Eikonal Type Approximations for Potential Scattering

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(Received December 22, 1977)

Approximation methods based on the systematic expansions of the modulating function are presented for potential scattering. It is shown that three types of the systematic methods, which are related to the eikonal type approximations, can be developed. These three expansions are expected to give the excellent approximations for the respective ranges of forward, backward and intermediate scattering angles at high energies. The functional integral representations for various propagators are given in closed form on the basis of the method proposed by Fradkin. The relations among systematic expansions and an approximation method for the functional integral are illustrated.

§ 1. Introduction

Applications of Glauber's eikonal approximation to the scattering problems at high energies have led to many interesting results. In view of these successes, many authors have sought to modify the approximations so as to extend its applicability to large scattering angles at high energies. Several improved forms, which are valid for large angles, have been proposed by several authors. For example, in their papers, Chen and Hoock presented an interesting formula for a description of high energy scattering by a potential at a fixed large angle, and they have shown that their formula works remarkably well for a fairly large angle of scattering by exponential potentials. It will be worth while to develop more generally systematic approximation techniques for obtaining scattering amplitudes which include the Chen-Hoock representation as a first order correction to the Glauber amplitude.

We would like to propose, here, high energy approximation methods for the potential scattering on the basis of systematic expansions of modulating function. For these purposes, we derive an integral equation for the modulating function with an inhomogeneous term which expresses the eikonal type propagation along any direction inside the potential. By applying an iteration procedures to this integral equation, the expansion of the modulating function is classified as the three types of series with the relevant propagators. The first type (eikonal-type) of these expansions, which emphasized a particle-like picture, works well for forward scattering angles and has been widely discussed in Ref. 3), so that in this paper, we shall consider the other two types of the expansions. First, we shall introduce
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longitudinal (eikonal-type) and transverse (Fresnel-type\textsuperscript{5}) approximations for the propagator. It is shown that a propagator which expresses propagation to backward direction can be constructed from the longitudinal approximation and the Chen-Hooch formula is also derived from this approximation. We shall refer it to the second type expansion. Moreover, we shall show that a propagator in the transverse approximation coincides with the Fresnel propagator discussed by Gottfried,\textsuperscript{9} and Frahn and Schüemann.\textsuperscript{6} The third type of the expansion is expected to give excellent approximation for intermediate scattering angles. In order to investigate the validity of this expansion, the numerical calculations are carried out for square-well potentials.

In § 2 we shall deal with the formal aspects of the integral equation for the modulating function. It is shown that various types of formal solutions of the integral equation can be obtained by iteration methods. In § 3 we shall apply the functional integration method to represent various propagators in closed forms. It is shown that the procedures developed in § 2 lead to physical pictures which correspond to the eikonal type and the Fresnel type approximations. We shall also show that, in the second type of the expansions, Chen and Hooch's result can be obtained from the first order approximation for the modulating function in the longitudinal approximation. The results of numerical calculations for the third type approximation are also shown in § 3. The relations among the systematic approximation methods developed in § 2 and an approximation method for the functional integral representation are discussed in § 4. Some comments for the Rytov expansion\textsuperscript{19} of the modulating function are also given.

§ 2. Formalism

In earlier papers,\textsuperscript{3,4} the modulating operator formulation of eikonal approximation was proposed. For completeness, we shall begin with a brief review of the formalism developed in Ref. 3. Let us consider a projectile of mass $m$ scattered by a local potential $V$ with an initial momentum $k_i$ and a final momentum $k_f$.\textsuperscript{5} Our formalism is based upon to introduce the modulating operator $ϕ^{(+)}$, which is defined from the solution $Ψ^{(+)}$ of the Lippmann-Schwinger operator equation,\textsuperscript{3} by the relation

$$Ψ^{(+)} = ϕ^{(+)}φ^{(+)} , \quad (2·1)$$

where $ϕ^{(+)}$ is the operator corresponding to the plane wave in initial state. It can be shown that the modulating operator $ϕ^{(+)}$ satisfies the following integral equation:

$$φ^{(+)} = \frac{1}{p^2/2m + v_α p - iε} V φ^{(+)} , \quad (2·2)$$

\textsuperscript{5} In this paper, natural units $\hbar = c = 1$ are used.
where $v_i$ is the incident velocity which is given by $v_i = k_i/m$, and $p$ is intermediate momentum. The conventional $T$ matrix element $T(k_f, k_i)$ can be expressed, in terms of $\phi^{(+)\dagger}$, as

$$T(k_f, k_i) = \langle \tilde{V} \phi^{(+)\dagger} \rangle,$$  \hspace{1cm} (2.3)

in which $\tilde{V}$ is defined by

$$\tilde{V} = \phi^{(-\dagger)} V \phi^{(+)} ,$$  \hspace{1cm} (2.4)

where $\Phi^{(-)}$ is the operator corresponding to the plane wave of final state. Here we have introduced the standard bra $<$ and ket $|$ with the properties $\langle r | = | r \rangle = 1$ and $\langle p | = | p \rangle = 0$. For our purposes, we shall now rewrite Eq. (2.2) into the integral equation with the inhomogeneous term which expresses propagation of the particle along a straight line path with an arbitrary direction inside potential. In fact, we have the integral equation for the modulating operator $\phi^{(+)\dagger}$ without any approximations:

$$\phi^{(+)} = \phi^{(+)} - \frac{1}{v_m \cdot p + V - i\xi} \left[ p^2 / 2m + (v_i - v_m) \cdot p \right] \phi^{(+\dagger)} ,$$  \hspace{1cm} (2.5)

