It is a rather curious fact that the exact quantal treatment of forced oscillation has only recently been discovered to be feasible. There may be a lot of general arguments\(^1\) which make the possibility of such a treatment very convincing, but I myself find the ground for it in an elementary observation that the wave equation of forced harmonic oscillation, including the case of subharmonic resonance, admits a solution of Gaussian type, i.e., of the form: an exponential of a quadratic. On this basis the complete solution and the related problems are analyzed here.

II. The forced harmonic oscillation

§ 1. Solutions of Gaussian type

Let the wave equation be

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{m}{2} \omega(t)^2 x^2 \psi - f(t) x \psi,
\]  

(1.1)

where \( f \), the impressed force, may be an arbitrary function of time variable \( t \). Moreover, we admit the circular frequency \( \omega \) also to be variable in time. It is evident, by mere inspection, that the equation can be satisfied by a function of the form:

\[
\psi = \exp \left[ \frac{i}{2\hbar} \left\{ a(t) x^2 + 2b(t) x + c(t) \right\} \right].
\]  

(1.2)

Inserting this expression into eq. (1.1) we get quadratic expression on both sides, and we have enough degrees of freedom (three coefficients \( a, b, c \)) to be able to equate the coefficients of the different power of \( x \) separately. We get in fact

\[
\begin{aligned}
\frac{1}{m} \frac{da}{dt} &= -\frac{a^2}{m^2} - \omega^2, \\
\frac{db}{dt} &= f - \frac{a}{m} b, \\
\frac{dc}{dt} &= \frac{i\hbar}{m} \frac{a}{m} - \frac{1}{m} b^2.
\end{aligned}
\]  

(1.3a)  

(1.3b)  

(1.3c)

If the first equation is solved, it is a simple matter to treat the second, which is linear, and the third equation.
The first equation (1·3a) is of Riccati type, and it is a well known fact that this
equation is an outcome of the classical equation of force free oscillation:

$$\frac{d^2X}{dt^2} = -\omega(t)^2X.$$  \hspace{1cm} (1·4)

In fact the ratio of momentum to coordinate

$$a = m\frac{X}{\dot{X}}$$  \hspace{1cm} (1·5)

satisfies the Riccati equation (1·3a). We may assume the classical equation has already
been solved. Then the solution of the second equation (1·3b) can be explicitly written
down:

$$b = \frac{1}{X} \left( \text{const} + \int fX \, dt \right),$$  \hspace{1cm} (1·6)

and the third equation gives

$$c = i\hbar \log X - \frac{1}{m} \int \delta^2 dx.$$  \hspace{1cm} (1·7)

The imaginary part gives rise to the amplitude.

The corresponding circumstance one finds also in the classical theory. If we try as
usual to solve (1·1) by putting

$$\psi = A \exp \left( iS/\hbar \right) \quad (A, S: \text{real}),$$ \hspace{1cm} (1·8)

we get by separating the real and the imaginary parts:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \frac{m\omega(t)^2x^2 - f(t)x - 2\hbar^2}{2m} \frac{1}{A} \frac{\partial^2 A}{\partial x^2},$$

$$-m\frac{\partial A}{\partial t} = \frac{\partial S}{\partial x} \frac{\partial A}{\partial x} + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} A.$$

If it is assumed that $A$ is at most a linear function of $x$, then the so-called quantal
potential $\propto \frac{1}{A} \frac{\partial^2 A}{\partial x^2}$ vanishes and the first equation becomes exactly equal to the
classical Hamilton-Jacobi equation, which is again soluble by a quadratic function of $x$.
One gets a similar set of ordinary differential equations as (1·3). The second equation
is compatible with the assumption that $A$ is linear and $S$ is quadratic in $x$. Actually
we have in the foregoing discussion taken $A$ to be dependent only on $t$.

\section*{§ 2. The reduction of the problem to the case $f = 0$}

Before proceeding it is convenient to reduce our problem to the simpler case of
vanishing $f$. The following canonical transformation has kindly been supplied by my
colleague Mr. T. Taniuti, Kobe University.

The coordinate of the particle may be referred to a moving origin; let the new
coordinate be

\[ x_1 = x - \xi(t), \quad (2.1) \]

where \( \xi(t) \) is to be specified later. In terms of \( x_1 \), the wave equation (1.1) becomes

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{2} m\omega^2 \cdot (x_1 + \xi)^2 \psi - f \cdot (x_1 + \xi) \psi, \]

which is transformed, by putting

\[ \psi = \exp(i m \xi \cdot \frac{x_1}{\hbar}) \varphi, \quad (2.2) \]

into the following:

\[ i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{1}{2} m\omega^2 x_1 \varphi + \left\{ m \frac{\ddot{x}}{\xi} + m \omega^2 \xi - f \right\} x_1 \varphi \]

\[ + \left\{ -\frac{1}{2} m \xi^2 + \frac{1}{2} m\omega^2 \xi^2 - f \xi \right\} \psi. \quad (2.3) \]

The first bracket vanishes if we choose \( \xi \) so that

\[ m \ddot{x} = f - m\omega^2 \xi \quad (2.4) \]

i.e. the classical equation. The second bracket, which is a function of \( t \) only, may be eliminated by introducing a further phase factor:

\[ \varphi = \chi \exp \left( \frac{i}{\hbar} \int \left\{ \frac{1}{2} m \ddot{x}^2 - \frac{1}{2} m \omega^2 x^2 + f \xi \right\} dt \right). \quad (2.5) \]

\( \chi(x, t) \) will then satisfy the wave equation of force-free oscillator. Thus our original wave function assumes the following form:

\[ \psi(x, t) = \chi(x - \xi(t), t) \exp \left\{ i m \ddot{x} (x - \xi) / \hbar + \frac{i}{\hbar} \int_0^t \dot{L} \, dt \right\} \quad (2.6) \]

where \( L \) is the classical Lagrangian:

\[ L = \frac{1}{2} m \ddot{x}^2 - \frac{1}{2} m\omega^2 x^2 + f \xi \quad (2.7) \]

\( \xi \) may, without loss of generality, be assumed to be that solution of the classical equation of motion, which initially vanishes together with its derivative \( \dot{\xi} \). Then the extra phase factor in (2.6) vanishes at the initial epoch.

§ 3. The treatment of the case \( f = 0 \)

The solution obtained in § 1 is only a special solution of the wave equation. However, we can obtain from it a fundamental solution, the so-called transformation function, by suitable choice of integration constants. The transformation function \( U \) is defined by
and is given by
\[ U_f(x, t \mid x_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar(t-t_0)}} \exp\left\{ im(x-x_0)^2/2\hbar(t-t_0) \right\} \]}

for a free particle. In this section we try to construct the transformation function for the case \( f \equiv 0 \) from the solution of Gaussian type of §1, by imitating \( U_f \) for \( t \sim t_0 \).

To imitate the behavior of the coefficient of \( x^2 \), it is necessary to choose \( X \) as that solution of (1.4) which is initially vanishing as \( t \rightarrow t_0 \); we set
\[ X=0, \dot{X}=1 \quad \text{for} \quad t=t_0. \]

Next, by comparing the coefficients of \( x \), we have to set the integration constant of \( b \) in (1.6) in such a way that
\[ b=-mx_0/X. \]

Finally we deal with the integral \( \{dt/X^2 \) in (1.7). Let \( Y \) be a second solution of the classical equation (1.4) with the initial values
\[ Y=1, \dot{Y}=0 \quad \text{for} \quad t=t_0. \]

Then, as well known, we have
\[ \dot{X}Y-X\dot{Y}=1, \quad V/X=-\{dt/X^2 \].

