Detailed Wave Equation for Extended Model
Underlying the Dual Amplitude

String Model of Hadrons and Its Hybrid Extension—

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The theory of detailed wave equation (DWE), which represents relativistic quantum
dynamics of an extended system with external interaction for a specific value of Regge inter­
ccept, avoiding ghost states and leading to dual amplitude, is further developed in several
respects. The original formulation (the \( \sigma \)-formalism) for the string model is transcribed into
a few equivalent formalisms. In the \( \lambda \)-formalism the invariance of DWE against the “Hilbert
transform” becomes manifest. Finite “gauge transformations”, under which the theory is
invariant, are explicitly obtained for the consideration of their structural meaning. The orig­
inal string model is generalized to the hybrid model, where each element constituting the
string is endowed with internal “Fermi oscillator variables”, in the framework of DWE,
which now fixes the Regge intercept at the value lower than the original one by \( \frac{1}{2} \) unit or
its multiples. In particular the hybrid model with vector Fermi variables, which implies
“double-string” consisting of Dirac-like elements, is treated, where our vector Fermi variables
differ from those in similar models by other authors with respect to their reality and Lorentz
properties and are free from the introduction of additional indefinite metric. Moreover our
theory is invariant under space reflection.

§ 1. Introduction

In pursuing the theory of extended system containing relativistic internal
movement for the purpose of a unified model of hadrons, we have introduced
the general postulate that the complete relativistic quantum mechanics for such a
system, including the case with external interaction, should be represented by a
‘detailed wave equation’ which holds at every element composing the system.\(^1\),\(^2\)
This seems to be a useful conception underlying duality and is motivated from
the following reason. Namely for the theory of an extended system the time
coordinates, each of which is assigned to each element, are necessary for the covar­
iant formalism in the presence of internal force and external interactions but
nevertheless, not representing proper dynamical degrees of the system, should be
effectively suppressed except a single one, and this fact is just in accord with the
necessity of avoiding unphysical states. In particular we have explicitly given
our theory for a ‘string model’, which is taken as a unified model for mesonic
resonances, to show that it results in the multiparticle dual amplitude for a
particular case where the trajectory intercept equals one. In the present paper
first we briefly summarize our original theory, in making additional remarks about
some subtle points and in giving reformulations of the theory in several equivalent forms, and then, based on all of them, we generalize the original model so as to relax the above very strict limitation as to the intercept value. In particular we do this by constructing within our framework the string model consisting of elements endowed with 'Fermi oscillator variables'. We call this model 'double Dirac string' since it is viewed as the coalescence of two strings consisting of Dirac-like particles, and we find that the corresponding detailed wave equation incorporates several interesting properties.

§ 2. The method of detailed wave equation for the original model

The configuration of the original 'finite string model' is represented by the positional coordinates $x_{\sigma}(\sigma)$ satisfying $[x(\sigma), x(\sigma')]=0$, where $\sigma$ labels each element constituting the string and is adjusted to range over $[0, \pi]$. We impose the 'open-end boundary condition'

$$\frac{dx_{\sigma}(\sigma)}{d\sigma}|_{\sigma=0,\pi} = 0. \quad (2.1)$$

The conjugate variables $p_{\sigma}(\sigma)$ satisfy

$$[x_{\sigma}(\sigma), p_{\nu}(\sigma')] = ig_{\mu\nu}(\delta(\sigma - \sigma') + \delta(\sigma + \sigma' - 2\pi) + \delta(\sigma + \sigma' - 2\pi)), \quad (\mu, \nu = 1, 2, 3, 0; \ \text{diag}(g_{\mu\nu}) = (1, 1, 1, -1)), \quad (2.2)$$

where on the right-hand side of (2.2) the last two $\delta$-functions survive only for the particular cases $\sigma = \sigma' = 0$ and $\sigma = \sigma' = \pi$, but are retained to ensure the compatibility of (2.2) with (2.1). The translation operator of the system must be $P_{\sigma} = \int_{\sigma}^{\sigma'} p_{\sigma}(\sigma') d\sigma$ because of (2.2), while the geometrical center of mass is defined as $X_{\sigma} = \int_{\sigma}^{\sigma'} x_{\mu}(\sigma') d\sigma/\pi$. For the string the free relativistic wave equation in the usual sense will be

$$(\mathcal{A} + \omega_0) \Psi = 0; \ \mathcal{A} = \int_{\sigma}^{\sigma'} F(\sigma) : d\sigma, \ F(\sigma) = \frac{1}{2} \left( \frac{p(\sigma)^2}{\kappa} + \kappa \left( \frac{dx}{d\sigma} \right)^2 \right). \quad (2.4)$$

This gives mass levels lying on parallel trajectories with the slope $(2\pi\kappa)^{-1}$. We employ in this paper the unit system in which $\kappa = (2\pi)^{-1}$. Then $\omega_0$ represents the (mass)$^3$ of the ground state, namely the scalar particle on the leading trajectory, whose intercept therefore equals $-\omega_0$.

The model (2.4) has been assumed for deriving the dual amplitude.$^{3,4,5}$ Our method is to replace (2.4) by a more stringent equation defined at every element of the string

$$H(\sigma) \Psi[x] = 0, \quad (2.5)$$

implying infinitely many equations, which are mutually coupled because the 'in-
variant hamiltonian density operator \( H(\sigma) \) in (2·5) contains \( dx(\sigma)/d\sigma \). Thus the compatibility of (2·5) must be ensured by the existence of a closed algebra for \( H(\sigma) \). This requirement and the correspondence requirement that (2·4) should be recovered from (2·5) restrict the structure of \( H(\sigma) \), under few simplifying assumptions, essentially to the form

\[
H(\sigma) = \pi \left( : F(\sigma) : + \tilde{G}(\sigma) \right) + \omega_0, \quad (2·6)
\]

where \( \tilde{G}(\sigma) \) denotes the finite Hilbert transform of \( G(\sigma) = \frac{1}{2} \{ dx/d\sigma, \rho(\sigma) \} \):

\[
\tilde{G}(\sigma) = -i \int_0^\pi G(\sigma') \frac{\sin \sigma' d\sigma'}{\cos \sigma' - \cos \sigma}. \quad (2·7)
\]

With (2·6), the compatibility of (2·5) with (2·1) is verified, while (2·4) is recovered since \( \int_\pi^\pi \tilde{G}(\sigma) d\sigma = 0 \). Also (2·6) really satisfies the closed algebra

\[
[H(\sigma_1), H(\sigma_2)] = \frac{2(1 - \cos \sigma_1 \cos \sigma_2)}{(\cos \sigma_1 - \cos \sigma_2)^2} (H(\sigma_1) - H(\sigma_2)) + \frac{1}{(\cos \sigma_1 - \cos \sigma_2)} \left( \sin \sigma_1 \frac{dH(\sigma_1)}{d\sigma_1} + \sin \sigma_2 \frac{dH(\sigma_2)}{d\sigma_2} \right),
\]

which clearly holds irrespective of the value of \( \omega_0 \) in (2·6), though later when interaction is introduced we find that \( \omega_0 \) must be restricted to a specific value. (2·8) is derived by the aid of the algebra

\[
[F(\sigma_1), F(\sigma_2)] = \{ G(\sigma_1), G(\sigma_2) \} = i(G(\sigma_1) + G(\sigma_2)) \delta'(\sigma_1 - \sigma_2), \quad (2·9)
\]

\[
[F(\sigma_1), G(\sigma_2)] = i(F(\sigma_1) + F(\sigma_2)) \delta'(\sigma_1 - \sigma_2), \quad (2·10)
\]

to be called the 'string algebra'. Thus \( F(\sigma) \) and \( G(\sigma) \) themselves satisfy a closed algebra. It is, however, to be noted that one cannot assume the double equations \( F(\sigma) \Psi = G(\sigma) \Psi = 0 \), since they do not recover (2·4) exactly unless \( \omega_0 = 0 \). Also, if one replaces \( F(\sigma) \) by \( : F(\sigma) : \), (2·10) attains additional terms violating the closure of the algebra. Moreover the method of detailed wave equation originally implies the existence of wave operators one for each element and not two.

By Fourier expansion (2·5) is brought to the set of equations

\[
(A^r + \delta_{r,0} \omega_0) \Psi = 0; \quad A^r = \int_\pi^\pi (: F(\sigma) : \cos r\sigma + iG(\sigma) \sin r\sigma) d\sigma, \quad (2·11)
\]

\[
(r = 0, 1, 2, \ldots)
\]

or equivalently to the set

\[
(A^0 + \omega_0) \Psi = 0, \quad (A^r - \omega_0) \Psi = 0; \quad A^r = A^r - A^0. \quad (r = 1, 2, \ldots) \quad (2·12)
\]
The introduction of the ‘weighted normal mode operators’

\[ C_\rho^r = \sqrt{2} \int_0^\infty \left( p_\rho(\sigma) \cos \frac{r \sigma}{2\pi} \frac{dx_\rho}{d\sigma} \sin r\sigma \right) d\sigma, \quad (r=0, \pm 1, \pm 2, \cdots) \]

(2.13)
satisfying \( C^r = C^{*r} \) and \([C_\rho^r, C_\rho^{*s}] = r \delta_{r+s,0} g_{\rho\sigma} \), allows to separate the external motion and internal motion and yet to treat both uniformly. In fact, \( C_\rho^0 = C_\rho^{*0} = \sqrt{2} P_\rho \), while \( C_\rho^r/\sqrt{r} = a_\rho^r \) and \( C_\rho^{-r}/\sqrt{r} = a_\rho^{-r} (r>0) \) are the annihilation and creation operators of the \( r \)-th mode vibrations.\(^*)\) Then \( p(\sigma) = (\sqrt{2\pi})^{-1} \sum_{r=-\infty}^{\infty} C_r \cos r\sigma \), and \( x(\sigma) = X(x) + \overline{x}(\sigma) \) with \( \overline{x}(\sigma) = i\sqrt{2} \sum_{r=-\infty}^{\infty} C_r \cos r\sigma/r \), where the extra variable \( X \) satisfies \( [X, C_r] = i\sqrt{2} \delta_{r,0} g_{\rho\sigma} \). Thus \([x_\rho(\sigma), C_r^*] = i\sqrt{2} g_{\rho\sigma} \cos r\sigma \), whence

\[ C_\rho^r e^{ikx(\sigma)} = e^{ikx(\sigma)} (C_\rho^r + \sqrt{2} k_\rho \cos r\sigma). \]