where $v_m$ is an arbitrary velocity of the particle and the inhomogeneous term $\phi^{(+\dagger)}$ is defined by the relation

$$\phi^{(+\dagger)} = \langle v_m \cdot p + V - i\xi \rangle ,$$  \hspace{1cm} (2.6)

which expresses the eikonal type propagation of the particle. It is worth while to note that Eq. (2.5) just corresponds to the Lippmann-Schwinger integral equation in the formal scattering theory. In our formalism, the propagator $- (v_m \cdot p + V - i\xi)^{-1}$ in Eqs. (2.5) and (2.6), as we shall see in § 3, means that the particle propagates along the $v_m$-direction in the potential and the phase of its wave function is accumulated. On the other hand, the term $[ p^2 / 2m + (v_i - v_m) \cdot p ]$ in Eq. (2.5) causes the large angle scattering without accumulating phase. The formal solutions of Eq. (2.5) can easily be written as

$$\phi^{(+)} = \phi^{(+\dagger)} - \frac{1}{p^2 / 2m + v_i \cdot p + V - i\xi} \left[ p^2 / 2m + (v_i - v_m) \cdot p \right] \phi^{(+\dagger)} ,$$  \hspace{1cm} (2.7a)

$$= \phi^{(+\dagger)} + \frac{1}{p^2 / 2m + v_i \cdot p + V - i\xi} \left[ p^2 / 2m + (v_i - v_m) \cdot p \right] \frac{1}{v_m \cdot p + V - i\xi} \langle v_m \cdot p + V - i\xi \rangle .$$  \hspace{1cm} (2.7b)

The inhomogeneous term $\phi^{(+\dagger)}$ can be chosen so as to improve the accuracy of the scattering amplitude in the zeroth order approximation. For examples, the Abarbanel-Itzykson\textsuperscript{10} and the modified Glauber approximations\textsuperscript{10} can be obtained by $v_m = (v_i + v_f) / 2$ and $y_m = |v_i| (v_i + v_f) / |v_i + v_f|$, respectively, where $v_f = k_f / m$.

The formal solutions (2.7a) and (2.7b) show that the simplest one will be
for $v_m = v_i$ in which case the solutions (2.7) gives:

$$\phi^{(+)} = \phi^{(+)}_a = \frac{1}{p^2/2m + v_i \cdot p + V - i\varepsilon} \left[ p^2/2m \right] \phi^{(+)}_a,$$

(2.8a)

$$= \phi^{(+)}_a + \frac{1}{p^2/2m + v_i \cdot p + V - i\varepsilon} \left[ p^2/2m \right] \frac{1}{v_i \cdot p + V - i\varepsilon} V,$$

(2.8b)

where the term $\phi^{(+)}_a$, hereafter called the Glauber approximation of $\phi^{(+)}$, is defined by

$$\phi^{(+)}_a = \frac{1}{v_i \cdot p + V - i\varepsilon} V.$$

(2.9)

It is, here, noted that the relation between $\phi^{(+)}_a$ and $\phi^{(+)}_m$ can easily be found from Eqs. (2.7b) and (2.8b) to be

$$\phi^{(+)}_m = \phi^{(+)}_a - \frac{1}{v_i \cdot p + V - i\varepsilon} (v_m - v_i) \cdot p \phi^{(+)}_a,$$

(2.10a)

and

$$\phi^{(+)} = \phi^{(+)}_m - \frac{1}{v_m \cdot p + V - i\varepsilon} (v_i - v_m) \cdot p \phi^{(+)}_a.$$

(2.10b)

For the sake of simplicity, we shall hereafter treat only the type of solution (2.8). We shall now consider three types of approximation methods for $\phi^{(+)}$, Eq. (2.8a), resulting from the three types of expansions of the propagator $(p^2/2m + v_i \cdot p + V - i\varepsilon)^{-1}$. The first type of the eikonal approximation will be obtained from the eikonal series expansion of $\phi^{(+)}$, Eq. (2.8a), in the form

$$\phi^{(+)} = \phi^{(+)}_a + \sum_{n=0}^\infty [g(v_i \cdot p) p^2/2m]^n g(v_i \cdot p) [p^2/2m] \phi^{(+)}_a,$$

(2.11)

where $g(v_i \cdot p)$ is given by

$$g(v_i \cdot p) = -\frac{1}{v_i \cdot p + V - i\varepsilon}.$$

(2.12)

The systematic approximation methods for the scattering amplitude based on this expansion of $\phi^{(+)}$ were discussed in somewhat detail in Ref. 3).

The second type is based on the following expansion of Eq. (2.8a):

$$\phi^{(+)} = \phi^{(+)}_a + \sum_{n=0}^\infty \left[ h(v_i \cdot p) V \right]^n [p^2/2m] h(v_i \cdot p) \phi^{(+)}_a,$$

(2.13)

in which $h(v_i \cdot p)$ is defined by

$$h(v_i \cdot p) = -\frac{1}{v_i \cdot p + p^2/2m - i\varepsilon}.$$

(2.14)

*) The original Glauber approximation will work well for all scattering angles as the energy goes to infinity.
It will be possible to develop similar arguments on systematic approximation procedures for this case to those in Ref. 3. The argument on the convergence of this expansion is almost the same as the Born series expansion.\(^{10}\) We shall not, however, go into any further details for it here.

