathcal{N}\rangle, |\mathcal{N}+1\rangle, |\mathcal{N}+2\rangle, \ldots\). Each of these states will be denoted by \(^6\)

\[|\mathcal{N}\mathcal{M}\rangle, \quad (\mathcal{N}>0)
\]

(2.5)

where the eigenvalue of \(K_\mathcal{N}\mathcal{M}\), is given by \(\mathcal{N}+n (n=0, 1, 2, \ldots)\). As will be discussed later, \(\mathcal{N}\) is in general real positive. The eigenvalue of Casimir operator can be written in terms of \(\mathcal{N}\) as

\[I_2 = \mathcal{N} (\mathcal{N} - 1).
\]

(2.6)

Each multiplet is labeled by \(\mathcal{N}\) and a state belonging to this multiplet is labeled by \(\mathcal{M}\).

The matrix elements of the generators of the group can be determined through (2.2) and (2.6) as

\(^4\) Throughout the present paper, the unitary representation will be denoted by German letters \(\mathcal{N}\) and \(\mathcal{M}\), while the finite dimensional representation will be denoted by \(J\) and \(M\). The representation of the SU(2) will be described by lower case letters \(j\) and \(m\).
The last equation (2.7d) is the definition of the positive discrete series.
The negative discrete series consist of the infinite chain of states: \(|-\mathfrak{m}\rangle, \quad |\mathfrak{m}-1\rangle, \quad |\mathfrak{m}-2\rangle, \quad \cdots\). Each of them will be denoted by

\[ |\mathfrak{m}, \mathfrak{m}\rangle. \quad (0 > \mathfrak{m} \geq \mathfrak{m}; \mathfrak{m} + \mathfrak{m} = \text{integer}) \quad (2.8) \]

With this definition of \(\mathfrak{m}\), the eigenvalue of Casimir operator and the matrix elements of the generators can be represented by precisely the same forms as the ones corresponding to the positive discrete series given by (2.6) to (2.7c). Equation (2.7d) should be replaced by

\[ K_+|\mathfrak{m}, -\mathfrak{m}\rangle = 0, \quad (2.9) \]

which is the definition of the negative discrete series.

Finally, we shall mention the possible values of \(\mathfrak{m}\). In contrast to the case of the compact group such as \(SU(2)\), any restrictions on the possible values of \(\mathfrak{m}\) do not follow from the Lie algebra of the noncompact group for the unitary representations, except for reality of \(\mathfrak{m}\). From the topological property of the group manifold, it is known that \(\mathfrak{m}\) should be integer for the \(O(2, 1)\) and integer or half-integer for the \(SU(1, 1)\). For the unitary irreducible representations of the universal covering group \(SU(1, 1) \simeq SL(2L)\), we have a possibility that \(\mathfrak{m}\) may be any real numbers.

§ 3. The d-function of the \(SU(1, 1)\) group

We shall derive the \(d\)-function of the \(SU(1, 1)\) group for both unitary infinite dimensional representations and nonunitary finite dimensional representations. The technique employed here are analogous to those in the elementary algebraic treatment of the \(SU(2)\) group.

We shall start from the identity

\[ e^{i\beta K_y} K_y e^{-i\beta K_y} = \cosh \beta \cdot K_y - \frac{1}{2} \sinh \beta \cdot (K_+ + K_-), \]

which can be derived by using the well-known formula

\[ e^{aA} Be^{-aA} = B + a [A, B] + \frac{a^2}{2!} [A, [A, B]] + \cdots, \]

together with the defining commutation relations (2.1) of the \(SU(1, 1)\). Taking the matrix elements of
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\[ K_{e^{-i\beta K_3}} = \cosh \beta \cdot e^{-i\beta K_3} K_3 - \frac{1}{2} \sinh \beta \cdot e^{-i\beta K_3} (K_+ + K_-), \]  

(3.1)

between \(|JM\rangle\) and \(|JM'\rangle\) and using (2.3), we obtain the recurrence formula of the \(d\)-function for the finite \((2J+1)\)-dimensional representation:

\[ (M' - \cosh \beta \cdot M) d_{2M+1}^{(2J)}(\beta) + \frac{1}{2} \sinh \beta \{ \sqrt{(J-M)(J+M+1)} d_{2M-1}^{(2J)}(\beta) + \sqrt{(J+M)(J-M+1)} d_{2M-1}^{(2J)}(\beta) \} = 0. \]  

(3.2)