(2.14)

Also we get

\[ A^r = \frac{1}{\lambda} \lim_{N \to \infty} \sum_{n=-N}^{N} C_n C_r^{*-n}. \]

(2.15)**

In particular \( A^0 = P^2 + R, \) \( R = \sum_{n=1}^{\infty} C_n^* C_n \). Thus (2.11) is a set of infinite-component wave equations for the case of infinite number of internal degrees of freedom; the 0-th one is the global wave equation (2.4) while those for \( r=1, 2, \cdots \) represent the subsidiary conditions on it to suppress ‘relative-time vibrations’ corresponding to \( C_0^r \) \((r>0)\) by the aid of indefinite metric. \( \{A^r\} \) or \( \{A^r\} \) satisfy the closed algebra

\[ [A^r, A^s] = (r-s) A^{r+s}, \quad (r, s=0, 1, 2, \cdots) \]

(2.16)

\[ [A^r, A^s] = (r-s) A^{r+s} - r A^s + s A^r. \quad (r, s=1, 2, \cdots) \]

(2.17)

Thus we have, e.g.,

\[ A^r x^n = z^{n+r} A^r, \quad A^r (A^0 + \omega_0)^{-1} = (A^0 + \omega_0 + r)^{-1} A^r. \]

(2.18)

\( A^r \) for \( r<0 \) are defined through \( A^{-r} = A^r \), and the whole set \( \{A^r\} \) becomes no longer closed under commutation: \([A^r, A^s] = (r-s) A^{r+s} + \delta_{r+s,0} (r^2-1)/3.\)\(^***\) We note that the theory is invariant under the internal reflection \( x_\rho(\sigma) \to x_\rho(\pi-\sigma) \), \( p_\rho(\sigma) \to p_\rho(\pi-\sigma) \). Thereby (2.1) and (2.2) are invariant and \( H(\sigma) \to H(\pi-\sigma), \) \( A^r \to (-1)^r A^r \), \( C_\rho^r \to (-1)^r C_\rho^r \). This is induced by the unitary operator \((-1)^r\).

The model of strong interaction is given by the string coupled to the external scalar field \( \phi(x) \). The system must still be described by a detailed wave equation with the invariant hamiltonian density supplemented with a coupling term.

\(^*)\) These \( C_\rho^r \)'s were introduced by the author.\(^1\) There is another method due to Fubini and Veneziano\(^5\) to treat \( P_\rho \) as the 0-th mode by means of a limiting procedure.

\(^**\) \( C_n C^* m = C^n C^* m - 4\pi n (n) \delta_{n+m,0}. \)

\(^***\) This algebra was given in Ref. 2. The existence of \( c \)-number term in this algebra is essential because this forces to pick out the closed subalgebra \( \{A^r\}, r \geq 0 \) in our theory. Cf. (2.11) and the argument preceding it.
The compatibility of this equation requires first that this coupling occurs locally at the end, \( \sigma = 0 \) or \( \pi \), of the string and second that the coupling term behaves in a specific way under the gauge generators \( A' \). Thus, if we first assume the scalar coupling, the equation is either of the following:

\[
[H(\sigma) + g\delta(\sigma) \phi(x_\mu(0))] \mathcal{Y} = 0, \\
[H(\pi) + g\delta(\pi - \sigma) \phi(x_\mu(\pi))] \mathcal{Y} = 0.
\] (2·19) (2·20)

If we adopt (2·19), this is equivalently rewritten as

\[
[A^\rho + \omega_0 + g\phi(x(0))] \mathcal{Y} = 0, \\
(A' - \omega_0) \mathcal{Y} = 0, \quad (r=1, 2, \ldots)
\] (2·21a, b)

which essentially agrees with the result obtained by Virasoro. The compatibility of (2·21) must be assured by the conditions

\[
\{[A_r', \phi(x(0))] - r\phi(x(0)) \} \mathcal{Y} = 0. \quad (r=1, 2, \ldots)
\] (2·22)

When the external field is a plane wave of 4-momentum \( k_\mu \), we have \( \phi(x(0)) = e^{ikx(0)}(0) = e^{ikx(0)}(X) \). In fact we take the normal product for the vertex operator, namely \( V(k) = e^{ikx(0)} ; \) or \( \overline{V}(k) = e^{ikx(0)} ; \), which differs from the original operators by a diverging c-number factor and are no longer unitary. As seen from (2·22), the existence of a consistent detailed wave equation in the interaction case requires the existence of that vertex operator which simply makes a complex 'dilatation' (including phase transformation) under the gauge generators \( A' \). This is a general condition, and in the present case is written as

\[
\exp(\zeta A') V(k) \exp(-\zeta A') = e^{it \zeta} V(k)
\]

as implied by (2·22). By applying a transformation induced by \( A' \)'s on a physical state \( \mathcal{Y} \) satisfying (2·21) we obtain

\[
\mathcal{Y}' = \exp(\sum_{r=1}^{\infty} \zeta A' \cdot \mathcal{Y}
\]

which is again a physical state, provided the condition (2·22) be satisfied, because of the closed algebra for the totality of \( A' \)'s and \( V \). Thus the 'gauge invariance' is an immediate consequence of the existence of the detailed wave equation for interaction case. Now the condition (2·22) means a restriction on the external (mass)\(^3\), \(-k^2\). To see this we calculate \([A'_r, e^{ikx(0)}] \) by resorting to the following trick. First we take \( A'_r \) of (2·15) without going to the limit \( N \to \infty \), namely \( A'_r = \frac{1}{2} \sum_{n=-N}^{N} C^n C'^{-n} \); and by the aid of (2·14) we get

\[
[A_N^r, e^{ikx(0)}] = e^{ikx(0)} \left\{ \sqrt{2} k \sum_{n=-N}^{N} C^n + k^2 (2N + r + 1) \right\},
\]

whence

\[
[A_N^r - A_N^s, e^{ikx(0)}] \mathcal{Y} = \sqrt{2} e^{ikx(0)} k \sum_{n=-N+1}^{N} C^n \mathcal{Y} + r k^2 e^{ikx(0)} \mathcal{Y}.
\]

At this final stage we take the limit \( N \to \infty \). Noting the series convergence and the fact that \( C^n(n>0) \) are annihilation operators, we may assume \( \lim_{n \to \infty} [C^n \mathcal{Y}] \to 0 \), to get
Thus (2.22) is assured if $k^2 = 1$ (tachyon). By identifying this external scalar meson with the state lying on the leading trajectory $\alpha(s) = s - \omega_0$, we have $\omega_0 = -k^2 = -1$ (i.e., intercept = 1). If one employs $e^{ikx(0)}$, (2.23) is re-expressed in a slightly more complicate form$^9$ $A^\prime(P) \cdot e^{ikx(0)} = e^{ikx(0)} \cdot (A^\prime(P-k) + rk^2)$. It is known that the propagator $(A^\prime + \omega_0)^{-1} = (P^2 + R + \omega_0)^{-1} D(R, \omega_0)$ and the vertex operator $e^{ikx(0)}$ lead to the dual amplitude as the Born term of tree diagrams, for the above case $\omega_0 = -1$. The propagator satisfies $(A^\prime - \omega_0)D(R, \omega_0) = D(R + r, \omega_0) \times (A^\prime - \omega_0 - r)$ due to (2.18) while the vertex satisfied (2.23). Using these relations one can verify that if $\mathcal{T}$ satisfies (2.21b), the vector $V(k) \mathcal{T}$ satisfies instead $(A^\prime - \omega_0 - rk^2) V(k) \mathcal{T} = 0$, while the vector $D(R, \omega_0) V(k) \mathcal{T}$ satisfies (2.21b) under the condition $k^2 = 1$. These results are relevant to eliminate unphysical intermediate states in the calculation of the amplitude. Finally we consider the alternative wave equation (2.20) which represents the case where the coupling occurs at the other end $\sigma = \pi$. (2.20) is equivalent to the set

$$[A^\prime + \omega_0 + g\phi(x(\pi))] \mathcal{T} = 0; \quad A^\prime \mathcal{T} = 0, \quad (r = 1, 2, \ldots) \quad (2.24)$$

where $A^\prime = (-1)^r A^\prime - A^\prime$ satisfy among themselves the same algebra as (2.17). Equations (2.24) are compatible if $[[A^\prime, \phi(x(\pi))], -r \phi(x(\pi))] \mathcal{T} = 0$, but we have $[A^\prime, e^{ikx(\pi)}] \mathcal{T} = rk^2 e^{ikx(\pi)} \mathcal{T}$ which is analogous to (2.23). Therefore (2.20) is consistent under the same condition $k^2 = 1$. Also this leads to the dual amplitude in a similar way as before. These are natural results since both ends are originally equivalent.

§ 3. Reformulations and gauge transformations

Before going into further discussions it is convenient to state a few reformulations of the theory each of which is completely equivalent to and immediately follows from our original formalism.