The third type of expansion of \(\psi^{(+)}\) can be obtained from Eq. (2.8a) in the form of a power expansion of \((v_i \cdot p)\),

\[
\psi^{(+)} = \phi_{a}^{(+)} + \sum_{n=0}^{\infty} [I(V) v_i \cdot p]^{n} I(V) [p^2 / 2m] \phi_{a}^{(+)} ,
\]

(2.15)

where \(I(V)\) is given by

\[
I(V) = \frac{1}{p^2/2m + V - i\varepsilon} .
\]

(2.16)

We may expect that the expansion (2.15) gives an excellent approximation method for the scattering amplitude in the range of scattering angles for which \((v_i \cdot p)\) can be regarded as small operator, that is, the transverse direction.

Now we proceed to write down explicitly the first few terms in these expressions (2.11), (2.13) and (2.15). The first order approximations for \(\psi^{(+)}\) will be given in the respective forms

\[
\psi^{(+)} \approx \psi_{B}^{(+)} = \phi_{a}^{(-)} + g(v_i \cdot p) [p^2 / 2m] \phi_{a}^{(+)} ,
\]

(2.17)

\[
\psi^{(+)} \approx \psi_{G}^{(+)} = \phi_{a}^{(+)} + [p^2 / 2m] h(v_i \cdot p) \phi_{a}^{(+)} ,
\]

(2.18a)

\[
= -(v_i \cdot p) h(v_i \cdot p) \phi_{a}^{(+)}
\]

(2.18b)

and

\[
\psi^{(+)} \approx \psi_{T}^{(+)} = \phi_{a}^{(+)} + I(V) [p^2 / 2m] \phi_{a}^{(+)} ,
\]

(2.19a)

\[
= -(1 - G_0 V)^{-1} G_0 V \phi_{a}^{(+)} .
\]

(2.19b)

In obtaining Eq. (2.19b), use has been made of the relation

\[
I(V) [p^2 / 2m] = -(1 - G_0 V)^{-1} ,
\]

(2.20)

where \(G_0 = -(p^2 / 2m - i\varepsilon)^{-1}\). Thus we have three types of correction terms to the Glauber amplitude \(\phi_{a}^{(+)}\). It is shown that Eq. (2.17) gives the Baker approximation,\(^{10}\) and that Eq. (2.18) can be reduced to the approximation which is proposed recently by Chen and Hoock for backward angles. On the other hand, it may be expected that \(\psi_{T}^{(+)}\) gives a good approximation for intermediate scattering angles, since the series expansion (2.15) will converge rapidly when \((v_i \cdot p)\) is small.

\(^{10}\) The usual Born approximation may be obtained from the following series expansion of Eq. (2.2):

\[
\psi^{(+)} = \sum_{n=0}^{\infty} [h(v_i \cdot p) V]^{n} .
\]
§ 3. High energy approximations

In this section we shall illustrate explicit forms of both the modulating function and the scattering amplitude in the second and the third type approximations. Before obtaining explicit forms of the scattering amplitude, let us consider physical meanings of various propagators \( \langle r | \frac{p^2}{2m} + v \cdot p + V - i\varepsilon \rangle^{-1} | r' \rangle \), \( \langle r | g(v \cdot p) | r' \rangle \), \( \langle r | h(v \cdot p) | r' \rangle \) and \( \langle r | I(V) | r' \rangle \) in coordinate space. For these purposes, it will be convenient to express the various propagators in terms of the functional integrals.

A functional integral expression for \( \langle r | \frac{p^2}{2m} + v \cdot p + V - i\varepsilon \rangle^{-1} | r' \rangle \) can be given in a closed form in terms of the functional integral with respect to trajectories:

\[
\langle r | \frac{p^2}{2m} + v \cdot p + V - i\varepsilon \rangle^{-1} | r' \rangle = i \int_0^\infty dt \left( \exp \left[ -i \int_0^t dt' V(r-vt') - \int_0^t \tau d\zeta \right] \delta(r-r' - vt - \int_0^t \tau d\zeta) \right)^{\text{(i)}},
\]

in which \( \langle \cdot \rangle^{\text{(i)}} \) signifies the quantity obtained by the following path integral

\[
\langle F[\tau] \rangle^{\text{(i)}} = \int [\delta^3 \tau]_0^i F[\tau],
\]

where \( F[\tau] \) is an arbitrary functional of \( \tau \) and

\[
[\delta^3 \tau]_0^i = \Pi d^2 \zeta(\xi) \exp \left[ i \frac{m}{2} \int_0^i \tau^2(\xi) \, d\xi \right] \cdot \int \cdot \Pi d^2 \zeta(\xi) \exp \left[ i \frac{m}{2} \int_0^i \tau^2(\xi) \, d\xi \right].
\]

The propagator \( \langle r | g(v \cdot p) | r' \rangle \), as was shown in Ref. 3), could be expressed in the form

\[
\langle r | g(v \cdot p) | r' \rangle = (-i) \int_0^\infty dt \delta(r-r' - vt) \exp \left[ -i \int_0^t dt' V(r-vt') \right],
\]

which is, of course, obtained from Eq. (3.1) by neglecting \( \tau \) in the arguments of the potential \( V \) and \( \delta \)-function. Since the delta function in Eq. (3.4) means that the wave propagates along the incident direction, this propagator corresponds to a propagation along the straight line and is accumulating a phase \( \exp[-i\int_0^t dt' V(r-vt')] \). This is an analogous situation to that in geometrical optics. Accordingly, the use of the propagator \( \langle r | g(v \cdot p) | r' \rangle \), in the expansion of the modulating function, may correspond to an emphasis of a particle-like picture of a scattering wave.