Likewise, we have

\[ (M' - \cosh \beta \cdot M) d_{2M}^{(2J)}(\beta) + \frac{1}{2} \sinh \beta \{ \sqrt{(M+2J)(M-2J+1)} d_{2M-2}^{(2J)}(\beta) + \sqrt{(M-2J)(M+2J-1)} d_{2M-2}^{(2J)}(\beta) \} = 0 \]  

(3.3)

for the unitary representation in the positive discrete series.\(^\text{a)\)}

Before solving these recurrence relations explicitly, we present here the corresponding formula of the \(d\)-function of the SU(2) group:

\[ (m' - \cos \beta \cdot m) d_{2m+1}^{(2J)}(\beta) + \frac{1}{2} \sin \beta \{ \sqrt{(j-m)(j+m+1)} d_{2m-1}^{(2J)}(\beta) + \sqrt{(j+m)(j-m+1)} d_{2m-1}^{(2J)}(\beta) \} = 0, \]  

(3.4)

which—combined with a suitable normalization condition—is sufficient to determine \(d_{2m}^{(2J)}(\beta)\) uniquely up to phase. Explicitly,

\[ d_{2m+1}^{(2J)}(\beta) = \left\{ \frac{\Gamma(j-m+1) \Gamma(j+m+1) \Gamma(j-m'+1) \Gamma(j+m'+1)}{\Gamma(j+m+1) \Gamma(j-m+1) \Gamma(j-m'+1) \Gamma(j+m'+1)} \right\}^{1/2} \frac{1}{\Gamma(m'-m+1)} \]  

\[ \times \left( \frac{\cos \beta}{2} \right)^{2j+1-m-m'} \left( -\sin \frac{\beta}{2} \right)^{m-m} \times {}_2F_1(-m-j, m'-j; m'-m+1; -\tan \frac{\beta}{2}), \quad (m' \geq m) \]  

(3.5)

where \( {}_2F_1 \) is the hypergeometric function defined by the formal series

\[ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a) \cdot \Gamma(b)} \sum_k \frac{\Gamma(a+k) \cdot \Gamma(b+k)}{\Gamma(c+k) \cdot k!} z^k. \]  

(3.6)

Inspection on (3.5) and (3.4) leads to the following factorization of \(d_{2m}^{(2J)}(\beta)\) of the SU(1, 1):

\(^\text{a)\)} As is clear from § 2B, the same recurrence formula holds also in the case of the negative discrete series. Therefore, all the procedures followed can be used with minor modifications such as an obvious analytic continuation of \(M\) to its negative in Eq. (3.8). We do not explicitly treat the negative discrete series here, partly because the modifications are straightforward and mainly because the symmetry relation given in the footnote under Eq. (3.18b) can be used to obtain directly the \(d\)-functions associated with the negative discrete series.
\[ d_{M'-M}^{(2)}(\beta) = \left\{ \frac{\Gamma(J-M+1)\Gamma(J+M'+1)}{\Gamma(J+M+1)\Gamma(J-M'+1)} \right\}^{1/2} \frac{1}{\Gamma(M'-M+1)} \times \left( \cosh \frac{\beta}{2} \right)^{M+M'} \left( \sinh \frac{\beta}{2} \right)^{M-M'} f(M; \tanh \frac{\beta}{2}). \quad (M' \geq M) \quad (3.7) \]

Inserting this into (3.2), we get
\[ (J+M)(J-M+1)\tanh \frac{\beta}{2} \cdot f(M-1; \tanh \frac{\beta}{2}) + \cosh^{-2} \frac{\beta}{2} \cdot (M' - \cosh \beta \cdot M) \cdot f(M; \tanh \frac{\beta}{2}) - (M' - M + 1)(M' - M) \cdot f(M+1; \tanh \frac{\beta}{2}) = 0. \quad (3.8) \]

For the unitary representations, we shall adopt the following factorization:
\[ d_{\mathfrak{W}'-\mathfrak{W}}^{(3)}(\beta) = \left\{ \frac{\Gamma(\mathfrak{W}'+\mathfrak{W})\Gamma(\mathfrak{W}'-\mathfrak{W}+1)}{\Gamma(\mathfrak{W}'+\mathfrak{W}-1)\Gamma(\mathfrak{W}'+\mathfrak{W})} \right\}^{1/2} \frac{1}{\Gamma(\mathfrak{W}'-\mathfrak{W}+1)} \times \left( \cosh \frac{\beta}{2} \right)^{\mathfrak{W}'-\mathfrak{W}} \left( \sinh \frac{\beta}{2} \right)^{\mathfrak{W}-\mathfrak{W}} f(\mathfrak{W}; \tanh \frac{\beta}{2}). \quad (\mathfrak{W}' \geq \mathfrak{W}) \quad (3.9) \]

