DISTANCE IN AN EXPANDING UNIVERSE.

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(Communicated by W. H. McCrea)

1. Introduction.—The distance that an observer in space-time of general relativity assigns to a star was discussed in a previous paper.* Prof. E. A. Milne kindly pointed out to me recently that a result in this paper does not agree with his formula for the apparent brightness of a receding nebulæ, and I find that this is due to the fact that the assumed definition of distance does not give a true account of the astronomical method of calculating distance.

In the present paper is the corrected definition, giving the distance calculated from a measurement of the apparent brightness of the star, and it is found to depend on the motions of both the star and the observer. This distance is therefore unsuitable when we are dealing with such problems as the relation between distance and recessional velocity, for it is here required that for a given observer two stars at the same event should give the same distance.

It has been shown † that the distance calculated from measurements of apparent size satisfies this condition. We shall see that this distance (Δ) bears a simple relationship with the distance (Δ′) given by apparent brightness, and it is therefore convenient and advisable to reduce all observed distances Δ′ to the form Δ.

The present paper is mainly concerned with finding an expression for Δ in a quite general form of expanding universe. As soon as we know the exact expression for distance in terms of co-ordinates, we can consider a possible extension of Hubble's Law, and it will be seen that this may lead to a method for determining the cosmical constant and the sign of curvature in the generally accepted form of space-time.

2. Distance from Apparent Brightness.—Let $P_0$ be the position of the observer, $B$, when he receives light emitted by a star, $A$, at the position $P_1$, the points $P_0$, $P_1$ necessarily lying on a null geodesic, $C$, in order that light may pass from $P_1$ to $P_0$. When the observer measures the apparent brightness of the star, he actually measures the amount of energy falling on his photographic plate. We must therefore consider a thin pencil of null geodesics issuing from the star (assumed to be a point source) and meeting the observer's instantaneous space in the two-dimension cross-section defined by the size of his exposed plate. Then the amount of energy received per second by the observer is to be found from the rate at which energy is radiated by the star into the solid angle determined by the above pencil.

† I. M. H. Etherington, loc. cit.
Let \( D = 1 + \delta \lambda / \lambda \) be a measure of the Doppler effect, where \( \lambda \) is a wavelength. Then if \( N_\nu \) is the number of light-quanta, or photons, of frequency \( \nu \) radiated per second into the above solid angle according to some observer at rest relative to the star, the rate at which these same photons are received by the observer is \( N_\nu / D \). Also, as the observed energy of a photon of observed frequency \( \nu \) is \( h \nu \), the energy received by the observer from a photon radiated by the star at frequency \( \nu \) is \( h \nu / D \). Hence, the total energy received per second by the observer is

\[
E = (\Sigma h \nu N_\nu) / D^2. \tag{1}
\]

The observer therefore measures the intensity as \( E / V \) where \( V \) is the area of his receiving plate, and he defines the distance, \( \Delta' \), to be such that \( \Delta'^2 E / V \) is the specific intensity of radiation.

It is assumed that the observer can measure the specific intensity by some method, and so find \( \Delta' \). To see what this intensity is, we must consider an observer \( B' \) at rest relative to the star and sufficiently near the star for him to measure the distance directly. For convenience, let us place this observer so that his world-line intersects the null geodesic \( C \) at a point \( P' \). Then if \( V' \) is the area of cross-section of the above pencil of null geodesics made by the instantaneous space of \( B' \) at \( P' \), this observer receives an amount of energy \( \Sigma h \nu N_\nu \) on an area \( V' \), and if he measures directly the distance, \( \delta \), of the star, the specific intensity of radiation is given by

\[
I = (\Sigma h \nu N_\nu) \delta^2 / V'. \tag{2}
\]

Hence, for \( \Delta' \), we have

\[
I = (\Sigma h \nu N_\nu) \delta^2 / V' = (\Sigma h \nu N_\nu) \Delta'^2 / V D^2,
\]

i.e.

\[
\Delta' = D \delta \sqrt{V / V'}, \tag{3}
\]

where, in evaluating this expression, we can take the limit \( \delta \to 0 \). Omitting the factor \( D \), we get the result obtained by Etherington, which is equivalent to that found by the author in the previous paper.

Etherington has shown that the expression for distance found from the apparent size is equivalent to *

\[
\Delta = \frac{\delta}{D} \sqrt{\frac{V}{V'}} \tag{4}
\]

in the above notation, so we have the simple relation

\[
\Delta' = D^2 \Delta \tag{5}
\]

connecting the two distances.