(a) $\lambda$-formalism. Our basic variables $x_\lambda(\sigma)$ and $p_\lambda(\sigma)$ are defined for $0 \leq \sigma \leq \pi$, but we can go over to the new variable $u_\lambda(\sigma)$ defined in the range $-\pi \leq \lambda \leq \pi$ by

$$\begin{align*}
(u_\lambda(\sigma)) &= x_\lambda(\sigma) + 2\pi \int_0^\sigma p_\lambda(\sigma) d\sigma, \\
(u_\lambda(-\sigma)) &= x_\lambda(\sigma) - 2\pi \int_0^\sigma p_\lambda(\sigma) d\sigma,
\end{align*} \quad (0 \leq \sigma \leq \pi) \quad (3.1)$$

and formulate the theory in terms of $u_\lambda(\lambda)$ only. Clearly

$$x(\sigma) = \frac{1}{2} (u(\sigma) + u(-\sigma)), \quad u(+0) = u(-0) = x(0), \quad X = \frac{1}{2\pi} \int_{-\pi}^\pi u(\lambda) d\lambda.$$
Further, due to (2.1), \( v(\lambda) = du(\lambda)/d\lambda \) is also continuous at \( \lambda = 0 \), \( v(+0) = v(-0) = 2\pi p(0) \). Moreover, due to (2.1), \( v(-\pi) = v(\pi) \) so that we can extend the domain of \( v(\lambda) \) to all over \(-\infty < \lambda < \infty\) via the periodic condition \( v(\lambda + 2\pi) = v(\lambda) \). Then \( u(\lambda) = \int_0^{\pi} v(\lambda) d\lambda + u(0) \) becomes defined also over all \( \lambda \). Clearly \( p(\sigma) = (v(\sigma) + v(-\sigma))/4\pi \). The basic commutation relation is now

\[
[u_\mu(\lambda), u_\nu(\lambda')] = -2\pi ig_{\mu\nu} \sum_{n=-\infty}^{\infty} \delta(\lambda - \lambda' - 2n\pi),
\tag{3.2}
\]

from which also

\[
[u_\mu(\lambda), v_\nu(\lambda')] = 4\pi ig_{\mu\nu} \sum_n \delta(\lambda - \lambda' - 2n\pi).
\tag{3.3}
\]

The 4-momentum is given by \( P = (u(\pi) - u(-\pi))/(4\pi) \), with the relation \( x(\pi) = u(\pi) - 2\pi P \) and \( [u_\mu(\lambda), P_\nu] = ig_{\mu\nu} \), while the angular momentum tensor is given by

\[
M_{\mu\nu} = \frac{1}{8\pi} \int_{-\pi}^{\pi} u_{[\mu}(\lambda) v_{\nu]}(\lambda) d\lambda + \frac{1}{2} u_{[\mu}(\pi) P_{\nu]}.
\tag{3.4}
\]

Note the occurrence of the second term. The free detailed wave equation is expressed as

\[
\tilde{H}(\lambda) \Psi = 0, \quad \tilde{H}(\lambda) = \frac{1}{4} \left\{ v(\lambda)^2 + \frac{i}{2\pi} \int_{-\pi}^{\pi} v(\lambda') : \cot \frac{\lambda' - \lambda}{2} d\lambda' \right\} + \omega_0,
\tag{3.5}
\]

which holds at every \( \lambda \) and is evidently invariant against the Hilbert transform. \( \tilde{H}(\lambda) \) satisfies the closed algebra

\[
2[\tilde{H}(\lambda_1), \tilde{H}(\lambda_2)] = -\cot \left( \frac{\lambda_1 - \lambda_2}{2} \right) \left( \frac{d\tilde{H}(\lambda_1)}{d\lambda_1} + \frac{d\tilde{H}(\lambda_2)}{d\lambda_2} \right)
\]

\[
+ \cosec \left( \frac{\lambda_1 - \lambda_2}{2} \right) (\tilde{H}(\lambda_1) - \tilde{H}(\lambda_2)).
\tag{3.6}
\]

The original \( H(\sigma) \) is \( H(\sigma) = (\tilde{H}(\sigma) + \tilde{H}(-\sigma))/2 \). The periodic function \( v(\lambda) \) is expanded as

\[
v_\mu(\lambda) = \sqrt{2} \sum_{n=-\infty}^{\infty} C_n^\mu e^{-tn},
\tag{3.7}
\]

where the coefficients \( C_n^\mu = (2\sqrt{2}\pi)^{-1} \int_0^{\pi} v(\lambda) e^{-\xi \lambda} d\lambda \) are precisely our weighted normal mode operators (2.13). Likewise the Fourier coefficients of the periodic function \( v(\lambda) \) are precisely the quantities (2.15), namely

\[
\lambda^\prime = \frac{1}{8\pi} \int_{-\pi}^{\pi} v(\lambda)^4 : e^{i\lambda^\prime} d\lambda,
\tag{3.8}
\]
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i.e. \( v(\lambda)^2 = 4\sum_{r=-\infty}^{\infty} A^r e^{-ir\lambda} \)  In contrast \( \Omega(\lambda) \) is expanded as \( \Omega(\lambda) = A^0 + 2 \sum_{r=1}^{\infty} A^r e^{-ir\lambda} + \phi_0 \) where the ‘negative frequency part’ (i.e. terms with \( r<0 \)) disappears because the second term in (3.5) for \( \Omega(\lambda) \) plays the role of eliminating it, and this allows the closed algebra (3.6) and the equivalence of (3.5) to (2.11). \( u(\lambda) \) is expanded as \( u(\lambda) = X + 2\lambda P + i\sqrt{2} \sum_{r=-\infty}^{\infty} C^r e^{-ir\lambda}/r \). We have

\[
[A', \mu(\lambda)] = -ie^{\lambda r} \frac{du(\lambda)}{d\lambda}, \quad [A', v(\lambda)] = e^{\lambda r} \left( -i \frac{dv}{d\lambda} + rv \right). \tag{3.9}
\]

Thus in particular \( A^0 \) is the \( \lambda \)-displacement operator

\[
e^{ir\lambda} u(\lambda) e^{-ir\lambda} = u(\lambda + \tau). \tag{3.10}
\]

Under the internal reflection, whose operator \((-)^A\) differs from \( e^{ir\lambda} \) by the factor \( e^{-i\pi P} \), \( u(\lambda) \rightarrow u(\lambda + \pi) - 2\pi P \) and \( v(\lambda) \rightarrow v(\lambda + \pi) \). The detailed wave equation with scalar interaction, (2.19), is reexpressed as

\[
\left[ \Omega(\lambda) + g\phi(u(0)) (\delta(\lambda) - i \frac{\lambda}{2\pi} \text{cot} \frac{\lambda}{2}) \right] \Psi = 0, \tag{3.11}
\]

which continues to be invariant against Hilbert transform. To derive Eq. (2.23) in this formalism we resort to the following trick. In the calculation of \([A^r, e^{ir\lambda}(u)]\) we do not directly employ (3.8) but use

\[
A^r_N = \frac{1}{16\pi^2} \int \int_{-\infty}^{\infty} v(\lambda) v(\lambda') e^{ir(\lambda + \lambda')} DA^r_N(\lambda - \lambda') d\lambda d\lambda', \tag{3.12}
\]

where \( DA^r_N(\lambda) = \sin[(r+2N+1)\lambda/2]/\sin(\lambda/2) \), and only at the end of calculation take the limit \( N \to \infty \), where \( DA^r_N(\lambda) \to 2\pi \delta(\lambda) \). (See Appendix II.)

(b) \((\sigma, \tau)\)-formalism. From the basic variable \( u_\mu(\lambda) \) in the above formalism we can go over to

\[
x_\mu(\sigma, \tau) = \frac{1}{2}(u_\mu(\tau - \sigma) + u_\mu(\tau + \sigma)), \tag{3.13}
\]

which can be regarded as being defined over the range \( 0 \leq \sigma \leq \pi, -\infty < \tau < \infty \). Clearly this satisfies the ‘initial conditions’ \( x(\sigma, 0) = x(\sigma), [\partial x(\sigma, \tau)/\partial \tau]_{\tau=0} = 2\pi P(\sigma) \), the boundary condition \( [\partial x(\sigma, \tau)/\partial \sigma]_{\sigma=0, \pi} = 0 \), and the ‘equation of motion’ \( \partial^2 x/\partial \tau^2 = \partial^2 x/\partial \sigma^2 \), to which (3.13) represents the D'Alembert solution. The basic commutation relation is

\[
[x_\mu(\sigma_1, \tau_1), x_\nu(\sigma_2, \tau_2)] = -i\pi g_{\mu\nu} D(\sigma_1 - \sigma_2, \tau_1 - \tau_2) + D(\sigma_1 + \sigma_2, \tau_1 - \tau_2), \tag{3.14}
\]

where \( D(\sigma, \tau) = \frac{1}{2} \sum_r (\delta(\tau - \sigma - 2n\pi) + \delta(\tau + \sigma - 2n\pi)) \) is the singular solution of

\[\text{(*) The introduction of the quantities mathematically identical with our } u(\lambda) \text{ and } v(\lambda), \text{ together with the relation (3.8), has also been made by other authors.}^{[10,12]} \text{ They employ them either purely operationally or by interpreting } \lambda \text{ as “proper time” (cf. (c) of this section), and they do not consider the detailed wave equation. Cf. also Ref. 14.} \]
\((\partial^2/\partial \tau^2 - \partial^2/\partial \sigma^2) D = 0\), satisfying \([\partial D/\partial \tau]_{\tau=\sigma} = \delta(\sigma)\) and the periodicity, and is invariant under the ‘Lorentz transformation’ in the 2-dimensional \(\sigma-\tau\) space: \(\sigma' = e^{2\pi i}(\sigma \pm \tau)\). \((3.14)\) includes \([x(\sigma, \tau), x(\sigma', \tau')] = 0\) and \([x_\mu(\sigma, \tau), \partial x_\mu(\sigma', \tau)/\partial \tau] = 2\pi i g_\mu \delta(\sigma - \sigma')\). From \(x(\sigma, \tau), u(\lambda)\) and \(v(\lambda)\) are recovered by
\[
u(\tau) = \frac{\partial x(0, \tau)}{\partial \tau}, \quad u(\tau) = x(0, \tau).
\]
Evidently we can reformulate our theory based on the variable \(x_\mu(\sigma, \tau)\) alone (see Ref. 2)). This means that we go over from the original \(\sigma\)-formalism to the ‘interaction representation’ by the unitary transformation
\[
\phi = e^{i\tau \hat{\Psi}},
\]
where the physical quantities varies with \(\tau\) by the ‘free hamiltonian’ \(\hat{A}\), like \(i[\hat{A}, x(\sigma, \tau)] = \delta x(\sigma, \tau)/\partial \sigma\).