It is to be noted that this factorization is just the analytic continuation of \( J \) to its negative in (3.7). Inserting (3.9) into (3.3), we have
\[ (\mathfrak{W} - \mathfrak{W}) (\mathfrak{W} + \mathfrak{W} - 1) \tanh \frac{\beta}{2} f(\mathfrak{W}-1; \tanh \frac{\beta}{2}) + \cosh^{-2} \frac{\beta}{2} \cdot (\mathfrak{W}' - \cosh \beta \cdot \mathfrak{W}) \cdot f(\mathfrak{W}; \tanh \frac{\beta}{2}) + (\mathfrak{W}' - \mathfrak{W} + 1)(\mathfrak{W}' - \mathfrak{W}) \cdot f(\mathfrak{W}+1; \tanh \frac{\beta}{2}) = 0. \quad (3.10) \]

In order to solve Eqs. (3.8) and (3.10), we note one of the defining equations\(^3\) of the hypergeometric function:

\(^3\) As is well known, the hypergeometric function (3.6) is the solution, regular at \( z=0 \), of the differential equation:
\[ z(1-z) \frac{d^2}{dz^2} F + [c-(a+b+1)z] \frac{d}{dz} F - ab \cdot F = 0. \]

Using the identities\(^6\)
\[ \frac{d}{dz} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z) \] and
\[ \frac{d}{dz} [(1-z) {}_2F_1(a, b; c; z)] = \left( -\frac{a(c-b)}{c} \right) (1-z)^{a-1} {}_2F_1(a+1, b; c+1; z), \]
we get Eq. (3.11).
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\[(a + 1) (c - b + 1) z F_1(a + 2, b; c + 2; z) - (c + 1) \{c + (a - b + 1) z\} F_1(a + 1, b; c + 1; z) + c (c + 1) z F_1(a, b; c; z) = 0.\]  (3.11)

The recurrence formula (3.8) can be identified to (3.11), if we adopt the substitutions

\[a = -M - J, \quad b = M' - J, \quad c = M' - M + 1; \quad z = \tanh^2 \frac{\beta}{2}.\]  (3.12)

Likewise, (3.10) becomes identical to (3.11) by the substitutions,

\[a = -\mathfrak{M} + \frac{3}{2}, \quad b = \mathfrak{M}' + \frac{3}{2}, \quad c = \mathfrak{M}' - \mathfrak{M} + 1; \quad z = \tanh^2 \frac{\beta}{2}.\]  (3.13)

Thus, we get

\[d_{\mathfrak{M}' \mathfrak{M}}^{(\mathfrak{J} + \mathfrak{J'})}(\beta) = \left\{\frac{\Gamma(J - M + 1) \cdot \Gamma(J + M' + 1)}{\Gamma(J + M + 1) \cdot \Gamma(J - M' + 1)}\right\}^{1/2} \cdot \frac{1}{\Gamma(M' - M + 1)}
\times \left(\cosh \frac{\beta}{2}\right)^{M + M'} \left(\sinh \frac{\beta}{2}\right)^{M' - M}
\times F_1(-M - J, M' - J; M' - M + 1; \tanh^2 \frac{\beta}{2}) \quad (M' \geq M) \]  (3.14)

and

\[d_{\mathfrak{M}' \mathfrak{M}'}^{(\mathfrak{J} + \mathfrak{J'})}(\beta) = \left\{\frac{\Gamma(\mathfrak{M}' - \mathfrak{M} + 3) \cdot \Gamma(\mathfrak{M}' - \mathfrak{M} + 1)}{\Gamma(\mathfrak{M} + 3) \cdot \Gamma(\mathfrak{M} + 1)}\right\}^{1/2} \cdot \frac{1}{\Gamma(\mathfrak{M}' - \mathfrak{M} + 1)}
\times \left(\cosh \frac{\beta}{2}\right)^{-2\mathfrak{M} \cdot -2\mathfrak{M}'} \left(\sinh \frac{\beta}{2}\right)^{2\mathfrak{M}' \cdot -2\mathfrak{M}}
\times F_1(-\mathfrak{M} + \frac{3}{2}, \mathfrak{M}' + \frac{3}{2}; \mathfrak{M}' - \mathfrak{M} + 1; \tanh^2 \frac{\beta}{2}) \quad (\mathfrak{M}' \geq \mathfrak{M}) \]  (3.15)

Explicit algebraic formulas of these \(d\)-functions can now be written down easily using (3.6).