3. Calculation of Distance.—We shall use for convenience some results of the previous paper with a slight change of notation. Points of the null geodesic \( C \) joining \( P_1, P_0 \) have the co-ordinates \( x^i (i = 0, 1, 2, 3) \) as functions of an affine parameter \( s \) having the values \( s_0, s_1 \) at \( P_0, P_1 \) respectively, and the

* We are not using Etherington's notation in the present paper. He wrote \( \Theta \) for \( V \), \( \theta \) for \( V' \) and \( \Delta' \) for the distance from apparent size.
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tangent vector $dx^i/ds$ at points of $C$ is written $\lambda^i$. A unit vector $\lambda_{1i}$ orthogonal to $\lambda^i$ is given at points of $C$ by parallel transport along $C$, and another unit vector $\lambda_{2i}$ is taken to be orthogonal to and independent of $\lambda^i$ and $\lambda_{1i}$. The components of these vectors are functions of $s$, and if $R_{hik}$ are the components at points of $C$ of the curvature tensor, we can construct invariants

$$\Gamma_\rho^p = \Gamma_\rho^q = - R_{hikpq} \lambda^h \lambda^i \lambda_{1q} \lambda_{2i}, \; (p, q = 1, 2)$$

which are therefore functions of $s$.

Consider now the equations

$$\frac{d^2z^p}{ds^2} - \Gamma_\rho^p z^q = 0 \; (\rho = 1, 2)$$

giving $z^1, z^2$ in terms of $s$, and choose two of the four arbitrary constants so that the $z$'s vanish when $s = s_1$. Then we can write

$$z^p = \phi^p_0 \alpha^q \; (\rho = 1, 2)$$

where $\alpha^1, \alpha^2$ are arbitrary constants and the $\phi$'s are functions of $s$ and $s_1$, vanishing at $s_1$. Forming the function $\Theta$ of $s_0, s_1$, where

$$\Theta^2 = K \mid \phi_0 \mid = s_{s_0}$$

and

$$\frac{1}{K} = \lim_{s \to s_1} \frac{\mid \phi_0 \mid}{(s - s_1)^2}$$

it can be shown that $\Theta$ depends only on the positions $P_0, P_1$, and the choice of the parameter $s$.

If $A^i$ is the unit vector tangent to the star's world-line at $P_1$, and if $g_{ij}$ is the fundamental tensor of the space-time, it was shown that, in the notation of § 2,

$$\lim_{s \to 0} \sqrt{\frac{V}{V'}} = \Theta(g_{ij} A^i \lambda^j)_{s = s_1}.$$  

Now it has been shown * that if $B^i$ is the unit vector tangent to the observer's world-line at $P_0$, the Doppler effect, $D$, is given by

$$D = (g_{ij} A^i \lambda^j)_{s = s_1}.$$  

Hence, writing †

$$\mu = (g_{ij} B^i \lambda^j)_{s = s_0},$$

we have, from (4),

$$\Delta = \mu \Theta.$$  

This, then, is the distance calculated from the apparent size of the star. It depends only on the positions $P_0, P_1$ and the vector $B^i$ at $P_0$, the product $\mu \Theta$ being independent of the choice of $s$. The other distance, $\Delta'$, can now

† It is assumed that $s$ is chosen to increase in passing along $C$ from $P_1$ to $P_0$, in which case $\mu$ is positive.
be found from (5), (11) and (3), and we see that it depends on both the vectors $A^i$ and $B^i$, i.e. depends on the motions of both star and observer.

4. Distance and Luminosity in Special Relativity.—A result found in the previous paper and given at once by the above method is that in space of constant curvature, and in particular in space of special relativity, the distance invariant, $\Theta$, is

$$\Theta = s_0 - s_1.$$  \hspace{1cm} (14)

Consider the case in special relativity when the star is moving away from the observer with velocity $V$, the velocity of light being unity. Then, in the observer’s frame of reference, the metric is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2,$$  \hspace{1cm} (15)

where $t$ is the observer’s proper time, and the world-line of the star is given by

$$t = s_1 + \sigma \cosh a, \quad x = - s_1 + \sigma \sinh a, \quad y = z = 0,$$

where $\sigma$ is the arc and $\tanh a = V$. When the observer is at the origin $t = 0$, he receives light emitted by the star at the position $\sigma = 0$, and the path of this light is the null geodesic

$$t = s, \quad x = - s, \quad y = z = 0.$$

Substituting in (12) and (11) the values of $g_{ij}$, $\lambda^i$, $A^i$ and $B^i$, we find

$$\mu = 1, \quad D = \cosh a + \sinh a,$$

and hence by (13), (5) and (14), as $s_0 = 0$,

$$\Delta = - s_1, \quad \Delta' = - s_1 (\cosh a + \sinh a)^2.$$  \hspace{1cm} (16)