(c) \(\tau\)-formalism. If one puts \(\tau \to 0\) in the above formalism one comes back to the original \(\sigma\)-formalism. On the other hand if we put \(\sigma = 0\) in (b) we get the formalism based on \(x_\mu(0, \tau)\) and \(\partial x_\mu(0, \tau)/\partial \tau\). This means that even if we restrict our attention to the end point \(\sigma = 0\) alone, still we have a closed formalism because the external coupling occurs just at this point. Indeed, because of the relations \((3.15)\), this \(\tau\)-formalism must be identical with the \(\lambda\)-formalism of (a) via the simple change of notation \(\lambda \to \tau\), which is, however, associated with reinterpretation of the \(\lambda\)-formalism. Viewed in the \(\tau\)-formalism, the periodicity becomes the one with respect to \(\tau\), and one needs to assume that \(P_\sigma\) be given by the ‘pitch’ of \(x_\mu(0, \tau), P = (x(0, \tau + 2\pi) - x(0, \tau))/(4\pi)\). Our original detailed wave equation is replaced equivalently by the equation defined at every \(\tau\):
\[
\left[\frac{1}{4} \int_{-\pi}^{\pi} \left(\frac{\partial x(0, \tau')}{\partial \tau'}\right)^2 \delta(\tau') + i \frac{\cot \tau}{2}\right] \phi = 0
\]
on the state vector \((3.16)\) of the interaction representation. In this \(\tau\)-formalism the extended nature of the model is concealed or avoided and hence the complete set of commuting dynamical quantities are not manifest in its basic commutation relations.\(^\#\)

In this paper we work in the original \(\sigma\)-formalism as well as in those of (a) and (b).

Now, to analyse the structural meaning of gauge transformations,\(^**\) the \(\lambda\)-formalism is most convenient to start with. From \((3.9)\) we derive the finite

\(^\#\) The \(\tau\)-formalism are favoured by some authors.\(^{13},7,13\) The wave equation \((3.17)\) has not been given however.

\(^**\) For somewhat different interpretations of gauge transformations, cf. Refs. 6), 14), and the second paper of Ref. 3).
transformation by $A^r$ as

$$
\exp(\zeta A^r)u(\lambda)\exp(-\zeta A^r) = \sum_{n=0}^{\infty} (-\zeta)^n e^{i\pi \lambda} (n!)^{-1}d(d-r) \cdots (d-(n-1)r) u(\lambda),
$$

where $d = id/d\lambda$ and $\zeta$ is a complex parameter. This is reexpressed as

$$
\exp(\zeta A^r)u(\lambda)\exp(-\zeta A^r) = (1-r\zeta e^{i\pi \lambda})^{(1/r)d/d\lambda} u(\lambda) = u(\lambda'), \quad (3.18a)
$$

$$
\lambda' = \lambda + \frac{i}{r} \log(1-r\zeta e^{i\pi \lambda}). \quad (3.18b)
$$

Thus $\lambda$ is altered to complex $\lambda'$, corresponding to the fact that the transformation is non-unitary, and $(3.18b)$ represents a nonlinear realization of the 'infinite-parameter Lie group' generated by $\{A^r\}$. In the limit $r \to 0$, $(3.18b)$ reduces to $(3.10)$ with $\zeta = ir$. By means of $(3.18)$ we can derive also a finite transformation by the gauge operator $A^r$, by the aid of the relation $\exp(\zeta A^r) = e^{-i\zeta \lambda} \exp[(1/r) \times (1-e^{i\pi \lambda}) A^r]$. Namely we get $\exp(-i\zeta A^r)u(\lambda)\exp(i\zeta A^r) = u(\lambda')$ with

$$
\lambda' = \lambda + \frac{i}{r} \log(1-e^{i\pi \lambda} + e^{i\pi (\lambda + 1)}), \quad i.e. \quad \lambda' = \frac{i}{r} \log(1-e^{i\pi \lambda} + e^{i\pi (\lambda + 1)}), \quad (3.19)
$$

which, unlike $(3.18)$, keeps the end $\lambda = 0$ invariant, with $u(0) \to u(0)$ and $v(0) \to v(0)e^{-i\pi \lambda}$. This is natural because the theory is invariant under gauge transformation. In terms of the Koba-Nielsen parameter $z = e^{i\pi \lambda}$, $(3.18b)$ and $(3.19)$ are expressed as $z' = (z^r - r \zeta)^{1/r}$ and $z' = ((z^r - 1)e^{-i\pi \lambda} + 1)^{1/r}$. Finite transformations by $A_1^{(r)} = (A^r + A^r)/(2r)$ and $A_2^{(r)} = (i/2r)(A^r - A^r)$ are unitary and correspond to real transformations on $\lambda$. Its explicit form is obtained again from $(3.18)$, if we note that $A_1^{(r)}$, $A_2^{(r)}$ and $A_3^{(r)} = [(A^r + (r^2 - 1)/6]/r$ satisfy, for each $r$, an $O(2,1)$ algebra, and consequently, e.g.,

$$
\exp(i\zeta A_1^{(r)}) = \left(\sinh \frac{\zeta}{2r}\right)^{-(r^2-1)/3r} \exp\left[\frac{i}{2r} \sinh \frac{\zeta}{2} \cdot A^r\right] \cdot \exp\left[\frac{i}{r} \theta \sinh \frac{\zeta}{2} \cdot A^r\right].
$$

Thus we get $\exp(i\zeta A_1^{(r)})u(\lambda)\exp(-i\zeta A_1^{(r)}) = u(\lambda')$ with

$$
\lambda' = \lambda + \frac{i}{r} \log\left(\frac{1-i \theta (\zeta/2)e^{i\pi \lambda}}{1+i \theta (\zeta/2)e^{-i\pi \lambda}}\right), \quad (3.20)
$$

which in fact transforms $\lambda$ non-linearly into real $\lambda'$. $(3.20)$ is a projective transformation on $z'$:

$$
z' = \left[\left(\sinh \frac{\zeta}{2}e^{-i \theta \sinh \frac{\zeta}{2}}\right) / \left(\sinh \frac{\zeta}{2} + i \theta \sinh \frac{\zeta}{2}\sin \frac{\zeta}{2}\right)\right]^{1/r}.
$$

§ 4. Simple extensions of the model

In the original string model we reach the result that the trajectory intercept must be restricted to one. In order to overcome this limitation it is necessary
to modify the coupling to the external field and/or to extend the original model itself. Next we state about two simple examples in this direction. The first is to change the coupling with the external scalar field \(\phi\) from the scalar one to the vector one. Assuming again that it occurs at the string’s end, we introduce the coupling term \(g \partial (\partial \phi(x_{n}(0)) / \partial x_{n}(0)) (\partial \mathcal{V}[x_{n}] / \partial x_{n}(\sigma))\) to the original detailed wave equation. Then it is equivalent to the set of equations \((A_{0} + \omega_{0} + \phi U) \cdot \mathcal{T} = 0\) and \((A'_{0} - \omega_{0}) \cdot \mathcal{T} = 0\), where \(U = i[\partial \phi(x_{n}(0)) / \partial x_{n}(0)] \rho_{n}(0)\), or for external plane wave \(U = (2\pi)^{-1} e^{ik_{x}(0)k_{y}(0)}\). Compatibility requires that the new vertex operator \(e^{ik_{x}(0)k_{y}(0)}\) should satisfy against \(A'_{0}\) the same commutation relation as (3-9). This is fulfilled if \(k_{x}^{2} = 0\), namely \(A'_{0} = 0\), because we have \([A'_{0}, e^{ik_{y}(0)k_{y}(0)}]\mathcal{T} = r(r + 1) e^{ik_{x}(0)k_{y}(0)}\mathcal{T}\), using \([A'_{0}, \rho_{n}(0)] = r\rho_{n}(0)\). (See (3-9).) Similarly if one assumes tensor coupling to the scalar field, one obtains a consistent model having \(A'_{0} = 1\).

By the above method \(\omega_{0}\) is raised (intercept is lowered) by each one unit. Raising \(\omega_{0}\) by each \(\frac{1}{2}\) unit is realized by introducing to the original string model additional internal variables obeying anticommutation relations, namely a set of internal ‘Fermi oscillators’.\(^{*}\) Next we state such an example which is the simplest in the context of our detailed wave equation. We associate the original model with additional internal variables represented by a real scalar intensity \(b(J_{n}) = b(J_{n})t\) with the properties\(^{**}\)

\[
b(J_{n}) = b(J_{n} + 2\pi), \quad \{b(J_{n}), b(J_{n}')\} = 2\pi \sum_{m} \delta(J_{n} - J_{n}' - 2m\pi).
\]

The free detailed wave equation has now the invariant hamiltonian density consisting of \(\mathcal{H}(\lambda)\) of (3-5) and the additional term \(\mathcal{H}_{0}(\lambda) = (i/4) [b(J_{n}), db/d\lambda] - (1/(8\pi)) \sum_{n} [b(J_{n}'), db/d\lambda'] \cot((J_{n}' - \lambda)/2) d\lambda'\), and is equivalent to the set of equations \((A_{0} + \omega_{0} + L_{0}^{r}) \cdot \mathcal{T} = 0\) and \((A'_{0} - \omega_{0} + A_{r}^{r}) \cdot \mathcal{T} = 0\), where \(L_{0}^{r} = L_{0}^{r} - L_{0}^{-}\), \(L_{0}^{r} = \frac{i}{8\pi} \int_{-\pi}^{\pi} [b, db/d\lambda]: e^{i\tau_{r} d\lambda} \cdot (r = 0, 1, 2, \ldots)\)