It is, however, necessary to generalize the above formulas to the cases \(M > M'\) and \(\mathfrak{M} > \mathfrak{M}'\). This will be made in terms of the symmetry properties of the \(d\)-functions.

For the nonunitary representations, \(K_y\) has been taken to be anti-hermitian. Therefore, \(\exp (-i\beta K_y)\) is clearly hermitian. Since we have chosen the \(d\)-function as real, the hermiticity implies

\[d_{\mathfrak{M}' \mathfrak{M}}^{(\mathfrak{J})}(\beta) = d_{\mathfrak{M} \mathfrak{M}'}^{(\mathfrak{J})}(\beta).\]  (3.16a)

For the unitary representations, the corresponding relation should be

\[d_{\mathfrak{M}' \mathfrak{M}}^{(\mathfrak{J})}(\beta) = (-)^{\mathfrak{M} \cdot -\mathfrak{M}'} d_{\mathfrak{M} \mathfrak{M}'}^{(\mathfrak{J})}(\beta),\]  (3.16b)
because the unitarity implies

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(-\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) \tag{3.17a} \]

and that inspection on (3.15) leads to

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(-\beta) = (-)^{\mathfrak{fr}^{(3)k}} d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta). \tag{3.17b} \]

The same relation as (3.17b) holds also for the nonunitary representation:

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(-\beta) = (-)^{M - \mathfrak{fr}^{(3)k}} d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta). \tag{3.18a} \]

Further inspection on (3.14) reveals that(*)

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta). \tag{3.18b} \]

Combining (3.16a) to (3.18), we obtain the complete symmetry relation of the \( d \)-function for the nonunitary finite dimensional representations:

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta). \tag{3.19} \]

Finally, we note that, since \( a = -M - J \) \((a = -\mathfrak{fr}^{(3)} + \mathfrak{fr}^{(3)k} + \mathfrak{fr}^{(3)k})**\) is always negative integer,*** the hypergeometric functions appearing in (3.14) and (3.15) are Jacobi polynomials defined by****

\[ P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \cdot \Gamma(\alpha + 1)} \cdot \frac{1}{2} \cdot \frac{\Gamma(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{x}{2})}{\Gamma(\alpha + 1)} \]

\[ = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1) \cdot \Gamma(\alpha + 1)} \cdot \left(1 + x\right)^n \cdot \frac{1}{2} \cdot \frac{\Gamma(-n, -n - \beta; \alpha + 1; \frac{x}{2})}{\Gamma(\alpha + 1)} \cdot \frac{1}{x + 1}. \]

Therefore

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = \frac{\Gamma(J - M + 1) \cdot \Gamma(J + M + 1)}{\Gamma(J - M' + 1) \cdot \Gamma(J + M' + 1)} \cdot \left(\cosh \frac{\beta}{2}\right)^{M - M'} \]

\[ \times \left(\sinh \frac{\beta}{2}\right)^{M - M'} \cdot \frac{1}{\Gamma(J + M' - M, M - M') \cdot \cosh \frac{\beta}{2}}. \tag{3.20a} \]

*) Likewise, inspection on (3.15) leads to an important symmetry

\[ d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta) = d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta), \]

by which we obtain the \( d \)-function, \( d_{\mathfrak{fr}^{(3)k}}^{(3)}(\beta), \) of the negative discrete series from that of the positive discrete series.

**) It should be noted that \(-\mathfrak{fr}^{(3)} + \mathfrak{fr}^{(3)k} + \mathfrak{fr}^{(3)k} + \mathfrak{fr}^{(3)k}\) is always negative integer even for any projective representation — a unitary irreducible representation of the universal covering group \( SU(1,1) \) — corresponding to the positive discrete series.

*** From this fact, the series in (3.11) terminate in our case. It then follows that the \( d \)-functions determined here are the unique solution of the relevant recurrence formula. Here, by unique we mean that there are no additional solutions of the recurrence formula which are linearly independent of ours.

For the unitary representation in the continuous series, this is not the case.