On the other hand, the propagator \( \langle r | h(v \cdot p) | r' \rangle \) can be obtained from Eq. (3.1) by putting \( V = 0 \):
The expression (3.5) tells us that the propagator \( \langle r | h(v \cdot p) | r' \rangle \) can be expressed as a functional average of the deviation from a straight line path in the incident direction. Thus the propagator \( \langle r | h(v \cdot p) | r' \rangle \) may correspond to the description in a wave-like picture, according to Feynman's path integral formulation of the quantum mechanics. Here, it should be noted that an eikonal propagator for free particle is obtained by the relation

\[
\langle r | h(v \cdot p) | r' \rangle = (-i) \int_0^\infty dt \delta(r - r' - vt),
\]

(3.6)

from which we see that \( p^2/2m \)-term in \( h(v \cdot p) \) and on the left-hand side of Eq. (3.1) concerns the quantum mechanical wave nature.

In order to clarify the physical meaning of \( h(v \cdot p) \), we shall now rewrite the propagator (3.5) in terms of the parametric integral

\[
\langle r | h(v \cdot p) | r' \rangle = (-i) \int_0^\infty dt \exp \left[ i \frac{\mathbf{r}^2}{2m} t \frac{\delta^2}{\delta \mathbf{r}^2} \right] \delta(r - r' - vt),
\]

(3.7)

which is obtained by representing \( \delta \)-function in terms of Fourier integral and using the following formulas

\[
\langle F[\tau] \rangle = \exp \left[ i \int_0^\infty dt \frac{d^2}{d \mathbf{r}^2} \frac{\delta^2}{\delta \mathbf{r}^2} \right] F[\tau] |_{r=0}
\]

(3.8)

and

\[
F \left[ \frac{\delta}{\delta \mathbf{r}} \right] G[\tau] |_{r=0} = G \left[ \frac{\delta}{\delta \mathbf{r}} \right] F[\tau] |_{r=0},
\]

(3.9)

where \( F \) and \( G \) are functionals of \( \mathbf{r} \).

Choosing \( v \) in the \( z \)-direction, we split the operator \( \mathbf{r} \cdot 2m \) in the integrand of Eq. (3.7) into transverse (index \( b \)) and longitudinal (index \( z \)) components with respect to the \( v \)-direction. Thus we shall express \( \langle r | h(v \cdot p) | r' \rangle \) in the form

\[
\langle r | h(v \cdot p) | r' \rangle = \left( \frac{-i}{\nu} \right) \int_0^\infty d\nu t(b, b' | \nu) l(z, z' | \nu),
\]

(3.10)

where

\[
t(b, b' | \nu) = \exp \left[ i \frac{\mathbf{r}^2}{2k} \nu \delta(b - b') \right]
\]

(3.11)

and

\[
l(z, z' | \nu) = \exp \left[ i \frac{\mathbf{r}^2}{2k} \nu \delta(z - z' - \nu) \right],
\]

(3.12)

where the impact parameter \( b \) and \( b' \) are the respective components of \( r \) and \( r' \).
perpendicular to \( \mathbf{v} \), and \( k = |\mathbf{k}| \). In the language of optics, the term \( t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) \) concerns the wave optical deviation in the transverse direction—diffractive deviations from geometric optical propagation. On the other hand, the term \( l(z, z'|\mathbf{v}) \) represents contributions describing longitudinal distortions of the geometric optical pattern. It is worth while to note that backward propagation of the wave may be described by the \( l(z, z'|\mathbf{v}) \) term.

In order to see the properties of \( t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) \) and \( l(z, z'|\mathbf{v}) \) more precisely, let us now perform Fourier transforms for these terms in the respective variables of the transverse momentum \( p_0 \) and the longitudinal momentum \( p_z \); we then have various expressions for \( t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) \) and \( l(z, z'|\mathbf{v}) \)

\[
t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) = \frac{1}{(2\pi)^2} \int d\mathbf{b} \exp \left[ ip_0 \cdot (\mathbf{b} - \mathbf{b}') - i \frac{p_0^2}{2k} \right],
\]

\[
= \frac{k}{2\pi i^2} \exp \left[ i \frac{k}{2^2} |\mathbf{b} - \mathbf{b}'|^2 \right],
\]

\[
= \frac{1}{2\pi} \int^\infty_0 p_0 d p_0 J_0(p_0 |\mathbf{b} - \mathbf{b}'|) \exp \left[ -i \frac{p_0^2}{2k} \right],
\]

where \( J_0 \) is the Bessel function of zeroth order, and

\[
l(z, z'|\mathbf{v}) = \frac{1}{2\pi} \int d\mathbf{p} \exp \left[ i p_z (z - z' - \nu) - i \frac{p_z^2}{2k} \right],
\]

\[
= \sqrt{\frac{k}{2\pi i^2}} \exp \left[ i \frac{k}{2^2} (z - z' - \nu)^2 \right].
\]