The latter satisfy among themselves the same algebra as (2-16), whence \(L_{0}^{r} = L_{0}^{r} + A_{r}^{r}\) also. Furthermore \([L_{0}^{r}, b(J_{n})] = e^{i\tau_{r}} (-i db/d\lambda + (r/2) b(J_{n}))\), analogous to (3-9), so that \([A_{r}^{r}, b(0)] = (r/2) b(0)\). Thus \(b(0)\) has the property of a vertex operator. Due to the periodicity \(b(J_{n})\) is written as \(b(J_{n}) = \sum_{m=0}^{\infty} b^{m} e^{-im\lambda}\), with \(b^{m} = b^{-m}\), \(\{b^{m}, b^{m'}\} = \delta_{m+m',0}\). Further we may use the ‘Pauli matrices’ \(\tau_{r}\) and the set of Fermi oscillator variables \(d^{n}(n = 1, 2, \ldots)\), which satisfy \(\{d^{n}, d^{m}\} = \delta_{n+m,0}\) and \([\tau_{r}, d^{n}] = 0\), in order to write \(b^{n} = \tau_{n}/\sqrt{2}, b^{m} = \tau_{m} d^{n}\), \((n > 0)\). Then \(b(J_{n}) = \tau_{n}/\sqrt{2} + \tau_{1} \sum_{m=1}^{\infty} (d^{n} e^{-im\lambda} + d^{m} e^{im\lambda})\).

\(^{*}\) The idea to construct a unified model of hadrons in terms of the set of internal variables consisting of both the Bose oscillator variables, \(a^{r}\) and \(a^{r}\), and the Fermi oscillator variables, \(b^{r}\) and \(b^{r}\), goes back to Ref. 16). Discussions of the use of internal Fermi oscillator variables and the “hybrid model” are further made in Ref. 17).

\(^{**}\) In Ref. 11) a similar model is given, where, however, \(b(J_{n}) = b(J_{n})\). Thus it has internal degrees twice as many as ours.
In particular $L_0 = \sum_{n=1}^{\infty} n d^n d^{-n}$, which is responsible for new mass degeneracy by the dichotomic eigenvalue of each $d^n d^{-n}$, but $b(\lambda)$ does not contribute to spin. Now we assume the coupling with the external field $\phi$, again at the string's end. Then the term $gb(0)^* e^{tku(0)} \phi(0)$ is added to the global wave operator. This vertex satisfies, against the gauge generators, the relations

$$[A^* + A^*, b(0)^* e^{tku(0)}] = r (k^2 + \nu/2) b(0)^* e^{tku(0)}.$$ 

Thus the detailed wave equation is consistent when $-k^2 = \nu/2 - 1$, which implies that intercept $= \frac{1}{2}$ for $\nu = 1$.

§ 5. Double Dirac string

In the example of § 4 the Fermi-like scalar internal variable $b(\lambda)$ is associated to the string. We now want to consider a model which associates the Fermi-like vector variable $d_\mu(\lambda)$ in place of $b(\lambda)$. However, instead of doing this merely formally we want to achieve this on a more realistic basis and for that purpose we start with a discrete model. Indeed the original string model can be regarded as the limit of a one-dimensional chain composed of structureless elements with positional coordinates $x_\mu(\alpha)$ ($\alpha = 1, 2, \ldots, N$) under specific constructive forces among them. Generalizing this we assume now an array of elements, each of which has its coordinates $x_\mu(a)$ and the internal degrees represented by a 4-vector Fermi oscillator variable $b_\mu(a)$ obeying

$$\{b_\mu(a), (b_\mu(a))^t\} = 0, \quad (\alpha = 1, 2, \ldots, N) \quad (5.1)$$

Note that if we regard $b_\mu(a)$ as a contravariant 4-vector its hermitian conjugate $(b_\mu(a))^t$ must be a covariant 4-vector. Thus (5.1) accords with the Lorentz transformation property and it shows at the same time that every component of $b_\mu(a)$ and $(b_\mu(a))^t$ form a pair of standard Fermi variables. For different elements, the $b_\mu(a)$'s are independent whence

$$[b_\mu(a), b_\nu(\beta)] = [b_\mu(a), (b_\nu(\beta))^t] = 0. \quad (\alpha \neq \beta) \quad (5.2)$$

Further we assume the boundary condition that $b_\mu(a)$ reduces to a 'real' vector at each end, i.e.,

$$(b_\mu(1))^t = \varepsilon_1 b_\mu(1), \quad (b_\mu(N))^t = \varepsilon_0 b_\mu(N). \quad (\varepsilon_1 = 1, \varepsilon_0 = -1) \quad (5.3)$$

Thus, even though we discard possible unitary spin freedom for simplicity, each

*) For the precise meaning that $b_\mu(a)$ is vector, see the second footnote in the next page together with (5.5).
element, except the two elements at the ends, is more complex than a simple Dirac particle.\footnote{1}

Our $b_\mu^{(a)}$ variables can be replaced by the pairs of the standard $\gamma$-matrices, $\gamma_\mu^{(a)}$ and $\beta_\nu^{(a)}$, obeying

\[ \{\gamma_\mu^{(a)}, \gamma_\nu^{(a)}\} = \{\beta_\mu^{(a)}, \beta_\nu^{(a)}\} = 2g_{\mu\nu}, \]

\[ [\gamma_\mu^{(a)}, \gamma_\nu^{(b)}] = [\beta_\mu^{(a)}, \beta_\nu^{(b)}] = 0 \quad (\alpha \neq \beta); \quad [\gamma_\mu^{(a)}, \beta_\nu^{(b)}] = 0, \]

(5.4)

where $\gamma_\mu^{(a)}$ is assumed to be vector while $\beta_\mu^{(a)}$ axial vector.\footnote{2} Due to our convention (2.3) for $g_{\mu\nu}$, (5.4) implies that the 'real' vector $\gamma_\mu^{(a)}$ (or $\beta_\mu^{(a)}$) has hermitian space components and antihermiian time component, i.e. $(\gamma_\mu^{(a)})^\dagger = \epsilon_{a}^{\gamma} \gamma_\mu^{(a)}$, etc. In terms of them $b_\mu^{(a)}$ is written as

\[ b_\mu^{(a)} = \frac{1}{2} (\gamma_\mu^{(a)} + i\gamma_5^{(a)} \beta_\mu^{(a)}), \]

\[ (b_\mu^{(a)})^\dagger = \frac{1}{2} \epsilon_{a}^{\gamma} (\gamma_\mu^{(a)} - i\gamma_5^{(a)} \beta_\mu^{(a)}) = -\epsilon_{a}^{\gamma} \beta_\mu^{(a)} b_\mu^{(a)}, \]

(5.5)

which indeed fulfill all of the properties stated above for $b_\mu^{(a)}$. Conversely

\[ \gamma_\mu^{(a)} = b_\mu^{(a)} - \frac{i}{2} \beta_\mu^{(a)}, \quad \beta_\mu^{(a)} = -i (\gamma_5^{(a)} b_\mu^{(a)} + b_\mu^{(a)} \gamma_5^{(a)}), \]

(5.3) is reexpressed as

\[ \beta_\mu^{(1)} = \beta_\mu^{(N)} = 0. \]

Thus the model may be interpreted as the 'double Dirac string' (somewhat reminiscent of 'double helix' for DNA), where the two strings which are linked at their ends coalesce in their locations, leaving their degrees of $\gamma$-matrices of the constituents independent. (See Fig. 1(a)). Indeed this structure has already been suggested at the stage when we considered the ring model consisting of structureless elements.\footnote{3} That is, that model gave doubled number of normal modes represented by the operators $a_\mu^{(a)}$ and $b_\mu^{(a)}$ ($r = 1, 2, \cdots$), and in order to suppress one of them we impose the constraint that the ring (b) be folded into the form (a).

\[ (a) \]

\[ (b) \]

Fig. 1. The schematic representation of double-string model and ring model.

\footnote{1} Recently Ahronov et al.\footnote{2} assumed a rather similar structure which is, however, associated with a single Dirac $\gamma^{(a)}$, instead of our complex $b_\mu^{(a)}$ or the pair $\gamma_\mu^{(a)}$ and $\beta_\mu^{(a)}$. Incidentally the idea of a model of quark-antiquark string associated with spin-wave motion was talked in the second paper of Ref. 5).