Thus we may put
\[ \epsilon=i\hbar \log X+mx_0^2Y/X + \text{const.} \]

Summarizing the above results we finally obtain the required function:
\[ U(x, t \mid x_0, t_0) = \sqrt{\frac{m}{2\pi i\hbar X}} \exp\left\{ \frac{im}{2\hbar X}(\dot{X}x^2-2x\dot{x}_0+Yx_0^2) \right\}. \]
There is an apparent dissymmetry between the coefficients of $x^2$ and $x_0^2$. To clarify this point we observe the combination:

$$X(t, t') = X(t) Y(t') - X(t') Y(t).$$  \tag{3.9}

It, as function of the first argument $t$, satisfies the classical equation (1.4) as much as $X(t), Y(t)$ do, and has the initial values: $X(t', t') = 0, \frac{\partial X(t', t')}{\partial t} = 1$. Since it is defined completely by these properties, $X(t, t')$ does not depend on $t_0$. If $t' = t_0$, it reduces to $X(t)$:

$$X(t, t_0) = X(t).$$  \tag{3.10}

Likewise one sees that

$$\left( \frac{\partial X(t, t')}{\partial t'} \right)_{t'=t_0} = -Y(t).$$

Thus we have

$$\dot{X}(t) = \frac{\partial X(t, t_0)}{\partial t}, \quad Y(t) = -\frac{\partial X(t, t_0)}{\partial t_0}$$  \tag{3.11}

and a symmetry really exists between the coefficients of $x^2$ and $x_0^2$. (3.9) may now be written

$$X(t, t') = -X(t, t_0) \frac{\partial X(t', t_0)}{\partial t_0} + X(t', t_0) \frac{\partial X(t, t_0)}{\partial t_0}.$$  \tag{3.12}

It holds irrespective of the order of time epochs $t, t', t_0$. This law of composition will be essential if we want to verify directly the law of composition for the transformation function (3.8):

$$\int_{-\infty}^{x_0} U(x, t | x, t) U(x, t | x_0, t_0) dx_1 = U(x, t | x_0, t_0).$$  \tag{3.13}

It is further easy to verify the unitary character of $U$:

$$\int_{-\infty}^{x_0} U(x, t | x_0, t_0)*U(x, t | x_0, t_0) dx = \delta(x - x_0').$$  \tag{3.14}

It is usual on the basis of this relation to set

$$U(x, t | x_0, t_0)* = U^{-1}(x_0', t_0 | x, t),$$  \tag{3.15}

in the sense of matrix theory. But in view of (3.13), which should hold for any time order $t, t_1, t_0$, it is unnecessary to introduce the inverse matrix $U^{-1}(x_0, t_0 | x, t)$ besides $U(x_0, t_0 | x, t)$ with $t_0 < t$. Hence we may write

$$U(x, t | x_0, t_0)* = U(x_0, t_0 | x, t).$$  \tag{3.16}
This leads to a symmetry of the matrix elements of $U$. Let us introduce a complete system of orthogonal functions, say the eigenfunctions of the undisturbed harmonic oscillator. The matrix elements of $U$ with respect to this system will show, by virtue of (3.16),

$$U(t, t_0)_{nn} = U(t_0, t)_{nn}$$  \hspace{1cm} (3.17)

or, taking the squares of absolute values,

$$|U(t, t_0)_{nn}|^2 = |U(t_0, t)_{nn}|^2.$$  \hspace{1cm} (3.18)

This symmetry may be interpreted in terms of transition probabilities (This might be a misnomer, since the term transition probability appears usually in a special, technical sense.) Let us suppose the circular frequency $\omega$ changes in time, starting from a normal value $\omega_0$ at time $t_0$ and ending again in this value at a later time $t$ but otherwise in an arbitrary manner. Then, through this disturbance, the oscillator will make transitions from its original state. The transition probability from the $m$-th to the $n$-th state is then

$$P_{m\rightarrow n}(t, t_0) = |U(t, t_0)_{nm}|^2.$$  \hspace{1cm} (3.19)

Now we consider a second oscillator of a similar structure, but exposed to another disturbance due to the change of $\omega$ reversed in time course. In this case we designate the transition probabilities by $P(t_0, t)$. Then (3.18) teaches us the symmetry:

$$P_{n\rightarrow m}(t, t_0) = P_{m\rightarrow n}(t_0, t).$$  \hspace{1cm} (3.20)

This is a quite general rule, resulting from (3.13) and (3.14) alone. If the Hamiltonian does not involve the time variable explicitly, or if, in any case, there is no physical difference between future and past, then the time order would be irrelevant and we expect the familiar symmetry of transition probabilities: $P_{m\rightarrow n} = P_{n\rightarrow m}$ to be valid for one and the same course of events. A set of conditions for the validity of this symmetry will be the following: (1) the Hamiltonian $H(t)$ is a real expression and is symmetric in its temporal behavior: $H(-t) = H(t)$; (2) the complete orthogonal system $\{\phi_n(x)\}$ is all real. From the first assumption follows that, if $\psi(t)$ is a solution of the Schrödinger equation, then $\psi^*(-t)$ is another solution; especially, if $\psi(t)$ starts from a real function $\phi_n$, $\psi(0) = \phi_n$, then we have identically $\psi^*(-t) = \phi_n(t)$. Now we envisage the transition probability $P_{m\rightarrow n}(t) = |(\phi_m, \phi_n(t))|^2 = |(\phi_m, \phi_n^*(-t))|^2 = |(\psi_n^*(-t))|^2 = P_{m\rightarrow n}(-t)$, where we have used the reality of $\phi_m$. This equality reveals the expected time reversal symmetry. Combined with the general rule (3.19), this leads to the symmetry of transition probabilities

$$P_{m\rightarrow n}(t) = P_{n\rightarrow m}(t)$$  \hspace{1cm} (3.20)

as anticipated. I have intentionally repeated the probably well-known arguments for the symmetry of the transition probabilities in order to clarify the conditions for their validity.
We shall learn still another symmetry of transition probabilities in our special case of harmonic oscillators, quite outside the reign of the above theory.