We assume here that the distance over which the potential changes by an appreciable fraction of itself has the same order of magnitude as the size \( R \) of the potential. For \( |\mathbf{b} - \mathbf{b}'| \leq R \) and \( b' \geq R \) a range of transverse momentum \( p_0 \), which mainly contributes to \( t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) \), will be of order \( R^{-1} \). Thus we see that \( t(\mathbf{b}, \mathbf{b}'|\mathbf{v}) \) behaves like \( \delta^2(\mathbf{b} - \mathbf{b}') \) for \( \nu/kR^2 \ll 1 \), where \( \nu \) is essentially the longitudinal distance. This approximation will be referred to as a longitudinal approximation. In this case, we find that

\[
\langle r | \hat{h}(\mathbf{v} \cdot \mathbf{p}) | r' \rangle \approx \left( \frac{-i}{\nu} \right) \delta^2(\mathbf{b} - \mathbf{b}') \int^\infty_0 d\nu l(z, z'|\mathbf{v})
\]

\[
= \left( \frac{-i}{\nu} \right) \delta^2(\mathbf{b} - \mathbf{b}') \left[ \theta(z - z') + \theta(z' - z) e^{-2ik(z-z')} \right],
\]

in which the step function \( \theta(z - z') \) and \( \theta(z' - z) \) restrict propagations to the forward and backward directions, respectively. On the other hand, for the momentum such that \( p_z^2/2k \ll p_z \), we get
\begin{equation}
\langle r | h(v \cdot p) | r' \rangle \approx \frac{(-i)}{v} \theta(z - z') t(b, b' | z - z'), \tag{3.16a}
\end{equation}

\begin{equation}
\approx \frac{(-i)}{v} \theta(z - z') \frac{k}{2\pi i (z - z')} \exp\left[i \frac{k}{2 (z - z')} | b - b' |^2 \right]. \tag{3.16b}
\end{equation}

We shall refer to Eq. (3.16b) as a transverse approximation. The expression (3.16b) is essentially the Fresnel Green function discussed by Frahn and Schürmann. Further discussion is given by Matsuki and Ishihara in somewhat different ways.

We shall now consider the physical meaning of the operator \(v \cdot p\) \(h(v \cdot p)\) in the longitudinal approximation. By taking account of Eq. (3.5), we can write this operator in the form

\begin{equation}
\langle r | (v \cdot p) h(v \cdot p) | r' \rangle = -\frac{i}{m} (k \cdot \mathcal{F}) \langle r | h(v \cdot p) | r' \rangle , \tag{3.17}
\end{equation}

where \(\mathcal{F}\) operates on \(r\). By applying the approximation Eq. (3.15) to Eq. (3.17), it is found to be

\begin{equation}
\langle r | (v \cdot p) h(v \cdot p) | r' \rangle = \delta^3(b - b') \theta(z' - z) (2ik) \exp[-2ik(z - z')]. \tag{3.18}
\end{equation}

It is clear that this corresponds to the propagation to the backward direction. It should be noted that this property is contrast to the fact that the propagator \(\langle r | g(v \cdot p) | r' \rangle\) is closely related to the forward propagation, as we can see from Eq. (3.4) by rewriting it into the form

\begin{equation}
\langle r | g(v \cdot p) | r' \rangle = \frac{(-i)}{v} \delta^3(b - b') \theta(z - z') \exp\left[-\frac{i}{v} \int_{z'}^{z} d\zeta V(b, \zeta)\right]. \tag{3.19}
\end{equation}

Although the expression of propagator \(\langle r | I(V) | r' \rangle\) in the third type approximation can simply be obtained from Eq. (3.1) by putting \(v = 0\), it will be convenient to consider matrix elements of the operator \((1 - G_0 V)^{-1}\), as is suggested from Eq. (2.20), in coordinate space, since \(\langle r | (1 - G_0 V)^{-1} | r' \rangle\) can be expressed in a simple closed form:

\begin{equation}
\langle r | (1 - G_0 V)^{-1} | r' \rangle = \left[1 - (-i) \int_0^\infty dt \, e^{i(t^3/3n)} V(r)\right]^{-1} \delta(r - r'). \tag{3.20}
\end{equation}

Let us calculate explicit forms of the modulating function in the second type approximation. In the longitudinal approximation, the modulating function \(\phi_0^{(+)}(r)\) can easily be found by substituting Eq. (3.18) into Eq. (2.18b). The result is

\begin{equation}
\phi_0^{(+)}(r) = -2ik \int_z^{z'} dz' \exp[-2ik(z - z')] \phi_0^{(+)}(b', z'), \tag{3.21}
\end{equation}

where \(\phi_0^{(+)}(b, z) = \exp[i\chi(b, z)]\) with \(\chi(b, z) = -\int_0^z dt V(r - vt)\). By substituting Eq. (3.21) into Eq. (2.3), and after a change of order of integrations over \(z\) and \(z'\), the scattering amplitude can be written as
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\[ T(k_i, k_f) = -2ik \int d^3r \exp[2ik \cdot r + i\chi(r)] \times \int_{-\infty}^{\infty} dz' \exp[i(\mathbf{A} - 2\mathbf{k}) \cdot \mathbf{r}'] V'(r'), \]  

(3.22)

where \( \mathbf{r} = (x, y, z) \) and \( \mathbf{r}' = (x', y, z') \), and the momentum transfer \( \mathbf{A} = \mathbf{k}_i - \mathbf{k}_f \). This formula just corresponds to Eq. (15) in Chen and Hock’s paper. A numerical test of validity of the amplitude, Eq. (3.22), was also widely discussed by these authors. On the other hand, from Eqs. (3.16b) and (3.17), \( \phi_0^{(+)}(\mathbf{r}) \) in the transverse approximation is given by

\[ \phi_0^{(+)}(\mathbf{r}) = \phi_0^{(+)}(\mathbf{r}) + \frac{i}{2k} \int_{-\infty}^{\infty} dz' \mathbf{b}' \nabla_b^2 \mathbf{l}(\mathbf{b}, \mathbf{b}'|z-z') \phi_0^{(+)}(r'), \]  