**** We adopt Szego's definition and notation used in Bateman project, so that all the formulas tabulated by Bateman project can be used without any changes.
and
\[
d_{\frac{\lambda}{2} }^{(\lambda)} (\beta) = \left\{ \frac{\Gamma(\lambda' + 3)}{\Gamma(\lambda' + 3)} \cdot \Gamma(\frac{\lambda' - 3}{4} + 1) \right\}^{1/2} \cdot \left( \frac{\cosh \beta}{2} \right)^{-\lambda - \lambda'} \\
\times \left( \sinh \frac{\beta}{2} \right)^{\lambda' - \lambda} P_{\lambda' - 3}^{\lambda' - \lambda - 3} (\cosh \beta).
\] (3.20b)

The orthogonality of Jacobi polynomials
\[
\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_{\ell}^{(\alpha, \beta)}(x) P_{k}^{(\alpha, \beta)}(x) \, dx
\]
leads to
\[
\int_{0}^{\infty} d_{\omega}^{(\omega)} (\beta) \cdot d_{\omega'}^{(\omega')} (\beta) \sinh \beta \cdot d \beta = \delta(J, J') \frac{2^{\lambda + \lambda'} - 1}{2 \lambda + 1}.
\] (3.21a)

and
\[
\int_{0}^{\infty} d_{\omega}^{(\omega)} (\beta) \cdot d_{\omega'}^{(\omega')} (\beta) \sinh \beta \cdot d \beta = \delta(J, J') \frac{2^{\lambda - 3} - 1}{-2 \lambda + 1}.
\] (3.21b)

Various useful formulas such as differential and integral formulas of the \( d \)-functions can be obtained by using the known properties of Jacobi polynomials.\(^5\) We do not present these, since the derivations are straightforward.

Recurrence formulas for the both nonunitary and unitary \( d \)-functions, \( d_{\omega}^{(\omega)} (\beta) \) and \( d_{\omega}^{(\omega)} (\beta) \), can be obtained also from the known recurrence relations\(^6\) of Jacobi polynomials. It should be noted, however, that all the recurrence relations of \( d \)-functions of the \( SU(2) \) group are nothing but special cases of the Clebsch-Gordan formula of the \( SU(2) \) group presented in \( \S \) 1. We shall study in the next section the corresponding Clebsch-Gordan formula for the \( SU(1, 1) \) group.

Before ending this section, we note that, apart from a phase due to the two-valuedness of the square root, the unitary \( d \)-function \( d_{\omega}^{(\omega)} (\beta) \) is precisely an analytic continuation of the nonunitary \( d \)-function \( d_{\omega}^{(\omega)} (\beta) \). This situation can be easily seen from Eqs. (3.14) and (3.15), because the hypergeometric functions appearing in these formulas are simply related by analytic continuation of \( J \) to its negative and that all the other factors have been constructed by the same analytic continuation.

\section{4. Clebsch-Gordan formulas of the \( SU(1, 1) \) group}

This section is divided into three parts. In the first part, Clebsch-Gordan formula is derived for the nonunitary finite dimensional representations. The corresponding formula for the unitary representations belonging to the positive discrete series is given in the second part. Although the results in \( \S \S \) 4A and
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4B are not original, but rather—in a sense—definition, we present these for completeness and, moreover, as an introduction to § 4C. In the third part, we find that there exists an additional Clebsch-Gordan formula of the SU(1, 1), which connects the d-functions of the nonunitary finite dimensional and unitary infinite dimensional representations to get that of the unitary representation. It is of interest to note that the new kind of Clebsch-Gordan coefficient found in our previous paper appears naturally in this formula.

4A. Finite dimensional nonunitary representations

Let us introduce two independent sets of the SU(1, 1) generators \( \mathbf{K}^{(1)} \) and \( \mathbf{K}^{(2)} \). The finite dimensional bases of these sets are denoted respectively by \( |J_1 M_1\rangle \) and \( |J_2 M_2\rangle \). The eigenstate of the total SU(1, 1) angular momentum, \( \mathbf{K} = \mathbf{K}^{(1)} + \mathbf{K}^{(2)} \), will be designed as \( |JM\rangle \), which is taken also to be finite dimensional:

\[
K_3^{(1)} |J_1 M_1\rangle = M_1 |J_1 M_1\rangle ,
K_3^{(2)} |J_2 M_2\rangle = M_2 |J_2 M_2\rangle ,
\]

\[
K^\pm |J_1 M_1\rangle = \mp \sqrt{(J_1 \pm M_1)(J_1 \pm M_1 + 1)} |J_1 M_1 \pm 1\rangle \quad (i = 1 \text{ and } 2)
\]

and

\[
K_3 |JM\rangle = M |JM\rangle ,
K^\pm |JM\rangle = \mp \sqrt{(J \pm M)(J \pm M + 1)} |JM \pm 1\rangle ,
\]

where \( K_3 = K_3^{(1)} + K_3^{(2)} \) and \( K^\pm = K^\pm^{(1)} + K^\pm^{(2)} \). We shall define Clebsch-Gordan coefficient of the SU(1, 1) connecting the finite dimensional representations through

\[
|JM\rangle = \sum_{M_1 M_2} (J_1 J_2 M_1 M_2 |JM\rangle_M |J_1 M_1\rangle |J_2 M_2\rangle . \tag{4.1}
\]