§ 4. The determination of the eigenfunctions
for the stationary case

Since the knowledge of the transformation function should yield all the relevant informations concerning the behaviors of the mechanical system, it should also supply the knowledge of the eigenfunctions of the Hamiltonian in the time-independent case \( \omega = \text{const} = \omega_0 \). In this case \( X \) and \( Y \) of the preceding section are simply sine and cosine; to be explicit, we have

\[ X = \sin \frac{\omega_0 (t - t_0)}{\omega_0}, \quad Y = \cos \frac{\omega_0 (t - t_0)}{\omega_0}, \tag{4.1} \]

and the law of composition (3.12) is merely the addition theorem of trigonometrical functions. Since our transformation function \( U, \tag{3.8} \) depends on \( t \) only through \( X, \dot{X}, Y \), it would be a periodic function of period \( 2\pi / \omega_0 \). Actually the square root \( X^{-1/2} \) introduces complexity. In order to determine the sign changes of the square root, we remind us of the analyticity of \( U \) over the negative imaginary domain of \( t \), by virtue of parabolic differential equations. We suppose the time variable \( t \) has a negative imaginary part and set \( \exp(-i\omega_0 t) = z, \quad |z| < 1. \) Then

\[ U = \sqrt{\frac{m\omega_0}{\pi \hbar}} \sqrt{\frac{\pi}{1-z^2}} \exp\left\{ -\frac{m\omega_0}{2\hbar(1-z^2)} \left( (1+z^2)(x^2 + x_0^2) - 4zx_0 \right) \right\}. \tag{4.2} \]

As \( t \) increases, \( z \) describes a circle with a radius \( < 1 \) about the origin and should go through the two Riemann surfaces of \( \sqrt{z} \) alternately. Thus \( \exp(i\omega_0 t/2)U \) will have a period \( 2\pi/\omega_0 \).

Now from the bilinear expansion

\[ U(x, t|x_0, 0) = \sum_{n} \phi_n(x) \phi_n^*(x_0) \exp(-iE_n t/\hbar), \tag{4.3} \]

we can in general get the knowledge of the eigenvalues and the eigenfunctions by taking time averages.
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \exp(iEt/\hbar) U(x, t|x_0, 0) dt.
\]

In our case \(U\) is essentially periodic, and we have to regard (4.3) as a Fourier expansion. Each and every eigenvalue \(E_n\) should be of the form
\[
E_n = (n + \frac{1}{2}) \hbar \omega_0,
\]
where \(n\) is a certain integer, zero or natural numbers, since we know a priori \(E_n > 0\).

We can determine the eigenfunction product \(\psi_n(x)\psi_n^*(x_0)\) as a Fourier coefficient. This amounts to determine the coefficients of different powers of \(z\) in the power series expansion of (4.2), apart from the factor \(\sqrt{z}\). In this way we get the well-known generating function for the oscillator eigenfunctions:
\[
\frac{1}{\sqrt{\pi (1 - z^2)}} \exp\left\{ \frac{-1}{2} \frac{(1 + z^2) (\bar{x}^2 + \bar{x}_0^2) - 4sx_0}{2 (1 - z^2)} \right\} = \sum_{n=0}^{\infty} z^n \psi_n(x) \psi_n^*(x_0) \tag{4.4}
\]
where \(\bar{x} = x/\sqrt{\hbar/m\omega_0}\) etc. We should not like to pursue the calculation since the results are too well-known. But we shall indicate the way the more usual linear generating function for Hermite polynomials is related to the above bilinear generating function. Let us write \(z/x_0\) for \(z\) and make \(x_0 \to \infty\) in the resulting equation. The left-hand side becomes
\[
\frac{1}{\sqrt{\pi}} \exp\left( \frac{-2x^2 + 2x_0^2 - 4sx_0}{2} \right) \cdot \exp\left( -\frac{x_0^2}{2} \right).
\]
Thus we may infer
\[
\psi_n^*(x_0) \cdot N_n 2^nx_0^n \cdot \exp\left( -\frac{x_0^2}{2} \right), \tag{4.5}
\]
for large \(x_0\). The normalization factor \(N_n\) may then be determined to \((\sqrt{\pi} n! 2^n)^{-1/2}\), by letting \(x\) increase without limit. In this way we get the familiar generating function
\[
\exp\left\{ -\frac{1}{2} (2x^2 + x_0^2 - 4sx_0) \right\} = \sum_{n=0}^{\infty} \sqrt{\pi} n! 2^n z^n \psi_n(x), \tag{4.6}
\]

§ 5. The transition probabilities for the case \(f=0\)

Now we want to calculate transition probabilities in the sense explained at the end of § 3. For the sake of greater simplicity we shall adopt natural units: \(\hbar = 1, m = 1, \) and \(\omega_0 = 1\), where \(\omega_0\) is that normal frequency for which the eigenfunctions of the preceding section have been defined. In evaluating the transition probabilities
\[
P_{mn} = \left| \int \psi_n^*(x) U(x, t|x_0, t_0) \phi_m(x_0) dx \right|^2, \tag{5.1}
\]
we shall use the method of generating functions. The integral in
\[ P(u, v) = \sum_{mn} u^m v^m P_{mn} \]

\[ = \frac{1}{\pi \sqrt{1 - u^2} \sqrt{1 - v^2}} \int \cdots \int U(x', t | x_0', t_0) * U(x, t | x_0, t_0) \times \exp \left\{ -\frac{(1 + u^2)(x_0^2 + x_0'^2) - 4ux_0x_0' - (1 + v^2)(x^2 + x'^2) - 4vx_0x_0'}{2(1 - u^2)} \right\} dx \, dx' \, dx_0 \, dx_0' \]

(5.2)

is a four-fold Gaussian integral and may be evaluated by means of the well-known formula:

\[ \int \cdots \int \exp \left\{-\frac{1}{2} \sum_{m} a_{mm} x_m^2 \right\} \, dx_1 \cdots dx_n = \frac{(2\pi)^{n/2}}{\det(a_{mm})^{1/2}}. \]

(5.3)

The result runs astonishingly simple:

\[ P(u, v) = 2^{1/2} \left\{ Q(1 - u^2)(1 - v^2) + (1 + u^2)(1 + v^2) - 4uv \right\}^{-1/2}, \]

(5.4)

where

\[ Q = \frac{1}{2} \{ \dot{X}^2 + \dot{X}'^2 + \dot{Y}^2 + \dot{Y}'^2 \}. \]

(5.5)

This result may be checked in various aspects: first, we have

\[ P(u, 1) = (1 - u)^{-1} \]

(5.6)

equivalent to the law of total probability \( \sum_{mn} P_{mn} = 1 \); second, if there is no disturbance at all, \( \omega = \omega_0 = 1 \), then we have to put \( X = -\dot{Y} = \sin t \), \( \dot{X} = Y = \cos t \), and

\[ Q = 1 \]

(5.7)

so that

\[ P(u, v)_{\omega = 1} = (1 - uv)^{-1} \]

(5.8)

equivalent to \( P_{mn} = \delta_{mn} \), i.e. no transition; thirdly we have to observe

\[ P(-u, -v) = P(u, v) \]

(5.9)

which means:

\[ P_{mn} = 0 \text{ if } m, n \text{ are of different parity.} \]

(5.10)

This should have been expected on account of the strict selection rule

\[ m = n, \text{ or } n \pm 2 \]

in the words of perturbation theory.

Further an inspection reveals the symmetry

\[ P(u, v) = P(v, u) \]

(5.11)

which implies the symmetry of transition probabilities:

\[ P_{mn} = P_{nm}. \]

This peculiar symmetry lies outside the scope of the general arguments at the end of
§ 3 and will be discussed later.