(3.23)

where \( \nabla_b^2 \) is the transverse Laplacian operator. In this case, we find that the scattering amplitude can be expressed as

\[ T(k_i, k_f) = \int d^3r \ e^{i\mathbf{d} \cdot \mathbf{r}} V(\mathbf{r}) \phi_0^{(+)}(\mathbf{r}) + \frac{i}{2k} \int d^3r' \ e^{i\mathbf{d} \cdot \mathbf{r}'(\mathbf{b}, \mathbf{b}'|z-z')} \phi_0^{(+)}(\mathbf{r}'), \]  

(3.24)

It is interesting to note that the first term in Eq. (3.24) is just the Glauber amplitude for small-angle scattering. The second term can be understood as a correction term of the Fresnel type to the Glauber amplitude. The Fresnel diffraction effects were first pointed out by Gottfried in a study of high energy scattering from deutrium.

We now wish to express the scattering amplitude in terms of modulating function in the third-type approximation. To do this, for example, we shall need the explicit form of \( \phi_0^{(+)}(\mathbf{r}) \). From Eqs. (2.19b) and (3.20), it can be shown that the modulating function \( \phi_1^{(+)}(\mathbf{r}) \) can simply be written in the renormalized form of

\[ \phi_1^{(+)}(\mathbf{r}) \approx A(\mathbf{r}) \phi_0^{(+)}(\mathbf{r}), \]  

(3.25)

where the renormalization factor \( A(\mathbf{r}) \) is defined by

\[ A(\mathbf{r}) = -\frac{\psi(\mathbf{r})}{(1 - \psi(\mathbf{r}))}, \]  

(3.26)

in which \( \psi(\mathbf{r}) \) is given by

\[ \psi(\mathbf{r}) = -\frac{m}{2\pi} \int d^3r' \frac{V'(r')}{|r' - r|}. \]  

(3.27)

Thus, we can write the scattering amplitude as

\[ T(k_i, k_f) = \int d^3r \ e^{i\mathbf{d} \cdot \mathbf{r}} V(\mathbf{r}) A(\mathbf{r}) \phi_0^{(+)}(\mathbf{r}). \]  

(3.28)

\(^*\) See the Appendix.
The validity of this approximation is, of course, restricted by the condition $|\psi(r)| < 1$. The higher order approximations for $\phi^{(1)}(r)$ can be obtained by similar procedures, but we shall not go further to discuss it here. The numerical calculations

\begin{figure}[h]
\centering
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1a}
\caption{The imaginary part of the scattering amplitude for a square-well potential $V(r) = \begin{cases} -10.0 & r < R \\ 0.0 & r > R \end{cases}$, with $R = 1.0$. In Figs. 1, 2 and 3, the scattering amplitude $F$'s have been normalized as $F = f(\theta)/(4\pi^2m)$, in which the initial momentum $k_i = 5.0$ and the momentum transfer $|A| = 2k \sin \theta/2$ and we have used the units $2m = \hbar = c = 1$.}
\end{subfigure}
\begin{subfigure}[b]{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{fig1b}
\caption{The real part of the scattering amplitude of Fig. 1(a).}
\end{subfigure}
\end{figure}
for the various scattering amplitudes are carried out for square-well potentials. The results are shown in Figs. 1~3, where the $F_E$'s denote the exact amplitudes which are calculated from partial-wave analysis with $l_{\text{max}} = 18$ or 30, according as the radius of the potential $R = 1.0$ or 4.0, $F_G$'s are the original Glauber amplitudes, $F_{MG}$ the modified Glauber one and $F_I$'s the amplitudes obtained from the third type approximation, Eq. (3-28).

Fig. 2(a). The imaginary part of the scattering amplitude for a potential

$$V(r) = \begin{cases} -5.0 & r < R \\ 0.0 & r > R \end{cases}$$

with $R = 1.0.$

Fig. 2(b). The real part of the scattering amplitude of Fig. 2(a).
Fig. 3(a). The imaginary part of the scattering amplitude for a potential
\[ V(r) = \begin{cases} 
-2.5 & r < R \\
0.0 & r > R 
\end{cases} \]
with \( R = 4.0 \).

Fig. 3(b). The real part of the scattering amplitude of Fig. 3(a).

§ 4. Discussion

In order to obtain physical insight of our approximations, we have used the functional integral techniques for various propagators. It is possible to show that the functional integral representation of modulating function \( \phi^{(\ast)}(r) \) is explicitly obtained from Eq. (2.2) in the closed form

\[ \phi^{(\ast)}(r) = \exp \left( -i \int_{0}^{\infty} dt \left( r - v t - \int_{0}^{t} \tau \, d\tau \right) \right) \]  

\[ (4.1) \]
by using Eq. (3·1). The Glauber approximation \( \phi_0^{(+)}(r) \) will simply be obtained from the expression (4·1) by setting \( \tau = 0 \) in the argument of the potential \( V \).