Operating \( K_3 \) on both sides of the above equation, we have \( M = M_1 + M_2 \). Similarly, by operation of \( K^\pm \), we have the set of the recurrence formulas to determine the Clebsch Gordan coefficient:

\[
\sqrt{(J + M)(J - M + 1)} (J_1 J_2 M_1 M_2 |JM\rangle_M ) = \sqrt{(J_1 + M_1)(J_1 - M_1 + 1)} (J_1 J_2 M_1 - 1 M_2 |JM - 1\rangle_M ) + \sqrt{(J_2 + M_2)(J_2 - M_2 + 1)} (J_1 J_2 M_1 M_2 - 1 |JM - 1\rangle_M ) \tag{4.2a}
\]

and

\[
\sqrt{(J + M)(J - M + 1)} (J_1 J_2 M_1 M_2 |JM\rangle_M ) = \sqrt{(J + M)(J - M + 1)} (J_1 J_2 M_1 - 1 M_2 |JM - 1\rangle_M ) - \sqrt{(J_2 + M_2)(J_2 - M_2 + 1)} (J_1 J_2 M_1 M_2 - 1 |JM + 1\rangle_M ) . \tag{4.2b}
\]

It should be noted that the above set of the recurrence relations is precisely the same as that which determines the Clebsch-Gordan coefficient of the SU(2) group. This procedure would be the simplest demonstration of the fact that
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the Clebsch-Gordan coefficient connecting nonunitary finite dimensional representations of the SU(1, 1) can be taken to be identical to the corresponding Clebsch-Gordan coefficient of the SU(2) group.

The Clebsch-Gordan formula can now be obtained straightforwardly by taking the matrix element of

$$\exp(-i\beta K_y) = \exp\{-i\beta (K_y^{(1)} + K_y^{(2)})\}$$

(4.3)

between the states |JM⟩ and |JM’⟩;

$$d_{\mu\mu'}^{(\ell)}(\beta) = \sum_{\mu_1, \mu_2} (J_1 J_2 M_1 M_2 |JM⟩⟨JM'|J_1 J_2 M_1' M_2' |JM') d_{\mu\mu'}^{(\ell)}(\beta) \cdot d_{\mu_1\mu_2}^{(\ell)}(\beta),$$

(4.4)

from which we can calculate all the explicit forms of $d^{(\ell)}(\beta)$ starting from $d^{(1/2)}(\beta)$.

4B. Unitary representations

Clebsch-Gordan formula for the unitary $\ell$-functions belonging to the positive discrete series can be derived in analogy to § 4A. Let $|\ell; M⟩, (|\ell; M⟩)$ be a basis of the unitary representations in the positive discrete series. The eigenstate of the total $K$ will be denoted by $|\ell⟩(\ell)$ which is assumed to belong also to the positive discrete series.

Then, we have

$$|\ell⟩(\ell) = \sum_{M_1, M_2} (\ell_1 \ell_2 M_1 M_2 |\ell⟩(\ell) |\ell_1 M_1⟩|\ell_2 M_2⟩,$$

(4.5)

which may be taken as a definition of Clebsch-Gordan coefficient $(\ell_1 \ell_2 M_1 M_2 |\ell⟩(\ell))_U$ of the SU(1, 1) connecting two unitary representations in the positive discrete series to get the third. The set of recurrence formulas to determine this coefficient can be obtained easily in just the same way as in (4.2):

$$\sqrt{(M - \ell) (M + 1)} (\ell, M - 1) (\ell, M) |\ell⟩(\ell)$$

$$= \sqrt{(M_1 - \ell_1) (M_1 + 1)} (\ell_1, M_1 - 1) (\ell_1, M_1) |\ell_1 M_1⟩$$

$$+ \sqrt{(M_2 - \ell_2) (M_2 + 1)} (\ell_2, M_2 - 1) (\ell_2, M_2) |\ell_2 M_2⟩$$

(4.6a)

and

$$\sqrt{(M_1 - \ell_1) (M_1 + 1)} (\ell_1, M_1 - 1) (\ell_1, M_1) |\ell_1 M_1⟩$$

$$= \sqrt{(M - \ell) (M + 1)} (\ell, M - 1) (\ell, M) |\ell⟩(\ell)$$

$$- \sqrt{(M_2 - \ell_2) (M_2 + 1)} (\ell_2, M_2 - 1) (\ell_2, M_2) |\ell_2 M_2⟩.$$

(4.6b)

Since the explicit algebraic formulas and their properties of the solution have been studied in detail by Holman and Biedenharn,7) we need not discuss it further. We simply mention that this coefficient is the analytic continuation of all $j$'s to their negatives in the corresponding Clebsch-Gordan coefficient of the SU(2) group.