In spite of its apparent simplicity, the generating function (5·4) is not manageable if we want to expand it in powers of \( u \) and \( v \). Explicit expressions for \( P_{mn} \) will be given in the following section. Here we confine ourselves to simpler consequences. First we want to get the average quantum number \( \langle m \rangle_n \) of the final states \( m \) for a given initial state \( n \). For this purpose we construct

\[
\sum_n u^n \sum_m m P_{mn} = \left. \frac{\partial P(u, v)}{\partial v} \right|_{v=1} = 2^{-1} (1 - u)^{-2} \{ Q(1 + u) - (1 - u) \}
\]

and get immediately

\[
\langle m \rangle_n = \sum_m m P_{mn} = (n + 1/2) Q - 1/2,
\]

or, in terms of energy,

\[
\langle E_m \rangle_n = Q \cdot E_n.
\]

Since \( \langle m \rangle \geq 0 \), it is a necessary condition for the validity of (5·12), that

\[
Q \geq 1.
\]

This inequality will be discussed at the end of this section.

In the same manner we can proceed to evaluate the average \( \langle m(m-1) \rangle_n \), from which we can derive the dispersion

\[
(\Delta m)^2_n = \langle m^2 \rangle_n - \langle m \rangle_n^2 = \frac{1}{2} (Q^2 - 1) (n^2 + n + 1)
\]

or, in terms of energy,

\[
(\Delta E_m)^2_n = \langle E^2 \rangle_n - \langle E_m \rangle_n^2 = \frac{1}{2} (Q^2 - 1) (E^2 + 3/4).
\]

For large excitation \( Q \geq 1 \), the dispersion is of the same order of magnitude as the square of the average energy and one cannot expect any concentration of the probability distribution in the sense of the law of large numbers. Thus the distribution will be very flat and have no conspicuous features, as may be inferred from the special cases of \( P_{m0} \), \( P_{m1} \). These are relatively simply obtained from

\[
P(0, v) = 2^{1/2} \{(Q + 1) - (Q - 1) v^2\}^{-1/2},
\]

and

\[
\left( \frac{\partial P}{\partial u} \right)_{u=0} = 2^{3/2} \{(Q + 1) - (Q - 1) v^2\}^{-3/2} v.
\]

We have

\[
P_{m0} = \left( \frac{2}{Q + 1} \right)^{1/2} \left( \frac{-1/2}{m/2} \right) \left( \frac{Q - 1}{Q + 1} \right)^{m/2} \quad (m = 0, 2, 4, \ldots),
\]

\[
P_{m1} = \left( \frac{2}{Q + 1} \right)^{3/2} \left( \frac{-3/2}{(m-1)/2} \right) \left( \frac{Q - 1}{Q + 1} \right)^{(m-1)/2} \quad (m = 1, 3, 5, \ldots)
\]
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Since the classical phenomenon \( Q \geq 1 \) might give an impression of apparent irreversibility of a purely mechanical system, it would not be superfluous here to give an analysis of the inequality.

A simple proof of that inequality has kindly been afforded by Mr. Y. Suzuki of this institute. It suffices to write down just one line:

\[
(X + Y)^2 + (\dot{X} - \dot{Y})^2 = X^2 + \dot{X}^2 + Y^2 + \dot{Y}^2 - 2(\dot{XY} - X\dot{Y}) \geq 0. \tag{5.19}
\]

The inequality really has a statistical bearing. Let \( Z \) be a solution of the equation of motion

\[
\ddot{Z} = -\sigma(t)^Z Z
\]

with initial values \( Z_0, \dot{Z}_0 \). Then we have

\[
Z = \dot{Z}_0 X + Z_0 Y.
\]

If we build the "total energy" corresponding to the motion \( Z \),

\[
E = \frac{1}{2} (Z^2 + \dot{Z}^2) = \frac{1}{2} (\dot{Z}_0^2 X^2 + Z_0^2 Y^2 + \dot{Z}_0^2 X^2 + Z_0^2 Y^2 + 2\dot{Z} Z (\dot{XY} + X\dot{Y}))
\]

\[
\tag{5.20}
\]

and take average over the initial values consistent with the given initial energy \( E_0 = \frac{1}{2} (\dot{Z}_0^2 + Z_0^2) \), then we have obviously

\[
\langle E \rangle = \langle \frac{1}{2} (\dot{Z}^2 + Z^2) \rangle = Q \langle \frac{1}{2} (\dot{Z}_0^2 + Z_0^2) \rangle = Q \cdot E_0. \tag{5.21}
\]

Physically interpreted, \( Q \geq 1 \) means thus the following:

"Let the string of a pendulum, while in oscillation, be shortened or lengthened in an arbitrary manner and let it return to its initial length after a time. Then we expect, on the average, to find the energy of the pendulum increased."

Eq. (5.21) is an exact classical counterpart of the quantal one (5.13). There is also the classical counterpart of the dispersion formula (5.16). If we make use of

\[
\langle Z_0^4 \rangle = \langle Z_0^2 \rangle = 3 E_0^2 / 2, \quad \langle Z_0^2 \dot{Z}_0^2 \rangle = E_0^2 / 2
\]

and the identity

\[
(\dot{X}^2 + X^2) (\dot{Y}^2 + Y^2) = (\dot{XY} + X\dot{Y})^2 + (\dot{XY} - X\dot{Y})^2,
\]

we get, in just the same way we derived (5.21),

\[
\langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{2} (Q^2 - 1) E_0^2. \tag{5.22}
\]

In comparing the classical (5.21) and the quantal (5.16) it is to be noted that the mean final energy vanishes in the classical case if the initial energy is zero, whereas in the quantal case the mean final energy never vanishes on account of the zero point motion,
From the expression (5.20) of the final energy, which may be written in the form

\[ \frac{E}{E_0} = Q + \sqrt{Q^2 - 1} \cos (2\theta + \text{const.}) \quad (5.23) \]

with \( Z_0 = \sqrt{2E_0} \cos \theta \), \( \dot{Z}_0 = \sqrt{2E_0} \sin \theta \), the full distribution for the final energy can easily be derived:

\[ \frac{d\theta}{2\pi} = \frac{dE}{4\pi \sqrt{E_0^2 (Q^2 - 1) - (E - E_0Q)^2}}, \quad Q + \sqrt{Q^2 - 1} \leq \frac{E}{E_0} \leq Q + \sqrt{Q^2 - 1}. \quad (5.24) \]

As one sees, there is a fundamental difference between the classical distribution (5.24) and the quantal one (5.17) or (5.18) if one starts from relatively smaller values of the initial energy, in spite of their coincidence in mean and dispersion.