Here, it is interesting to know how to properly approximate the functional average (4·1) in order to get the eikonal type and the Fresnel type approximations. For these purposes, it will be convenient to make use of the following identity proposed by Barbashov:

\[
\langle \exp F[\tau] \rangle = \exp \langle F[\tau] \rangle \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (F[\tau] - \langle F[\tau] \rangle)^n, \quad (4·2)
\]

in which \( F[\tau] \) is an arbitrary functional of \( \tau \). In our discussion, the functional will be given by

\[
F[\tau] = (-i) \int_0^\infty dt \, V(r - vt - \int_0^t \tau \, d\tilde{z}). \quad (4·3)
\]

The functional average \( \langle F[\tau] \rangle^{(\omega)} \) can be evaluated by similar procedure for obtaining Eq. (3·7) from Eq. (3·5). The result is

\[
\langle F[\tau] \rangle^{(\omega)} = (-i) \int_0^\infty dt \, \exp \left[ i \frac{F^2}{2m} \right] V(r - vt), \quad (4·4a)
\]

\[
= \langle r | h(v \cdot p) V \rangle, \quad (4·4b)
\]

where we have used Eq. (3·7).

We shall now consider two types of the expressions for \( \langle F[\tau] \rangle^{(\omega)} \) corresponding to the longitudinal and the transverse approximations for \( h(v \cdot p) \). In the longitudinal approximation, \( \langle F[\tau] \rangle^{(\omega)} \) can be expressed by using Eq. (3·15) as

\[
\langle F[\tau] \rangle^{(\omega)} = \frac{(-i)}{v} \left[ \int_{-\infty}^z dz' V(b, z') + e^{-izk} \int_{-\infty}^0 dz' \, e^{izk} V(b, z') \right], \quad (4·5)
\]

in which the first term expresses the usual eikonal phase and the second term gives a phase for backward propagation. Since the second term contains a fast oscillating function \( \exp[2ik(z' - z)] \), it can be neglected at high energies. On the other hand, the transverse approximation, Eq. (3·16b), gives

\[
\langle F[\tau] \rangle^{(\omega)} = \frac{(-i)}{v} \int_{-\infty}^z dz' \frac{k}{2\pi i(z - z')} \int \exp \left[ i \frac{k}{2(z - z')} |b - b'| \right] V(b', z') \, db'. \quad (4·6)
\]

The zeroth order approximation of \( \phi^{(+)}(r) \), for example, can be defined by

\[
\phi_0^{(+)}(r) = \exp \langle F[\tau] \rangle^{(\omega)}. \quad (4·7)
\]

By substituting Eq. (4·5) into Eq. (4·7) and retaining the first term of the Taylor expansion of \( \exp[-(i/v)\int_{-\infty}^z dz' e^{izk} V(b, z')] \), we find that

\[
\phi_0^{(+)}(r) \approx \left[ 1 - \frac{i}{v} e^{-izk} \int_{-\infty}^z dz' \, e^{izk} V(b, z') \right] \exp \left[ \frac{-i}{v} \int_{-\infty}^z dz' V(b, z') \right]. \quad (4·8)
\]
It is easily shown that this expression of \( \phi_0^{(+)}(r) \) gives just the scattering amplitude obtained by Chen and Hoock, Eq. (3·22). In the transverse approximation, however, the relation between Eqs. (4·6) and (3·23) is not so clear.

We shall mention the Baker approximation for \( \phi^{(+)}(r) \). From Eq. (4·2), for example, the first order approximation of \( \phi^{(1)}(r) \) may be expressed as

\[
\phi^{(1)}(r) = \exp \langle F[\tau] \rangle^{(n)} [1 + D/2],
\]

where \( D \) is defined by

\[
D = \langle F^2[\tau] \rangle^{(n)} - \langle F[\tau] \rangle^{(n)} \times \langle F[\tau] \rangle^{(n)}.
\]

In this case, it will be sufficient to approximate \( \langle F[\tau] \rangle^{(n)} \) by

\[
\langle F[\tau] \rangle^{(n)} \approx (-i) \int_0^\infty dt \, V(r - vt) + \frac{(-i)}{2m} \int_0^\infty dt \, tP'V(r - vt).
\]

After somewhat lengthy calculations, it can be shown that the correction term \( D \) can be expressed approximately as

\[
D \approx - \frac{i}{m} \int_0^\infty d\vec{v} \left[ 2 \int_0^\infty d\vec{x} \, \varphi V(r - v\vec{x}) \right]^2.
\]

Thus, we find that

\[
\phi^{(1)}(r) \approx \left( 1 - \frac{1}{2mV} \int_0^\infty d\vec{v} \left[ \varphi V(b, \vec{x}) + i \langle \varphi \rangle \langle \varphi \rangle \right] \right) \phi^{(+)}(b, z).
\]

This expression of \( \phi^{(1)}(r) \) is just obtained by Baker, and discussed in Ref. 3).