\footnote{2} By this we mean more precisely the following. We start with $\gamma_\mu^{(a)}$ and $\gamma_\nu^{(a)}$ ($\mu = 1, 2, \cdots, N$) satisfying $\{\gamma_\mu^{(a)}, \gamma_\nu^{(a)}\} = 2g_{\mu\nu}$, $[\gamma_\mu^{(a)}, \gamma_\nu^{(a)}] = 0$. We say $\gamma_\mu^{(a)}$ and $\gamma_\nu^{(a)}$ behave as vectors in the sense that $\vec{F} \gamma_\mu^{(a)} \vec{F} = \gamma_\nu^{(a)}$ and $\vec{F} \gamma_\mu^{(a)} \vec{F} = 0$. We now define $\beta_\mu^{(a)}$ by $\beta_\mu^{(a)} = \gamma_5^{(a)} \gamma_\mu^{(a)}$ (whence $\beta_\mu^{(a)} = \gamma_5^{(a)}$), then $\gamma_\mu^{(a)}$'s satisfy (5.4), and $\vec{F} \beta_\mu^{(a)} \vec{F}$ is pseudovector.
Next, since the set of \( \gamma_{\mu}^{(a)} \) (or \( \beta_{\mu}^{(a)} \)) has a definite ordering by the label \( \alpha \), we can transform them into the following new quantities in an essentially unique way:
\[
\gamma_{\mu}^{(a)} = \prod_{i=1}^{\alpha-1} \gamma_{i}^{(a)} \beta_{i}^{(a)\nu} \gamma_{\nu}^{(a)}, \quad \beta_{\mu}^{(a)} = \prod_{i=1}^{\alpha} \gamma_{i}^{(a)} \prod_{i=1}^{\alpha-1} \beta_{i}^{(a)\nu} \beta_{\nu}^{(a)},
\]
which are both vectors and satisfy the complete anticommutation rule
\[
\{ \gamma_{\mu}^{(a)}, \gamma_{\nu}^{(b)} \} = \{ \beta_{\mu}^{(a)}, \beta_{\nu}^{(b)} \} = 2\delta_{\mu\nu} g_{\mu\nu}, \quad \{ \gamma_{\mu}^{(a)}, \beta_{\nu}^{(b)} \} = 0, \quad (5.7)
\]
and the reality relation \( (\gamma_{\mu}^{(a)})^{*} = \varepsilon_{n} \gamma_{\mu}^{(a)} \), etc. Then, by taking the limit \( N \to \infty \) we go over to the continuum case, where with the use of \( \sigma = \pi x/N \) in place of \( \alpha \), \( x_{\mu}^{(a)}, \gamma_{\mu}^{a} \) and \( \beta_{\mu}^{a} \) go over to \( x_{\mu}(\sigma), \gamma_{\mu}(\sigma) \) and \( \beta_{\mu}(\sigma) \), \( (0 \leq \sigma \leq \pi) \), respectively, and (5.8) goes over to \( \delta \)-function anticommutation relations. Since we also impose the boundary conditions
\[
\frac{d\gamma_{\mu}(\sigma)}{d\sigma} \bigg|_{\sigma=0,\pi} = 0, \quad \beta_{\mu}(0) = \beta_{\mu}(\pi) = 0, \quad (5.9)
\]
the latter of which being (5.6), the anticommutation relations are
\[
\{ \gamma_{\mu}(\sigma), \gamma_{\nu}(\sigma') \} = 2\pi g_{\mu\nu} \delta(\sigma - \sigma') + \delta(\sigma + \sigma') + \delta(2\pi - \sigma - \sigma'), \quad \{ \beta_{\mu}(\sigma), \beta_{\nu}(\sigma') \} = 2\pi g_{\mu\nu} \delta(\sigma - \sigma') - \delta(\sigma + \sigma') - \delta(2\pi - \sigma - \sigma'),
\]
\[
\{ \gamma_{\mu}(\sigma), \beta_{\nu}(\sigma') \} = 0. \quad (5.10)
\]
The free wave equation for our double Dirac string may be assumed as
\[
\left[ \int_{0}^{\pi} \left( \gamma(\sigma) \beta(\sigma) + \frac{1}{2\pi} \beta(\sigma) \frac{dx}{d\sigma} \right) d\sigma + \mu_{0} \right] F = 0, \quad (5.11)
\]
where the constant \( \mu_{0} \) is normally pure-imaginary because then the conserved current exists,* and \( F \) is a functional of \( x_{\mu}(\sigma) \) and at the same time infinite-component multispinor. According to our general postulate we replace (5.11) by the following detailed wave equation:
\[
K(\sigma) F = 0, \quad K(\sigma) = \pi(F_{1}(\sigma) + \tilde{F}_{2}(\sigma)) + \mu_{0}, \quad (5.12)
\]
where
\[
F_{1}(\sigma) = \gamma(\sigma) \beta(\sigma) + \frac{1}{2\pi} \beta(\sigma) \frac{dx}{d\sigma}, \quad F_{1}(\sigma) = \beta(\sigma) \gamma(\sigma) + \frac{1}{2\pi} \gamma(\sigma) \frac{dx}{d\sigma},
\]
and \( \tilde{F}_{2}(\sigma) \) denotes the Hilbert transform of \( F_{2}(\sigma) \). (C.f. (2.7)). Note that (5.12) is compatible with (5.9) and that it is invariant under space-inversion. Via Fourier expansion, (5.12) is equivalent to the set
\[
(K^{r} + \mu_{0}) F = 0, \quad K^{r} F = 0, \quad (r=1, 2, \cdots) \quad (5.13a, b)
\]
* Cf. our convention for the metric \( g_{\mu\nu} \) and the definition of \( \gamma \)-matrices.
with
\[ K' = \int_0^\pi (F_1(\sigma) \cos r\sigma + iF_2(\sigma) \sin r\sigma) d\sigma \quad (r = 0, 1, 2, \ldots) \] 
(5.13a) recovers (5.11). Under the internal reflection, we have \( \gamma(\sigma) \to \gamma(\pi - \sigma) \), \( \tilde{\gamma}(\sigma) \to -\tilde{\gamma}(\pi - \sigma) \), under which (5.9) and (5.10) are invariant. Also the detailed wave equation is invariant since \( K(\sigma) \to K(\pi - \sigma) \).

(5.12) corresponds to taking the 'Dirac root' of both the kinetic and the tension terms of (2.6) simultaneously by the aid of the two Dirac matrix functions \( \gamma_\mu(\sigma) \) and \( \tilde{\gamma}_\mu(\sigma) \). Conversely we iterate (5.12) into \( \{i^r K(\sigma')d\sigma', K(\sigma)\} \Phi = 0 \).

This becomes
\[ (H(\sigma) + H_\delta(\sigma)) \Phi = 0, \quad (5.14) \]
where \( H(\sigma) \) is the operator (2.6) with the identification \( \omega_\delta = -\mu_\delta \), while \( H_\delta(\sigma) \) consists of \( \gamma_\mu \) and \( \tilde{\gamma}_\mu \) alone. If we employ \( L_\delta' = (1/\pi) \int_0^\pi H_\delta(\sigma) \cos r\sigma d\sigma = (i/4\pi) \times \left\{ \int_0^\pi (\gamma(\sigma) \cdot d\tilde{\gamma}(\sigma)/d\sigma - d\gamma(\sigma)/d\sigma \cdot \tilde{\gamma}(\sigma)) \cos r\sigma d\sigma + (1/8\pi) \left\{ (d\gamma/\partial\sigma, \gamma(\sigma)) + (d\tilde{\gamma}/\partial\sigma, \tilde{\gamma}(\sigma)) \right\} \sin r\sigma d\sigma \right. \), \( (r = 0, 1, 2, \ldots) \), and \( L' = A' + L_\delta' \), (5.14) is reexpressed as
\[ (L' + \omega_\delta) \Phi = 0, \quad L' \Phi = 0. \quad (r = 1, 2, \ldots) \quad (5.15) \]
The \( L_\delta' \)'s satisfy among themselves the same algebra as (2.16), whence \( L' \)'s also:
\[ [L', L^*] = (r - s) L'^{r+s}. \quad (5.16) \]
On the other hand \( L' \) is produced from \( K' \) by iteration, satisfying
\[ \{K', K''\} = 2L'^{r+s}, \quad [L', K'] = \left( -\frac{1}{2} r - s \right) K'^{r+s}, \quad (5.17) \]
and (5.16) results from (5.17). The algebra (5.16) and (5.17) for \( (K', L') \) ensures the compatibility of the detailed wave equation.

In place of \( \gamma_\mu(\sigma) \) and \( \tilde{\gamma}_\mu(\sigma) \) defined over \( 0 \leq \sigma \leq \pi \), we can employ \( d_\mu(\lambda) \) defined over \( -\pi \leq \lambda \leq \pi \), such that
\[ d_\mu(\sigma) = \gamma_\mu(\sigma) + \tilde{\gamma}_\mu(\sigma), \quad d_\mu(-\sigma) = \gamma_\mu(\sigma) - \tilde{\gamma}_\mu(\sigma). \]
Then owing to (5.9)
\[ d_\mu(0) = \gamma_\mu(0), \quad d_\mu(+\pi) = d_\mu(-\pi) = \gamma_\mu(\pi). \]
Hence \( d(\lambda) \) is continuous at \( \lambda = 0 \) and also its domain can be extended by \( d(\lambda + 2\pi) = d(\lambda) \) over the whole \( -\infty < \lambda < \infty \). Also by (5.9), \( dd_\mu(\lambda)/d\lambda \) is continuous at \( \lambda = 0 \). The anticommutation relations become
\[ \{d_\mu(\lambda), d_\nu(\lambda')\} = 4\pi g_{\mu\nu} \sum_{n} \delta(\lambda - \lambda' - 2n\pi). \]
Further \( d_\mu(\lambda) \), like \( \gamma_\mu(\sigma) \) and \( \tilde{\gamma}_\mu(\sigma) \), satisfies \( d_\mu(\lambda)' = \varepsilon_\mu d_\mu(\lambda) \). Note that this fixed reality property is consistent because the parameter \( \sigma \) or \( \lambda \) means the label for each element and not the time and because we are working in the 'Schrödinger picture' where physical quantities do not depend on time. The detailed equation (5.12) is now reexpressed as
\[ K(\lambda) \Psi = 0, \quad K(\lambda) = \frac{1}{2} d(\lambda) v(\lambda) + \frac{i}{4\pi} \int_{-\pi}^{\pi} d(\lambda') v(\lambda') \cot \frac{\lambda'-\lambda}{2} d\lambda' + \mu_0, \]

and correspondingly the iterated equation (5·14) as

\[ (\bar{H}(\lambda) + \bar{H}_d(\lambda)) \Psi = 0, \]

\[ H_d(\lambda) = i \int_{-\pi}^{\pi} \left[ d(\lambda'), \frac{d(d(\lambda'))}{d\lambda'} \right] \left( \delta(\lambda'-\lambda) + \frac{i}{2\pi} \cot \frac{\lambda'-\lambda}{2} \right) d\lambda'. \]

\( K' \) and \( L_d' \) are now expressed as

\[ K' = \frac{1}{4\pi} \int_{-\pi}^{\pi} d(\lambda) v(\lambda) e^{ir\lambda} d\lambda, \]

\[ L_d' = \frac{i}{16\pi} \int_{-\pi}^{\pi} \left[ d(\lambda), \frac{d(d(\lambda))}{d\lambda} \right] e^{ir\lambda} d\lambda = \frac{-1}{8\pi} \int_{-\pi}^{\pi} d(\lambda) \left( -i \frac{d(d(\lambda))}{d\lambda} + \frac{r}{2} d(\lambda) \right) e^{ir\lambda} d\lambda. \]

Under the gauge generators \( d_\sigma(\lambda) \) behaves as \([L_d', d_\sigma(\lambda)] = (-i [dd_\sigma(\lambda)/d\lambda] + (r/2) d_\sigma(\lambda)) e^{ir\lambda} \), whence \([L_d'-L_d^0, d_\sigma(0)] = (r/2) d_\sigma(0) \), and so \( d_\sigma(0) = \bar{\gamma}_\sigma(0) \) has the property of a vertex operator. As for \( K' \), \([K', u(\lambda)] = -id(\lambda) e^{ir\lambda}, [K', e^{i\lambda(\sigma)}] = kd(0) e^{ik\pi(0)}, [K', d_\sigma(\lambda)] = v_\sigma(\lambda) e^{ir\lambda}, [K' - K', d_\sigma(0)] = 0 \), etc.