Taking the matrix element of $\exp(-i\beta K_y)$ between $|\ell⟩(\ell)$ and $|\ell⟩(\ell)$, we get the Clebsch-Gordan formula for the unitary infinite dimensional representations
This formula is, however, not necessarily convenient to the practical purpose of obtaining any explicit form of the \(d\)-functions, since it involves infinite summation over \(\mathcal{M}\).

4C. **New kind of Clebsch-Gordan formula**

First, we note that the recurrence formula (3·3) cannot be classified either to the Clebsch-Gordan formulas (4·4) or (4·7) of the \(SU(1, 1)\) group. Moreover, every recurrence formula of \(d^{(3)}(\beta)\) connecting different \(\mathcal{M}\)'s, which can be derived by using known recurrence relations of Jacobi polynomials, is actually found unable to be identified to (4·7). This situation may be easily realized by the fact that the recurrence formulas of Jacobi polynomials involve usually only finite sum over its indices. We, therefore, expect that there must exist another type of the Clebsch-Gordan formula which connects the finite and infinite dimensional representations.

We shall tentatively\(^*\) define the coupling rule of basis states of unitary and nonunitary representations through

\[
\langle \mathcal{M}_1 \mathcal{M} \rangle = \sum_{\mathcal{M}_1, \mathcal{M}_2} (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}} \cdot | \mathcal{M}_1 \mathcal{M}_2 \rangle \cdot | \mathcal{M} \rangle ,
\]

where \((\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}}\) denotes the relevant Clebsch-Gordan coefficient to be determined below.

Exactly the same procedure as in 4A and 4B leads to the set of recurrence relations to determine this coefficient:

\[
\sqrt{\mathcal{M} - \mathcal{M}_1} (\mathcal{M} + \mathcal{M}_1 - 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}} \\
= \sqrt{\mathcal{M}_1 - \mathcal{M}_1} (\mathcal{M}_1 + \mathcal{M}_1 - 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 - 1 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}} \\
+ \sqrt{(J + M)} (J - M + 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 - 1 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}}
\]

and

\[
\sqrt{\mathcal{M}_1 - \mathcal{M}_1} (\mathcal{M}_1 + \mathcal{M}_1 - 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}} \\
= \sqrt{\mathcal{M} - \mathcal{M}_1} (\mathcal{M} + \mathcal{M}_1 - 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 - 1 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}} \\
+ \sqrt{(J + M)} (J + M + 1) (\mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3 - 1 | \mathcal{M}_1 \mathcal{M}_2 \mathcal{M}_3)_{\mathcal{M}},
\]

together with

\[
\mathcal{M} = \mathcal{M}_1 + \mathcal{M} .
\]

\(^*\) It is to be noted that such a coupling rule of state vectors is merely artificial, since the finite dimensional representations —necessarily nonunitary— of any noncompact groups never appear as state vectors in our physical world. On the other hand, we emphasize that the transformation properties of any operators should be and actually have been always classified in terms of the finite dimensional bases such as four-vector, etc., in Lorentz group. What we need in classification of tensor operators is irreducibility. (See, for a typical example, the succeeding paper.)
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It should be noted that the above recurrence formulas are precisely the same as those of the new kind of Clebsch-Gordan coefficient obtained in a previous paper.\(^2\) Let us summarize briefly how this Clebsch-Gordan coefficient appears in problems of physics. As emphasized repeatedly, the tensor operators should be classified according to the finite dimensional irreducible representations. Indeed, the tensor operator in the $SU(1, 1)$ group is found to be defined most conveniently through

\[
[K_\lambda, T_q^{(K)}] = QT_q^{(K)}
\]

and

\[
[K_\lambda, T_q^{(K)}] = (\pm) \sqrt{(K \mp M)(K \pm M + 1)} T_q^{(K)}_{\pm 1},
\]

which are consistent with our choice of phases. This definition of the $SU(1, 1)$ irreducible tensors may be compared to the well-known Racah's definition of the tensor operators in the $SU(2)$ group. For the rigorous derivation of (4·10), see reference 2).