§ 5. Adiabatic theorem

We can verify the adiabatic theorem for our quantum oscillator on the basis of the same theorem within the classical theory. Let the angular frequency \( \omega(t) \) of our oscillator changes infinitely slowly from an initial value \( \omega_0 \) to a final value \( \omega_1 \). It is a well-known fact of the classical mechanics that the quantity

\[ (X^2 + \omega^2(t)X'^2)/\omega(t) \quad (5a.1) \]

is an adiabatic invariant, where \( X(t) \) is any solution of the classical equation. If \( X(t) \) and \( Y(t) \) are defined as in the preceding section, we have then the two adiabatic invariants (5a.1) = \( \omega_0^{-1} \) and

\[ (Y^2 + \omega^2Y'^2)/\omega(t) = \omega_0 \quad (5a.2) \]

If we explicitly introduce the angular frequency \( \omega \) in our oscillator eigenfunctions, the generating function is modified to

\[ \sqrt{\frac{\omega}{\pi(1 - s^2)}} \exp \left\{ -\frac{(1 + s^2)(x^2 + x'^2) - 2xx'}{2(1 - s^2)} \right\} \omega \quad (5a.3) \]

In proving the adiabatic theorem, it is necessary to adapt the eigenfunctions to the instantaneous value of the angular frequency. Thus we have to ask the transition from the state \( \omega_0^{1/2} \phi_0(\omega_0^{1/2}x) \) to the state \( \omega_1^{1/2} \phi_m(\omega_1^{1/2}x) \). The transition probabilities defined in this sense have kindly been calculated by Mr. T. Dodo according to the method of § 5. The generating function for them is this:

\[ P(u, v) = \sum u^n v^m P_{mn} = 2^{1/2} \{ Q^* (1 - u^2) (1 - v^2) + (1 + u^2)(1 + v^2) - 4uv \}^{-1/2} \quad (5a.4) \]

where

\[ 2Q^* = \{ \omega_0^2(\omega_1^2X^2 + \dot{X}^2) + (\omega_1^2Y^2 + \dot{Y}^2) \}/\omega_0\omega_1 \quad (5a.5) \]

For a adiabatic process we have (5a.1) and (5a.3) and hence we get \( Q^* = 1, P(u, v) \)
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\[(1 - uv)^{-1}, P_{mn} = \delta_{mn}, \text{proving the adiabatic theorem for the quantum oscillator.}\]

§ 6. The matrix elements of the transformation function

We have probably been too much hasty in deriving the transition probabilities. We have lost sight in finding the nature and origin of the unexpected symmetry of transition probabilities. We now turn back to get an explicit expressions for the matrix elements

\[U_{mn}(t, t_0) = \int \psi_n^* (x) U(x, t \mid x_0, t_0) \psi_n (x_0) dx dx_0. \tag{6.1}\]

Again we shall make use of generating functions, this time, the linear generating function (4.5). We then have to evaluate

\[U(u, v) = \sum_{n,m} \left\{ \frac{\pi^{2n+2m}}{n! m!} \right\}^{1/2} u^n v^m U_{mn}(t, t_0)\]

\[= \frac{e^{-(n^2 + m^2)}}{\sqrt{2\pi i X}} \int dxdx_0 \exp \left[ -\frac{1}{2} \left\{ \frac{1}{X} \right\} X^2 + \frac{2ixx_0}{X} + \left( 1 - \frac{IV}{X} \right)x_0^2 - 4vx - 4ux_0 \right]. \tag{6.2}\]

The integral may be computed by the formula

\[\int dxdy \exp \left\{ -\frac{1}{2} (ax^2 + 2bxy + cy^2 - 2px - 2qy) \right\} = \frac{2\pi^2}{\sqrt{ac - b^2}} \exp \left\{ \frac{aq^2 - 2bpq + cp^2}{2(ac - b^2)} \right\}.\]

The result runs

\[U(u, v) = \sqrt{\frac{2\pi}{i\sigma}} \exp \left\{ \frac{4uv + \rho^* \sigma^*}{2} \right\}, \tag{6.3}\]

where

\[\rho = X - iX + iY - \hat{Y}, \quad |\rho|^2 = 2(Q - 1), \]
\[\sigma = X + iX - iY - \hat{Y}, \quad |\sigma|^2 = 2(Q + 1). \tag{6.4}\]

In this case the expansion in powers of \(u\) and \(v\) is immediate.

\[U_{mn}(t, t_0) = \left\{ \frac{n! m! \rho^\mu \rho^* \mu}{2^{n + m - 1} \sigma^n \sigma^m + 1} \right\}^{1/2} \sum_{l \geq 0} \frac{(-2i\sqrt{2/(Q - 1)})^l}{l! \left[ (n - l)/2 \right]! \left[ (m - l)/2 \right]!}. \tag{6.5}\]

where \(l\) runs over even numbers only, if \(m, n\) are even, and over odd numbers only, if \(m, n\) are odd. Explicitly,

\[U_{\mu, \nu} = \left\{ \frac{(2\nu)! (2\mu)! \rho^{2\nu + 1} \rho^* \mu^{2\nu + 1}}{2^{2\nu + 2\mu + 1} \sigma^{2\nu + 2\mu + 1}} \right\}^{1/2} \sum_{\lambda = 0, 1, \ldots} (-2\sqrt{2/(Q - 1)})^\lambda \frac{(-2\nu/(Q - 1))^{\lambda}}{(2\lambda)! (\nu - \lambda)! (\mu - \lambda)!}. \tag{6.6}\]

\[U_{\mu + 1, \nu + 1} = \left\{ \frac{(2\nu + 1)! (2\mu + 1)! \rho^{2\nu + 2} \rho^* \mu^{2\nu + 2}}{2^{2\nu + 2\mu + 2} \sigma^{2\nu + 2\mu + 3}} \right\}^{1/2} \sum_{\lambda = 0, 1, \ldots} (-2\nu/(Q - 1))^{\lambda} \frac{(-2\nu/(Q - 1))^{\lambda}}{(2\lambda + 1)! (\nu - \lambda)! (\mu - \lambda)!}. \tag{6.7}\]
The corresponding transition probabilities become now

\[
P_{2\nu,2\nu} = \left(\frac{2}{Q+1}\right)^{3/2} \left(\frac{Q-1}{Q+1}\right)^{\nu+\nu} \frac{(2\nu)! (2\mu)!}{2^{2\nu+2\mu}} \left\{\sum_{\lambda=0}^{\min(\nu,\mu)} \frac{(-2^{\lambda}/(Q-1)^\lambda}{(2\lambda)! (\nu-\lambda)! (\mu-\lambda)!}\right\}^2,
\]

\[
P_{2\nu+1,2\nu+1} = \left(\frac{2}{Q+1}\right)^{3/2} \left(\frac{Q-1}{Q+1}\right)^{\nu+\nu+1} \frac{(2\nu+1)! (2\mu+1)!}{2^{2\nu+2\mu}} \left\{\sum_{\lambda=0}^{\min(\nu,\mu)} \frac{(-2^{\lambda}/(Q-1)^\lambda}{(2\lambda+1)! (\nu-\lambda)! (\mu-\lambda)!}\right\}^2.
\]

The symmetry property of transition probabilities is seen to originate in a rather intricate structure of \( U_{mn} \). The matrix \( U_{mn} \) is neither hermitic nor antihermitic, but consists of three parts: the hermitic part \((\rho^* \rho)^{1/2}\), the symmetric but complex part \((\sigma^* + m + 1)^{1/2}\) and the remaining real (or purely imaginary) symmetric part. \( U_{mn} \) should be of course unitary; a direct proof for this will require the following properties of the parameters \( \rho \) and \( \sigma \). In view of (3.11) we have

\[
\rho = \left(1 - i \frac{\partial}{\partial t}\right) \left(1 - i \frac{\partial}{\partial t_0}\right) X(t, t_0),
\]

\[
\sigma = \left(1 - i \frac{\partial}{\partial t}\right) \left(1 + i \frac{\partial}{\partial t_0}\right) X(t, t_0),
\]

so that

\[
\rho(t, t_0) = -\rho(t_0, t),
\]

\[
\sigma(t, t_0) = -\sigma(t_0, t),
\]

from which we can derive the remarkable symmetry

\[
Q(t, t_0) = Q(t_0, t).
\]

The unitary character of (6.5) is guaranteed if we can verify the universal symmetry (3.17), which appears to hold good by virtue of these properties (6.11), (6.13), except for ambiguity of sign. This disagreeable point may have its origin in our rather uncareful treatment of square root (cf. § 4).