It is useful to connect the expression (4·1) with the Rytov expansion 15) which is the power series expansion of the logarithm of the modulating function with respect to the coupling constant \( \lambda \):

\[
\phi^{(+)}(r) = e^{i\gamma(r)} = \exp [i \sum_{n=1}^{\infty} \lambda^n \gamma_n].
\]

From Eq. (4·1) and the expression obtained by putting \( \lambda V \) instead of the potential \( V \) in Eq. (4·1), we get

\[
\exp [i \sum_{n=1}^{\infty} \lambda^n \gamma_n] = \exp \left[ -i \lambda \int_0^\infty d\vec{v} \, V(r - v\vec{v} - \int_0^\tau d\vec{x}) \right] \langle F[\tau] \rangle^{(n)}.
\]

The coefficient \( \gamma_n \) of this power expansion can easily be determined by the relations, for examples,

\[
i \gamma_1 = \frac{\delta}{\delta \lambda} e^{i\lambda \gamma_1|_{\lambda=0}} = \langle F[\tau] \rangle^{(n)},
\]

\[
i \gamma_2 = \frac{1}{2} \left[ \frac{\delta^2}{\delta \lambda^2} e^{i\lambda \gamma_2|_{\lambda=0}} - (i \gamma_1)^2 \right]
\]

\[
= \langle \frac{1}{2} (F[\tau] - \langle F[\tau] \rangle^{(n)})^2 \rangle^{(n)},
\]
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\[ i\xi_k = \frac{1}{3!} \left[ \frac{\partial^2}{\partial \lambda^2} e^{ix}\big|_{\lambda=0} + 6\xi_k \lambda + i\xi_k \right] \]
\[ = \left( \frac{1}{3!} \left( F[\tau] - \langle F[\tau] \rangle^{(\infty)} \right) \right)^{(n)} \]

and so on.

It should be noted that the coefficient \( \gamma_n \) in the Rytov expansion coincides with the term of the \( n \)-th order in the expansion (4·2) only up to \( n = 3 \), and for the coefficients higher than \( n = 4 \), the coincidence cannot be obtained by the sequent manner.

In Figs. 1, 2 and 3, we have shown that the amplitude \( (F_I) \) resulting from Eq. (3·28) gives an appropriate correction to the Glauber amplitude \( (F_0) \) for a range of intermediate angles, and that, in the case of \( k_i R = 20 \) (Fig. 3), the scattering concentrate on forward angles, which is the characteristic features of the high energy behaviour, the corrections are much smaller than in the cases of \( k_i R = 5 \) (Figs. 1 and 2), which indicates that the original Glauber amplitude becomes better at all scattering angles as the energy goes to infinity.

The numerical calculations were carried out by FACOM 230 of the Tokyo Metropolitan University.

The authors would like to thank Professor M. Sasaki for his hospitality and the members of his laboratory for the constructive discussions of the numerical calculations.

Appendix

In this appendix we would like to give an explicit form of \( \phi^{(\infty)} \) in the coordinate representation. Before obtaining this form, we shall give more explicit form of \( \langle r | G_0 V | r' \rangle \):  
\[ \langle r | G_0 V | r' \rangle = (-i) \int_0^\infty dt \exp \left[ \frac{i \mathbf{r} \cdot \mathbf{r}'}{2m} t \right] \delta (r - r') V (r') , \quad (A·1) \]
\[ = - \frac{m}{2\pi} \frac{1}{|r - r'|} V (r') , \quad (A·2) \]

where we have used the relation
\[ (-i) \int_0^\infty dt \exp \left[ \frac{i \mathbf{r} \cdot \mathbf{r}'}{2m} t \right] \delta (r - r') = - \frac{m}{2\pi} \frac{1}{|r - r'|} . \quad (A·3) \]

Here it should be noted that Eq. (A·3) coincides with \( g (r - r' | k) \big|_{k=0} \) when \( g (r - r' | k) \) is ordinary Green's function. By expanding \((1 - G_0 V)^{-1}\) in power series of \( G_0 V \), Eq. (2·19b) becomes
\[ \langle r | \phi^{(\infty)} \rangle = - \int d^3 r' \sum_{n=0}^{\infty} \int d^3 r_1 \int d^3 r_2 \cdots \int d^3 r_n \langle r | G_0 V | r_1 \rangle \langle r_1 | G_0 V | r_2 \rangle \cdots \]
\[ \cdots \langle r_{n-1} | G_0 V | r_n \rangle \langle r_n | G_0 V | r' \rangle \phi^{(\infty)} (r') . \quad (A·4) \]
By substituting Eq. (A·2) into Eq. (A·4), we get
\[
\langle r | \phi_f^{(+)} \rangle = -\sum_{n=0}^{\infty} \left( -\frac{m}{2\pi} \right)^{n+1} \int \frac{d^3r_1 \cdots \int d^3r_n \int d^3r' \frac{1}{|r-r'|} V(r_i)}{\left| r_1 - r_2 \right| \left| r_2 - r_3 \right| \cdots \left| r_n - r' \right|} \phi_0^{(+)}(r') .
\] (A·5)

Let us consider the integral defined by
\[
I = \int \frac{d^3r_1}{|r-r_1|} V(r_i) \frac{1}{|r_1 - r_2|} .
\] (A·6)

Under the adequate conditions on the potential, Eq. (A·6) will be approximated as follows:
\[
I \approx \left\{ \int d^3r_1 \frac{1}{|r-r_1|} V(r_i) \right\} \frac{1}{|r-r_2|} .
\] (A·7)

Applying this approximation to the multiple integrations in Eq. (A·5), we obtain
\[
\phi_f^{(+)}(r) \approx -\sum_{n=0}^{\infty} \left( -\frac{m}{2\pi} \right)^{n+1} \left\{ \int d^3r' \frac{1}{|r-r'|} V(r') \right\}^{n+1} \phi_0^{(+)}(r)
\]
\[
= -\frac{\psi(r)}{1-\psi(r)} \phi_0^{(+)}(r) ,
\] (A·8)
in which \(\psi(r)\) is defined by Eq. (3·27).

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