Now, consider the matrix element of the operator between physical states, which constitute unitary—necessarily infinite dimensional—representations of the $SU(1, 1)$ group. We can prove Wigner Eckart theorem on this matrix element. That is, all dependences on three projective quantum numbers $\mathcal{M}$, $\mathcal{M}_1$, and $M$ can be factorized into a single coefficient:

\[
(\mathcal{M} | T_q^{(K)} | \mathcal{M}_1, \mathcal{M}) = (\mathcal{M}_1, J_1, J_2, M_1, M_2 | J_1, J_2, M_1, M_2) (\mathcal{M} | T^{(K)} | \mathcal{M}_1). \tag{4·11}
\]

By using the definition (4·10) and the matrix elements of generators, this $M$-dependent coefficient has been proved\(^4\) to obey precisely the same set of the recurrence relations as (4·9a) and (4·9b). Explicit algebraic formulas and their properties have been studied in detail. It has been proved that this coefficient can be obtained from the Clebsch-Gordan coefficient $(J_1 J_2 M_1 M_2 | JM)$ of the $SU(2)$ group by analytic continuation of $J_1$ and $J_2$ to their negatives or, equivalently, by analytic continuation of $\mathcal{M}_1$ to its negative in the Clebsch-Gordan coefficient $(\mathcal{M}_1, \mathcal{M}_2, M_1, M_2 | \mathcal{M})$ connecting three unitary representations of the $SU(1, 1)$ in the positive discrete series.

Once the new kind of Clebsch-Gordan coefficient is known, the corresponding Clebsch-Gordan formula can be obtained straightforwardly by taking the matrix element of $\exp(-i\beta K_\lambda)$ between $|\mathcal{M} \rangle$ and $|\mathcal{M}' \rangle$ in (4·8):

\[
d_{g^{(2)}}^{(\mathcal{M}, \mathcal{M}')} (\beta) = \sum_{\mathcal{M}_1} (\mathcal{M}_1, J_1, J_2, M_1, M_2 | \mathcal{M} \rangle \langle \mathcal{M}_1, J_1, J_2, M_1, M_2 | \mathcal{M}' \rangle) d_{\mathcal{M}_1, \mathcal{M}}^{(2)} (\beta) \cdot d_{\mathcal{M}_1, \mathcal{M}}^{(3)} (\beta), \tag{4·12}
\]

where the summation runs over very restricted terms in comparison to (4·7).

Finally, it is interesting to note that our defining recurrence formula (3·2) of the nonunitary $d$-function $d_{g^{(2)}}^{(\mathcal{M}, \mathcal{M}')} (\beta)$ can be regarded as a special case of the usual Clebsch-Gordan formula (4·4). Indeed, using the explicit forms of $d^{(3)} (\beta)$ and those of Clebsch-Gordan coefficients $(J_1 M_1 M_2 | JM)$, one sees immediately that (3·2) is nothing but
On the other hand, the defining equation (3·3) of the unitary $d$-function $d_{M\rightarrow M}^{(\ell)}(\beta)$ is a special case of the new type of Clebsch-Gordan formula (4·12), as can be verified by using the new kind of Clebsch-Gordan coefficient tabulated in reference 2.

The other three examples of (4·12), which arise as recurrence formulas of the unitary $d$-functions, will be presented in the succeeding paper. 9)

§ 5. Concluding remark

We found three kinds of Clebsch-Gordan formulas of the $SU(1, 1)$ group. They are presented in (4·4), (4·7) and (4·12). Since the Clebsch-Gordan coefficients and the $d$-functions appearing in these formulas are related with each other by appropriate analytic continuation of their indices, the three Clebsch-Gordan formulas are clearly interrelated by the analytic continuations. That is, the analytic continuation of $J_1$ and $J$ to their negatives in (4·4) leads us to our new type of Clebsch-Gordan formula (4·12). Further analytic continuation of $J_2$ to its negative gives us the Clebsch-Gordan formula (4·7) connecting three unitary $d$-functions in the positive discrete series.

References

2) H. Ui, Ann. of Phys. 49 (1968), 69.
3) For example, J. Strathdee, J. F. Boyce, R. Delbourgo and A. Salam, "Partial Wave Analysis", Chap. 6, IC/67/9 (Lecture Notes at International Center for Theoretical Physics, Trieste).
6) L. C. Biedenharn, Lecture on Theoretical Physics, Cargese Summer School, Corsica, 1965.
9) H. Ui, the succeeding paper, Prog. Theor. Phys. 44 (1970), 703.