§ 7. The transformation function in the general case;
\( \omega, f \) both variable

According to the result obtained in § 2, the two cases of free \((f=0)\) and forced \((f \neq 0)\) oscillations are intimately connected. Once a solution of the free case has been obtained, the corresponding solution under arbitrary external disturbances \( f \) can immediately be constructed by means of the simple formula (2.6). This procedure may in particular be applied to the transformation function and we get
where the last two terms in the exponential are functions of $t$ only and we have
\[
\int_t^t Ldt - \dot{\xi}^2 = \int_t^t \left\{ \frac{1}{2} \dot{\xi}^2 - \frac{\omega^2}{2} \xi^2 + f \xi \right\} dt - \dot{\xi}^2 = \frac{1}{2} \int_t^t f \xi dt - \frac{1}{2} \dot{\xi}^2, \tag{7.2}
\]
on account of the equation of motion
\[
\dot{\xi} = -\omega^2 \xi + f
\]
and the initial values
\[
\xi(t_0) = 0, \quad \dot{\xi}(t_0) = 0.
\]
\(\xi\) may be explicitly written down as
\[
\xi(t) = \int_{t_0}^t X(t, t') f(t') dt'. \tag{7.3}
\]
in terms of the function \(X(t, t')\) defined by (3.9). Then the integral appearing in (7.2) is quadratic in \(f\):
\[
\int_{t_0}^t f \ddot{\xi} dt = \int_{t_0}^t f(t') \ddot{\xi} dt' = -\int_{t_0}^t f(t') \ddot{\xi} dt' = -\int_{t'}^t f(t') \dot{\xi} dt' \int_{t'}^t X(t', t') \dot{\xi} dt'.
\]
As one sees, it alternates its sign on the interchange of \(t\) and \(t_0\). It will be appropriate to introduce the initial time epoch \(t_0\) in the arguments of \(\xi\) and to write \(\xi(t, t_0)\):
\[
\xi(t, t_0) = \int_{t_0}^t X(t, t') f(t') dt'. \tag{7.3a}
\]
Then
\[
\xi(t_0, t) = \int_{t_0}^t X(t_0, t') f(t') dt'.
\]
represents a reversed motion. And the above relation may be expressed as:
\[
\int_{t_0}^t \xi(t', t_0) f(t') dt' = -\int_{t'}^t \xi(t', t) f(t') dt'. \tag{7.4}
\]
Upon expansion of \((x - \xi)^2\) in (7.1), we are led to treat the following expression:
\[
\dot{X}^2 - \dot{\xi}^2 = \int_{t_0}^t \left\{ \frac{\partial X(t, t_0)}{\partial t} X(t, t') - \frac{\partial X(t, t')}{\partial t} X(t, t_0) \right\} f(t') dt'.
\]
\[
\int_{t_0}^t X(t_0, t') f(t') dt' = -\dot{\xi}(t_0, t). \tag{7.5}
\]
In this way we arrive at the following expression for our transformation function:

\[ U(z, t'|x_0, t) = \frac{e^{itT}}{\sqrt{2\pi iX(t, t_0)}} \exp \left( \frac{i}{2X} \left\{ \dot{X}z^2 - 2xz + Yz_0^2 + 2\dot{z} (t_0, t)x + 2\dot{z} (t, t_0)z \right\} \right), \]

where

\[ T(t, t_0) = -\frac{\dot{z} (t_0, t)}{2X(t, t_0)} + \frac{1}{2} \int_{t_0}^{t} f(t')\dot{z} (t', t_0) dt' = - T(t_0, t). \]

Now we proceed to compute the matrix elements of \( U \) in quite an analogous way as in § 6. The generating function for them is

\[ U(u, v) = \sum \{ \frac{\pi^{2n+1}}{n! m!} \} \frac{u^n v^m}{n! m!} U_{nm}(t, t_0) \frac{e^{-u^2 - v^2 + 2\dot{u}u + 2\dot{v}v}}{\sqrt{2\pi X}} \int dx dx_0 \]

\[ \exp \left\{ \frac{1}{2} \left( 1 - \frac{iX}{X} \right) z^2 + \frac{2ixz_0}{X} + \left( 1 - \frac{iY}{X} \right) z_0^2 - 2 \left( 2v + \frac{i\eta}{X} \right) z - 2 \left( 2u + \frac{i\bar{\eta}}{X} \right) z_0 \right\} \times \]

\[ \exp \left\{ \frac{(mu^2 - 4iuv + p^2v^2 + 2i(\bar{\xi} - i\bar{\eta})u + 2i(\eta + i\bar{\eta})v)}{\sigma} \right\}, \]

where

\[ \eta = \dot{z} (t_0, t), \quad \eta' = \partial \dot{z} (t_0, t) / \partial t_0 \]

and where we have utilized the relation (7.5), i.e.

\[ X\dot{\xi} + \eta = X\dot{\xi} \]

(7.10)

and the inverse relation

\[ Y\dot{\eta} + \dot{\xi} = -X\dot{\xi} \]

(7.11)

so that we have also

\[ Y\dot{\eta} + 2\dot{\eta} + \dot{X}\dot{\xi} = X(\dot{\xi} - \eta \dot{\eta})\]

(7.12)

Next we want to construct a direct generating function for the transition probabilities, with aid of (5.2),

\[ P(u, v) = \sum_{m,n=0}^{\infty} u^n v^m P_{nn} \]

\[ = 2^{1/2} \left\{ Q (1 - u^2) (1 - v^2) + (1 + u^2) (1 + v^2) - 4uv \right\}^{-1/2} \times \exp \left\{ \frac{(1 - u^2) (1 - v^2)}{2Q (1 - u^2) (1 - v^2) + (1 + u^2) (1 + v^2) - 4uv} \left\{ (\eta^2 + \eta'^2) \frac{1 - \eta}{1 + \eta} + (\xi^2 + \bar{\xi}^2) \frac{1 - u}{1 + u} \right\} \right\}. \]

(7.13)

The possibility of a symmetry of \( P_{nn} \) depends on the equality or inequality of \( (\xi^2 + \bar{\xi}^2) \).
§ 8. The special case: $\omega = \text{const.} = 1$

The results of the foregoing section will be greatly simplified in the elementary case $\omega = \text{const.} = 1$. We have to set

$$X(t_0, t_0) = \sin(t - t_0), \quad \rho = \rho^* = 0, \quad \sigma = 2i \exp(it - it_0),$$

$$\xi - i\xi' = -i\epsilon \int_{t_0}^{t} e^{it} f(t') dt',$$

$$\eta + i\eta' = -i \epsilon e^{-it_0} \int_{t_0}^{t} e^{it} f(t') dt',$$

$$\frac{2i}{\sigma} (\xi - i\xi') = \eta - i\eta', \quad \frac{2i}{\sigma} (\eta + i\eta') = \xi + i\xi',$$

$$\xi'' + \xi'^2 = \eta' + \eta'' = 2W,$$

$$(\eta - i\eta')(\xi + i\xi') = -2e^{it_0 - t} W$$

$$2W = \int_{t_0}^{t} \int_{t_0}^{t} \cos(t' - t'') f(t') f(t'') dt' dt'' = \frac{1}{X}(\xi\eta - \xi'\eta').$$

It is a remarkable characteristic of a classical harmonic oscillator that the work $W$ done against the oscillator initially at rest is the same for the reversed motion as for the direct motion.