The internal degrees \( \bar{\gamma}_\sigma(\sigma) \) and \( \bar{\beta}_\sigma(\sigma) \) do not contribute to the 4-momentum, while the angular momentum tensor is the sum of \( M_\mu \) formerly given and \( S_\mu = \frac{8\pi i}{l_0} \int d_\sigma(\lambda) d_{\sigma 1}(\lambda) d\lambda \), which contains half-integer spin states. This \( S_\mu \) is the limit of \((4i)^{-1} \sum_{n=0}^{\infty} (\bar{\gamma}_\sigma^{(n)} + \bar{\beta}_\sigma^{(n)}) \).

In accordance with (5·9), \( \bar{\gamma}(\sigma) \) and \( \bar{\beta}(\sigma) \) are expanded as

\[ \bar{\gamma}(\sigma) = \bar{\gamma}^0 + \sqrt{2} \sum_{n=1}^{\infty} \bar{\gamma}_n^m \cos n\sigma, \quad \bar{\beta}(\sigma) = \sqrt{2} \sum_{n=1}^{\infty} \beta_n^m \sin n\sigma, \]

with \( \{ \bar{\gamma}_n^m, \bar{\beta}_n^m \} = \{ B_n^m, B_n^m \} = 2g_{\mu n} \delta_{n,m}, \{ \bar{\gamma}_n^m, B_n^m \} = 0 \). Correspondingly, \( d_\sigma(\lambda) \) is expanded as

\[ d_\sigma(\lambda) = \sqrt{2} \sum_{n=-\infty}^{\infty} d_\sigma^n e^{-in\lambda}, \quad \{ d_\sigma^n, d_\sigma^m \} = \delta_{n+m,0} g_{\mu n}, \quad (5\cdot18a, b) \]

where \( d_\sigma^n \) are expressed by \( \bar{\gamma}_n^m \) and \( B_n^m \) as

\[ d_\sigma^0 = \frac{1}{\sqrt{2}} \bar{\gamma}_n^0, \quad d_\sigma^n = \frac{1}{2} (\bar{\gamma}_n^0 + iB_n^0), \quad d_\sigma^{-n} = \frac{1}{2} (\bar{\gamma}_n^0 - iB_n^0), \quad (n>0) \quad (5\cdot19) \]

and (5·18a) is rewritten as

\[ d_\sigma(\lambda) = \bar{\gamma}^0 + \sqrt{2} \sum_{n=1}^{\infty} (d_\sigma^n e^{-in\lambda} + d_\sigma^{-n} e^{in\lambda}). \quad (5\cdot20) \]

It is again important to note that since \( \bar{\gamma}_n^{(n)} = \varepsilon_\sigma \bar{\gamma}_n^{\sigma}, B_n^{(n)} = \varepsilon_\sigma B_n^{\sigma} \), (5·19) implies

\[ (d_\sigma^n)^0 = \frac{1}{2} \varepsilon_\sigma (\bar{\gamma}_n^{0} - iB_n^{0}) = \varepsilon_\sigma d_\sigma^{-n} \quad (n\geq0). \]

Indeed for all \( n \geq 0 \)

\[ (d_\sigma^n) = \varepsilon_\sigma d_\sigma^{-n} = g^{s\sigma} d_\sigma^{-n}. \quad (5\cdot21) \]

In the special case \( n + m = 0 \), (5·18b) means \( \{ d_\sigma^n, d_\sigma^{-n} \} = g_{\mu n} \), which is \( \{ d_\sigma^n, (d_\sigma^{-n}) \} = \delta_{\mu n} \), indicating that every component of \( d_\sigma^n \) and \( (d_\sigma^{-n}) \) are standard Fermi oper-
ators, \( \{d^n_0, d^n_0\} = 1 \), involving no difficulty.) In terms of \( d^n_0 \), \( K^n = \sum_{n=-\infty}^{\infty} d^n C^n \). Accordingly the global equation \( (5 \cdot 11) \) is expressed as \[ [\tilde{\Gamma}^0 P + \sum_{n=-1}^{\infty} (d_0^n C^n + C^n d_0^n) + \mu_0] \psi = 0 \].

This has the form of the generalized Dirac equation, where \( \Gamma_0 \psi = (1/\pi) \int_0^\infty \tilde{\gamma}_\rho(\sigma) d\sigma \) plays the role of Dirac’s \( \gamma_\rho \). Note that \( \langle \tilde{\psi}, \psi \rangle = 0 \), and \( \Gamma_0 = \tilde{\Gamma}_0 \psi = \tilde{\psi} \). Further

\[
L^r_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( n + \frac{r}{2} \right) d^{r-n} d^n = \sum_{n=0}^{\infty} \left( n + \frac{r}{2} \right) d^{r-n} d^n + \sum_{n=1}^{\infty} \left( n + \frac{r}{2} \right) d^{r-n} d^n.
\]

In particular \( L^r_0 = \sum_{n=\infty}^{\infty} n d^{r-n} d^n = \sum_{n=\infty}^{\infty} n (d_0^n + d_0^n) \), so that the global equation or its iteration implies a new mass degeneracy. Further \( S_{\psi} = (1/4i) [\tilde{\Gamma}^0_0 \Gamma_0^2 + \sum_{n=0}^{\infty} n d_0^n d_0^n + (1/i) \sum_{n=0}^{\infty} d_0^n d_0^n] \). Under the internal reflection, \( d(\lambda) \to d(\lambda + \pi) \). This is induced by the unitary operator \( (-1)^{N+L_0^r} \), whereby \( d^n_0 \to (-1)^n d^n_0 \), \( L^r_0 \to (-)^r L^r_0 \).

At this point we remark that the detailed wave equation \( (5 \cdot 12) \) or its equivalent \( (5 \cdot 13) \) can be replaced by the following two equations only:

\[
(K^0 + \mu_0) \psi = 0, \quad K^r \psi = 0,
\]

since from these all of the remaining equations of \( (5 \cdot 13) \) result through the algebraic relations \( (5 \cdot 17) \). One may equally take the two equations, \( (K^0 + \mu_0) \psi = 0 \) and \( L^r \psi = 0 \), as the fundamental ones.

In order to introduce the coupling with the external scalar field we should rather start with the iterated detailed equation \( (5 \cdot 14) \) to insert the coupling term that occurs again at the string’s end. That is, we assume \( [H(\sigma) + H_0(\sigma) + g(\sigma) U] \psi = 0 \), which is equivalent to the set

\[
(L^0 + \omega_0 + g U) \psi = 0, \quad (L^r - L^0 - \omega_0) \psi = 0.
\]

Consistency requires \( [L^r - L^0, U] = r U \) as before. If we assume the vector coupling \( U = -id_\rho(0) \partial \phi / \partial x_\rho(0) \), i.e., \( U = k\tilde{\gamma}_\rho(0) e^{i\kappa x(0)} \phi(0) \), where \( d_\rho(0) = \tilde{\gamma}_\rho(0) \) means the Dirac matrix \( \gamma_\rho(1) \) of the element at the end. Since

\[
[L^r - L^0, k\tilde{\gamma}_\rho(0) e^{i\kappa x(0)}] = r (k^2 + \frac{1}{2}) k\tilde{\gamma}_\rho(0) e^{i\kappa x(0)}
\]

the consistency is fulfilled if \( k^2 = \frac{1}{2} \), meaning that intercept \( = -\omega_0 = \frac{1}{2} \). If we assume the tensor coupling, so that \( U = k^* k\tilde{\gamma}_\rho(0) \tilde{\gamma}_\rho(0) e^{i\kappa x(0)} \phi(0) \), the model is consistent for \( k^2 = 0 \), i.e., \( \omega_0 = 0 \).

In §§ 4 and 5 we have constructed in the framework of our detailed wave equation consistent models which imply the interpret differing from 1. Further

*) Operators like \( d_\rho(4) \) has recently been employed by other authors\(^{11},^{16},^{19} \) to consider similar models. Their operators, however, are different from ours with respect to their reality and/or Lorentz properties, since theirs, unlike ours, are always expanded in the form of the type \( \sum_{n} (d^n e^{-inx} + d^n e^{inx}) \) with \( [d^n_0, d^n_0] = \delta_{n=0} \delta_{n=0} \). In this case one has \( [d^n_0, d^n_0] = -1 \) contradicting the positive semi-definiteness of \( [d^n_0, d^n_0] \), and one needs the introduction of a new indefinite metric. In contrast, in our equation \( (5 \cdot 20) \) \( d^{-n} \) on the right side is rewritten as \( e_d d^{-n} d^n \) but not as \( d^n_0 \).