Assuming that at some period the star was with the observer, the interval of time $t$ between that event and the event when the observer receives the above signal is

$$\tau = - s_1 (\cosh a + \sinh a)/\sinh a.$$

Hence, substituting $\tanh a = V$, (16) can be written

$$\Delta = \frac{\tau V}{1 + V}, \quad \Delta' = \frac{\tau V}{1 - V}. \hspace{1cm} (17)$$

From the definition of the distance $\Delta'$, the intensity measured by the observer is proportional to $1/\Delta'^2 = (1 - V)^2/V^2 \tau^2$, and this is in agreement with Milne’s formula for apparent brightness.\(^*\)

5. Some of the properties of conformal spaces can be advantageously applied to the calculation of distance, for it can be shown that if a space $S'$ is conformal to the space $S$, then the null geodesics of $S'$ correspond to null geodesics of $S$. If $C'$, $C$ are two corresponding null geodesics, and if the two fundamental tensors are connected by

$$g_{ij}' = f^2 g_{ij},$$  \hspace{1cm} (18)

\(^*\) E. A. Milne, Zs. für Astrophysik, 6, 95, 1933.
where $f$ is a function of the co-ordinates, it can be shown that an affine parameter $s'$ of $C'$ can be chosen so that $ds' = f^3 ds$, where $s$ is the parameter of $C$, the point $s'$ corresponding to the point $s$. With this choice of $s'$, the tangent vector of $C'$ at $s'$ is given by

$$\lambda'^{i} = f^{-3} \lambda^{i},$$

where $\lambda^{i}$ is the tangent vector of $C$ at $s$.

It can also be shown that if $s_0', s_1'$ are the points of $C'$ corresponding to the points $s_0, s_1$ of $C$, and if $\Theta', \Theta$ are the corresponding distance invariants defined as in § 3, then

$$\Theta' = f_0 f_1 \Theta,$$

where subscripts $0, 1$ denote values at the points $s_0, s_1$ respectively.

6. Distance in an Expanding Universe.—A form of space-time including many well-known forms in general relativity has the metric

$$ds^2 = \xi^2 dt'^2 - R^2 (\eta^{-2} dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2 \theta' d\phi'^2),$$

where $\xi, \eta$ are some functions of $r'$ and $R$ is a function of $t'$. Defining new co-ordinates $t, r$ by $dt = dt'/R$, $r = r'/\xi$, the form (21) becomes

$$ds^2 = f^2 (dt^2 - (\gamma^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)),$$

where $\gamma$ is a function of $r$, given by $\gamma = (\xi \eta dr/dr')^2$, and $f$ is a function of $t$ and $r$, given by $f = R \xi$. Hence, by § 5, the distance in space-time of the form (21) can be deduced from the corresponding distance in space-time with the metric

$$ds^2 = dt^2 - (\gamma^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2),$$

where $\gamma$ is a function of $r$, and if this solution is known, we can at once find the distance when the metric is given by (22), $f$ now being any function of the co-ordinates. Let us therefore consider the form (23).

This form has spherical symmetry in the "spatial" co-ordinates $r, \theta, \phi$, and from the geodesic equations we find that any geodesic must lie in a plane through the origin. There is therefore no loss of generality if we consider only those null geodesics for which $\theta = \pi/2$. The remaining geodesic equations can be solved to give

$$\frac{dt}{ds} = 1, \quad \frac{dr}{ds} = \pm \psi \sqrt{\gamma}, \quad \frac{d\phi}{ds} = \frac{h}{r^2}, \quad \psi^2 = 1 - \frac{h^2}{r^2},$$

where $h$ is a constant which vanishes if the null geodesic passes through the origin $r = 0$. We now have the components of the vector $\lambda^{i}$.

One vector orthogonal to $\lambda^{i}$ and given by parallel transport along this null geodesic is found to be

$$\lambda^{i}_1 \equiv (0, 0, 1/r, 0),$$

and a vector orthogonal to and independent of $\lambda^{i}$ and $\lambda^{i}_1$ is

$$\lambda^{i}_2 \equiv (h/r\psi, 0, 0, 1/r\psi).$$
Calculating from (23) the components of the curvature tensor, and substituting in (6), we find $\Gamma_1^1 = \Gamma_1^2 = 0$, and by using (24), the remaining invariants can be written in the form

$$
\begin{align*}
\Gamma_1^1 &= \frac{1}{r} \frac{d^2 r}{ds^2} - \frac{h^2}{r^4} \\
\Gamma_2^2 &= \frac{1}{r} \frac{d^2 r}{ds^2} + \frac{h^2}{r^4} \frac{1}{\psi} \frac{d\psi}{ds^2}
\end{align*}
$$

(27)