The last factor of (7·8) now becomes

$$\exp\{2e^{i(t_0-t)} uv + (\eta - i\eta') u + (\xi + i\xi') v\}$$

$$= \exp\left\{ -\frac{1}{W} (\eta - i\eta') u (\xi + i\xi') v + (\eta - i\eta) u + (\xi + i\xi') v\right\},$$

which can be expanded in powers of $u$ and $v$ according to the formula

$$\exp(u + \beta - \alpha \beta/W) = \sum_{m,n=0}^{\infty} C(m, n \mid W) \frac{u^m}{m!} \frac{\beta^n}{n!}$$

where $C$'s are the well-known Charlier polynomials of the probability theory:

$$C(m, n \mid W) = \sum_{i=0}^{\infty} \frac{m! n!}{(m-i)! (n-i)!} (-W)^{-i}.$$  \hspace{1cm} (8·2)

It is a curiosity that Poisson's law of small numbers, to which the Charlier polynomials belong, appears side by side with Laplace's law of large numbers, to which the Hermite polynomials belong, in the quantum mechanics of a harmonic oscillator.

Summing up the above results we now write down the matrix elements of the transformation function:
where

\[ V = \int_0^\infty \int_0^\infty \sin(t' - t'')f(t')f(t'')dt'dt''. \]  

The transition probabilities are

\[ P_{mn}(t, t_0) = \frac{e^{-W}W_{m+n}}{m! n!} (C(m, n | W))^2 \]  

which are again symmetric with respect to \( m, n \), since the Charlier polynomials are symmetric. If \( n=0 \), \( C(m, 0 | W) \) reduces to unity, and we have the Poisson distribution

\[ P_{m0} = \frac{e^{-W}W^m}{m!}. \]  

The Charlier polynomials may be defined by another generating function

\[ G(m, n | W) = \left( 1 - e^{-W}W \right)^m e^{e^{-W}W} = \sum_{n=0}^{\infty} \frac{W^n}{n!} C(m, n | W). \]  

We construct

\[ \sum_{m=0}^{\infty} \frac{e^{-W}W^m}{m!} G(m, n | W) G(m, \beta | W) = e^{\beta W}, \]  

from which we can deduce

\[ \sum_{m=0}^{\infty} \frac{e^{-W}W^m}{m!} C(m, n | W) C(m, n' | W) = \delta_{nn'} n! W^{-n}. \]  

This proves the law of total probability: \( \sum_m P_{mn} = 1 \). If we construct

\[ \sum_{m=0}^{\infty} \frac{e^{-W}W^m}{m!} G(m, n | W) G(m, \beta | W) = \{ W - \alpha - \beta + \alpha \beta / W \} e^{\beta W}, \]  

we can deduce the average excitation:

\[ \langle m \rangle = \sum_{m=0}^{\infty} m P_{mn} = n + W. \]  

In the same way we arrive at

\[ \langle dm^2 \rangle = \langle m^2 \rangle - \langle m \rangle^2 = (2n + 1) W. \]  

In terms of energy these relations run:

\[ \langle E_m \rangle = E_n + W, \quad \langle dE_m \rangle = 2W E_n. \]  

Classically we expect the following distribution for the final energy \( E \)

\[ \frac{dE}{2\pi \sqrt{4E_0 W - (E - E_n - W)^2}}. \]
so that
\[ \langle E \rangle = E_0 + W, \quad \langle \Delta E^2 \rangle = 2WE_0 \]
in accordance with (8·12).

It will be remarked that the transition amplitude (8·3) can also be expressed as
\[ U_{mn}(t, t_0) = \frac{e^{i(t_0-t)(n+1/2)+4\sqrt{2}f}}{\left(2^{n+m+1}m!n!\right)^{1/4}} (-1)^n e^{-W/2} C(m, n | W)(\xi + i\dot{\xi})^m (\bar{\xi} - \ddot{\xi})^n. \]  

(8·13)

The phase factor \( e^{it_0-t(n+1/2)} \) exists also for an undisturbed motion, and may properly be ignored. The second phase factor \( e^{i\sqrt{2}f} \) and the sign \((-1)^n\) prevent \( U_{mn} \) to be a hermitian matrix, and make difficult an interpretation of the symmetry of the transition probabilities obtained above.

§ 9. The perturbation theory

In order to get an idea of how the symmetry of our transition probabilities comes out, we revert to the usual theory of perturbation. Thanks to the recent advancement in this theory, as far as the oscillator problem concerns, we can compute the terms of any order, and we may expect the series can be summed in the closed form we have obtained by the direct way.

In the Schrödinger equation
\[ i\frac{\partial \psi}{\partial t} = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2} x^2 \right) \psi - \bar{\chi} x \psi, \]
we expand the wave function \( \psi(x, t) \) in terms of the eigenfunctions of the unperturbed Hamiltonian in the bracket
\[ \psi(x, t) = \sum_{n=0}^{\infty} c_n(t) e^{-it(n+1/2)} \phi_n(x) \]  
and obtain for the equations governing the "variation of constants"
\[ i\frac{dc_n}{dt} = -\sum_{n=0}^{\infty} \epsilon_n f e^{i(n-m)x} \int \phi_n(x) x \psi dx, \]
or, in matrix notation
\[ i\frac{dc}{dt} = H(t)c, \]
the elements of the matrix \( H(t) \) being
\[ H(t)_{mn} = -f(t) e^{i(n-m)x} x_{mn}, \]  

(9·4)

\[ x_{mn} = \sqrt{\frac{n+1}{2}} \]  for \( m = n + 1, = \sqrt{\frac{n}{2}} \)  for \( m = n - 1, = 0 \) otherwise.

The differential equation becomes on formal integration
c(t) = c(0) + \frac{1}{i} \int_0^t H(t')c(t')dt'

and the method of iteration yields a formal solution in the form of infinite series

c(t) = \left\{ 1 + \frac{1}{i} \int_0^t H(t')dt' + \frac{1}{i^2} \int_0^t H(t')dt' \int_0^t H(t'')dt'' + \cdots \right\} c(0) = S(t, 0)c(0).