**) Wave equation apparently similar to this one has recently been obtained by Ramond\(^{12} \), but his equation is not invariant under space reflection and also implies indefinite metric regarding Fermi variables (see the preceding footnote), in contrast to ours.
properties of these models including the discussion about the duality property of the amplitude resulting therefrom and the further generalization of the model to more realistic cases will be investigated subsequently.

§ 6. Concluding remarks

To conclude we would like to add and repeat a few remarks.

(a) Our theory is based on the postulate that the relativistic quantum mechanics of an extended system with external interaction should be represented completely by a DWE. We reached this idea partly from analogy to the many-time theory of Dirac,\textsuperscript{20} although the physical implications of both theories are not the same. The concept of DWE is motivated mainly from the reason that the time coordinates, one for each element constituting an extended system, are necessary for covariant formalism but nevertheless, not being proper dynamical degrees, have to be suppressed, and that \textit{this should concern each elementary constituent of the system uniformly}. At the same time we are anticipating that this suppression of relative-time degrees implied by DWE should just correspond to the ghost elimination, though still this is a conjecture, which has been verified only with respect to lower excited levels for the original string model. However, from our viewpoint the postulate of DWE is a quite general one such that a complete quantum mechanics of any relativistic composite system should in principle be represented in this framework, irrespective of whether its degrees of freedom be infinite or finite, (apart from non-composite systems such as a single Dirac particle), and the existence condition of a consistent DWE with interaction selects permissible models, resulting in the quantization of the Regge intercept.

Even in the non-relativistic theory the DWE may be imposed consistently. Thus, for the non-relativistic string the basic variables are $x(\sigma)$ and $p(\sigma)$ and the wave functional is $\Psi[x(\sigma), t]$. By replacing the quantities $F(\sigma)$ and $G(\sigma)$ in § 2 with $F_{NR}=\frac{1}{2}[K(dx/d\sigma)^2+(1/\mu)p(\sigma)^2]$ and $G_{NR}=\frac{1}{2}K/\mu\{dx/d\sigma, p(\sigma)\}$, ($K$ and $\mu$ are constants), we can write down the DWE analogous to (2.5) associated with (2.6). In this case, however, the usual wave equation of the string, $i\partial \Psi/\partial t = -\sigma F_{NR}(\sigma)d\sigma$, defines all allowable physical motions for the string properly, while the DWE implies to exclude some of them. By contrast, in the relativistic theory DWE purports to exclude unphysical ghost states.

(b) The DWE is defined at each element (labelled by $\sigma$) by the invariant hamiltonian density. Via expansion with respect to a complete set of basis functions of $\sigma$, the basic variables of the model are replaced by the set of normal mode operators such as $\mathcal{C}_\sigma^*$ and correspondingly the DWE is brought to a set of countably infinite simultaneous equations defined by a countably infinite number of operators such as $\mathcal{A}_\sigma^*$, which are expressed in terms of the normal mode operators and form a closed algebra. One may start with this discrete set of equations, forgetting the original DWE completely. Then the continuous label $\sigma$ describing
the extension of the system disappears in favor of the index numbering the normal
mode operators. One may furthermore take the standpoint of treating these
operators purely operationally free from any structural meaning. Still the DWE
is originally essential in defining the underlying physical model and in fact it
determines how to identify the 4-momentum and external interactions.

On the other hand we can go over from the original \( \sigma \)-formalism to the com-
pletely equivalent \( \lambda \)-formalism. Then the basic variables are represented by \( u_\mu (\lambda) \)
alone which itself has the appearance of a positional coordinate, and the free
invariant hamiltonian density is represented in terms of \( (du_\mu (\lambda)/d\lambda)^2 \)
alone as if there existed tension only (or else the kinetic term alone if \( \lambda \) is reinterpreted as \( \tau \)).
However, \( u_\mu (\lambda) \) is not a simple positional coordinate for the system, as is
observed from the commutation relation and the constructions of momentum,
angular momentum and parity of the system; also the external interaction does
not occur in DWE just at \( \lambda = 0 \) but occurs according to the factor \( \delta_\tau (\lambda) = \delta (\lambda)
- \left( i/2\pi \right) \cot (\lambda/2) = - \left( i/2\pi \right) \cdot 1 / (\tan (\lambda/2) - i\varepsilon) \).

(c) In §§ 4 and 5 we considered the hybrid model in which Bose oscillators
and Fermi oscillators coexist. Especially we payed special attention for the case
of the 4-vector Fermi oscillator variables, which in the field-theoretic analogy
corresponds to an abnormal spin-statistics case. We assign to those Fermi vari-
ables, denoted like \( d_\mu (\lambda) \) or \( d^{\cdot \cdot} \), the property \( d_\mu (\lambda) \) = \( d^\ast (\lambda) \)
or \( (d_\mu \cdot \cdot) = d^{\cdot \cdot \cdot} \), in contrast to the vector Bose variables which have the property \( u_\mu (\lambda) \) = \( u^\ast (\lambda) \)
or \( (C_\mu \cdot \cdot) = C^{\cdot \cdot \cdot} \). It is to be noted that those Fermi variables \( d_\mu (\lambda) \) represent the
internal variables conveying Dirac-like intrinsic spin and do not mean the usual
field. (*) Indeed we regarded this model as the limit of an array of Dirac-like
elements associated with Fermi oscillator variables \( b_\mu (a) \), or equivalently \( \gamma_\mu (a) \)
and \( \beta_\mu (a) \). The multispinor wave functional \( \Psi \) will have to satisfy, in so far as it
describes an assembly of Dirac-like elements, the antisymmetry requirement against
permutation, and this will impose a delicate problem in the continuum limit.
However, we should note that the usual spin-statistics relation need not neces-
sarily apply to the assembly of elementary constituents (or say partons)
constituting our hadron model.

(d) The bilocal case \( N = 2 \) in the theory given in § 5 can be treated in an
interesting way. We thus obtain a very simple hybrid model which represents
baryonic states lying on linear trajectories. The wave equation is

\[
\left[ \gamma_\mu P^\mu + \frac{1}{\sqrt{2}} \gamma_\mu \gamma_\nu \left( \gamma_\nu P^\nu + \kappa \gamma_\nu x^\nu \right) \right] \Psi = 0 ,
\]

(*) The difference in the natures of internal Fermi variables and Bose variables is discussed
with regard to “the internal Lorentz group” \( L_{10}^n \) or to the “group of internal motion” \( G^{10} \), in ref.
17). Namely for the former the wave function corresponds to the base of non-unitary representa-
tion of \( L_{10}^n \) while for Bose variables it corresponds to the base of unitary infinite-dimensional rep-
resentation of \( L_{10}^n \).
where $x = (x^{(1)} - x^{(2)})/(2\sqrt{2})$, $p = \sqrt{2}(p^{(1)} - p^{(2)})$ with $[x^{(a)}_\mu, p^{(b)}_\nu] = i\theta_{ab}g_{\mu\nu}$, $m_0$ = real const, and $\gamma_\mu, \gamma'_\mu$ and $\gamma''_\mu$ are three independent $\gamma$-matrices satisfying the relations like (5.4). To this wave equation we can associate the subsidiary condition $(\gamma'_\mu + \gamma''_\mu)P^a\Psi = 0$. Owing to the existence of the three independent $\gamma$-matrices, which are analogous to the spins of three quarks, the equation gives only half-integer spin states, which lie on parallel trajectories. The leading trajectory is given by

$$a(s) = \frac{s}{\xi} + \left(\frac{1}{2} - \frac{m_0^2}{\xi}\right),$$

whose intercept must be less than $\frac{1}{2}$, in nice agreement with the empirical baryonic trajectories.

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**Appendix I**

We note a few mathematical formulas employed.

$$\int_0^\pi \frac{\sin \sigma}{\cos \sigma' - \cos \sigma \pm i\xi} d\sigma' = \mp i\pi, \quad P \int_0^\pi \frac{d\sigma'}{\cos \sigma' - \cos \sigma} = 0, \quad (0 < \sigma < \pi)$$

$$\int_0^\pi \frac{P \sin \sigma_1}{\cos \sigma' - \cos \sigma_1} \frac{P \sin \sigma_2}{\cos \sigma' - \cos \sigma_2} d\sigma' = \pi^2 \delta(\sigma_1 - \sigma_2), \quad (0 < \sigma_1, \sigma_2 < \pi)$$

$$P \int_{-\pi}^\pi \cot \frac{\lambda - \lambda'}{2} e^{\epsilon r'\lambda'} d\lambda' = -2\pi i e^{r'\lambda} \text{sgn } r.$$  

(\text{sgn } r = 1, 0, -1 \text{ for } r > 0, = 0, < 0)

**Appendix II**

In the $\lambda$-formalism the derivation of (2.23) is made as follows. Due to (3·3) we have $[v_\rho(\lambda), e^{i ku(\lambda')}] = -4\pi i(\delta e^{i ku(\lambda')}/\delta u(\lambda)) = 4\pi k\delta(\lambda - \lambda')e^{i ku(\lambda)}$. Using this and (3·12) we get

$$[A_N^r, e^{i ku(\lambda)}] = \frac{1}{2\pi} e^{i ku(\lambda)} \int k v(\lambda') e^{i r(\lambda + \lambda')/\xi} d\lambda'$$

$$+ k^2 e^{i ku(\lambda)} \int \delta(\lambda - \lambda') e^{i r(\lambda + \lambda')/\xi} dN^r(\lambda - \lambda) d\lambda'.$$

On taking $N \to \infty$, we have $A_N^r(\lambda - \lambda) \to 2\pi \delta(\lambda - \lambda)$ in the first term of the right side, while in the second term we must use $\lim_{v \to \infty} A_N^r(\lambda - \lambda) = r + 2N + 1$. Thus

$$[A_N^r, e^{i ku(\lambda)}] \approx e^{i r\lambda} e^{i ku(\lambda)} \left[ k \frac{d\mu}{d\lambda} + k^2 (r + 2N + 1) \right].$$

Noting $u(0) = x(0)$, this again leads to (2.23).
References