(9.5)

The \(k\)-th term of the operator \(S(t, 0)\) on the right-hand side may be written in the following form

\[
\frac{(-i)^k}{k!} \int_0^t \cdots \int_0^t \, dt' dt'' \cdots dt^{(k)} T(H(t')H(t'') \cdots H(t^{(k)})),
\]

(9.6)

where \(T\) is the chronologically ordered product of Wick\(^3\);

\[
x(t)_{mn} = e^{i\chi_{mn}}x_{mn}
\]

(9.7)

are the elements of the operator \(x\) in the interaction representation. They satisfy the commutation relations

\[
[x(t), x(t')] = \frac{1}{i} x(t' - t),
\]

(9.8)

As usual, we separate it into creation and annihilation operators:

\[
x(t) = \hat{b}(t) + \hat{b}^+(t)
\]

(9.9)

with

\[
[\hat{b}(t), \hat{b}^+(t')] = \frac{1}{2} e^{i\chi_{tt'}}
\]

(9.10)

and define the normal product

\[
\mathcal{N}\{x(t')x(t'') \cdots x(t^{(k)})\}.
\]

Then it holds Wick's theorem\(^3\):

\[
T\{x(t') \cdots x(t^{(k)})\} = \sum \mathcal{N}\{x(t^{(1)}) \cdots x(t^{(k)})\} \sum (x(t^{(a)})x(t^{(a')})) \cdots (x(t^{(n)})x(t^{(n')})),
\]

(9.11)

where

\[
\frac{t^{(1)} \cdots t^{(1')}}{l} + \frac{t^{(a)} \cdots t^{(a')}}{2\rho} = k
\]

is a rearrangement of \(t', \cdots, t^{(k)}\) and the summation is over essentially different expressions. Further

\[
(x(t)x(t')) = \frac{1}{2} \{\cos(t-t') - i \epsilon(t-t') \sin(t-t')\}
\]

(9.12)

with

\[
\epsilon(t-t') = +1 \quad \text{for} \quad t > t', \, = -1 \quad \text{for} \quad t > t'.
\]
In the integration of (9·6) we first meet with the integral (see (8·4))
\[ \int_0^t \int_0^t (x(t')x(t''))f(t')f(t'')dt'dt'' = W - iV. \tag{9·13} \]

For a given set \( t^{(1)} \ldots t^{(1)} \) and given \( \rho \), there arise from (9·11) \( (2\rho)!/\rho!2^\rho \) terms of the same magnitude \((W - iV)^p\), and there are \( \binom{k}{\ell} \) ways of choosing \( t^{(1)} \ldots t^{(1)} \). Thus (9·6) becomes
\[ \frac{(-i)^k}{k!} \sum (-1)^l \frac{k!}{(k-l)!} \int_0^t \cdots \int N\{x(t') \ldots x(t^{(1)})\} f(t') \cdots f(t^{(1)}) dt' \cdots dt^{(1)} \]
\[ \frac{(2\rho)!}{2^\rho \rho!} (W - iV)^p, \tag{9·14} \]
which goes, on summation over \( k = l + 2\rho \), into
\[ \sum_{l=0}^{\infty} \frac{t^l}{l!} e^{-W/2 + iV/2} \int_0^t \cdots \int N\{x(t') \ldots x(t^{(1)})\} f(t') \cdots f(t^{(1)}) dt' \cdots dt^{(1)}. \tag{9·15} \]

We first note
\[ \int_0^t b(t')f(t')dt' = b(0) \int_0^t e^{-it'}f(t')dt' = b(0)(+i)e^{-it(\xi - i\xi)}, \]
\[ \int_0^t b^+(t')f(t')dt' = b^+(0) \int_0^t e^{it'}f(t')dt' = b^+(0)(-i)e^{it(\xi + i\xi)}, \]
so that (9·15) becomes
\[ e^{-W/2 + iV/2} \sum_{l=0}^{\infty} \frac{1}{l!} N\{(-b(0)e^{-it(\xi - i\xi)} + b^+(0)e^{it(\xi + i\xi)})^l\}. \tag{9·17} \]

Let us seek the \( m-n \)-element of this \( S \)-matrix. A product of the form
\[ b_0(0)^m b_0(0)^n \] with \( n - \nu = m - \mu = \lambda \)
will contribute to the matrix element
\[ \sqrt{\frac{m}{2}} \sqrt{\frac{m-1}{2}} \cdots \sqrt{\frac{m-\mu+1}{1}} \sqrt{\frac{n-\nu+1}{2}} \cdots \sqrt{\frac{n}{2}} \]
\[ = \sqrt{\frac{m!n!}{2^{m+n}(m-\mu)!\nu!}} = \left( \frac{m!n!}{2^{m+n}(m-\mu)!\nu!} \right)^{1/2} \frac{1}{2^\lambda}. \]

We have then
\[ S_{mn} = e^{-W/2 + iV/2} \sum_{l=0}^{\min(m, n)} \frac{1}{(m+n-2\lambda)!} \]
\[ (m+n-2\lambda)! (\xi + i\xi)^{m-\lambda} (-1)^{n-\lambda} (\xi - i\xi)^{n-\lambda} \]
\[ (m!n!/\lambda! (m-\lambda)!(\nu!))^{1/2} 2^\lambda \]
\[ = e^{-W/2 + iV/2} \sum_{l=0}^{\min(m, n)} \frac{1}{m!n!} \frac{(-1)^n (\xi + i\xi)\nu (\xi - i\xi)\nu}{\lambda! (m-\lambda)!(n-\lambda)!}, \tag{9·18} \]
\[ \]
in complete agreement with our previous result (8·13). Viewed in the light of the perturbation theory, the common factor \( \exp(-W/2 + iV/2) \) comes out to be connected with the damping phenomena. The unexpected symmetry of the transition probabilities seems to depend essentially on the almost hermitian character of the normal product; in our case the presence of the sign \((-1)^n\) spoils the exact hermitian character of the normal product. Let us write the factor in the form: 
\[
(-1)^n H_{mn},
\]
where \( H_{mn} \) is hermitian. It may be written, as usual, in the form of a sum of a hermitian and an antihermitian part:
\[
(-1)^n H_{mn} = \frac{1}{2} \left\{ (-1)^n + (-1)^m \right\} H_{mn} + \frac{1}{2} \left\{ (-1)^n - (-1)^m \right\} H_{mn}.
\]
If \( m, n \) are of the same parity, the hermitian part only survives, and if, on the other hand, \( m, n \) are of the different parity, the hermitian part drops out. Thus the symmetry of the transition probabilities originates from this mixed hermitian character of our normal product. The either-or character may be traced to the selection rule \( n \rightarrow n + 1 \). This requires an even number of operators \( b, b^+ \) for the transitions between the states of the same parity and an odd number for the different parity. This state of affairs has been anticipated by Mr. M. Otuka.

In the calculation of this paragraph I have been much assisted by Dr. R. Utiyama, who also will extend the perturbation theory to the case \( \omega \) variable. I wonder whether this sort of calculation would lead to a reasonable interpretation of the symmetry of transition probabilities.

§ 10. Concluding remarks.

There are numerous problems which can be tested on our elementary example of the forced harmonic oscillator. The adiabatic theorem, the asymptotic equivalence of the classical and quantal treatments, and others. These problems will be taken up elsewhere. There are also a number of open questions, which have not been analyzed in the text. For example, the characteristic difference between (6·8) and (6·9) which are proportional to \( Q^{-1/2} \) and \( Q^{-3/2} \) respectively for large \( Q \), remains quite incomprehensible to me.

I owe much to my colleagues of this institute for their interests and advices on this work. Some of their names appear in the text. Mr. H. Arita independently calculated the results of § 5 by directly solving the Heisenberg equation of motion.

References

2) I do not know if there is any exact correspondence between Taniuti’s transformation here described and the canonical transformation devised by Bloch and Nordsieck, Phys. Rev. 52(1937), 96.