I. $h \neq 0$.

Substituting in (7), and writing $x^1 = r\alpha$, $x^2 = r\gamma$, we get

$$
\frac{d^2 x}{d\phi^2} + x = 0
$$

and

$$
\frac{d^2 y}{d\phi^2} = \frac{d^2 \psi}{d\phi^2} = 0.
$$

Hence, the required solutions are

$$
z^1 = \alpha^1 r \sin (\phi - \phi_1), \quad z^2 = \alpha^2 r\psi(U - U_1),
$$

(28)

where

$$
U = \int^{\psi}_{\phi_0} \frac{d\psi}{\psi^2}
$$

(29)

and subscripts $0$, $1$ denote values at the points $s_0$, $s_1$ respectively. We therefore find, from (9),

$$
\Theta = Kr_1^2 \psi_0(U_0 - U_1) \sin (\phi_0 - \phi_1),
$$

where

$$
\frac{1}{K} = \left( r^2 \psi \frac{dU}{ds} \frac{d\phi}{ds} \right)_{s = s_1},
$$

i.e., from (24),

$$
K = r_1^2 \psi_1 / h^2.
$$

Hence, we have

$$
\Theta = \frac{1}{h} r_1 \psi_1 \psi_0(U_0 - U_1) \sin (\phi_0 - \phi_1)^{1/2}.
$$

(30)

II. $h = 0$.

In this case

$$
\Gamma_1^1 = \Gamma_2^2 = \frac{1}{r} \frac{d^2 r}{ds^2}.
$$

Substituting in (7), the equations can be solved, and we find

$$
\Theta = r_1 | W_0 - W_1 |,
$$

(31)

where

$$
W = \int^{r} \frac{1}{\sqrt{\gamma r^2}} dr.
$$

(32)
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Returning now to the more general form (22), then from (19) a typical null geodesic is given by $\theta = \pi/2$ and

$$
\frac{dt}{ds} = \frac{1}{f^{2}} \quad \frac{dr}{ds} = \pm \frac{\psi \sqrt{\gamma}}{f^{2}}, \quad \frac{db}{ds} = \frac{h}{f^{2}r^{2}}, \quad \psi^{2} = 1 - \frac{h^{2}}{r^{2}}. \tag{33}
$$

The new invariant $\Theta$ for two points $s_0, s_1$ of this geodesic can be found by using (20). Hence from (30) and (13), the distance when $h = 0$ is

$$
\Delta = \frac{\mu}{h} \int_{s_0}^{s_1} r_{0} r_{1} \left[ \frac{1}{2} \left( U_{0} - U_{1} \right) \sin \left( \phi_{0} - \phi_{1} \right) \right]^{1/2}, \tag{34}
$$

where $\mu$ is given by (12) and $U$ by (29). When $h = 0$, then from (31) the distance is

$$
\Delta = \mu f_{0} f_{1} r_{0} r_{1} | W_{0} - W_{1} | \tag{35}
$$

where $W$ is given by (32).

A case of particular interest is when the observer is at the origin. Here $r_{0} = 0$, and we find in general

$$
\lim_{r \to 0} (rW) = -\frac{1}{\sqrt{\gamma}_0},
$$

where the subscript $o$ now denotes values at the origin $r = 0$. Hence, from (35), for a star at the position $r$,

$$
\Delta = \mu (f/\sqrt{\gamma})_0 f r. \tag{36}
$$

If we assume that the observer is instantaneously "at rest" with respect to the spatial co-ordinates, the vector $B^i$ is

$$
B^i \equiv (1/f_0, 0, 0, 0),
$$

and from (12), (33) and (22)

$$
\mu = 1/f_0.
$$

Hence, from (36),

$$
\Delta = \gamma_0^{-1/2} f r. \tag{37}
$$

7. The Modified Lemaître Universe.—The form for a homogeneous isotropic universe can be written *

$$
\frac{ds^2}{R^2(dt^2 - (\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)), \quad \gamma = 1 - kr^2, \tag{38}
$$

where $k = 1$, 0, or $-1$, and $R$ is a function of $t$ only. Comparing this form with (22), we see from (33) that the path of light received by an observer at the origin is given by

$$
dr/dt = -\sqrt{1 - kr^2}, \tag{39}
$$

and the distance of a star from the origin is, by (37),

$$
\Delta = Rr. \tag{40}
$$

* H. P. Robertson, Reviews of Modern Physics, 5, 62, 1932. For convenience, we have written $Rdt$ for the usual element $dt$. 

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If the star is assumed to be at rest with respect to the spatial co-ordinates, the Doppler effect can be found from (11), and we get $D = R_0/R$. Define an observable, $v$, by

$$v = D - 1 = \frac{R_0}{R} - 1.$$  (41)

Then for small values, $v$ is the apparent velocity of recession of the star.* Assuming that $R$ is known, we can by (39) and (41) expand $r$ and $R$ in terms of $v$ and substitute in (40) to obtain a relation between $\Delta$ and $v$. To the required order, we can write this relation in the form

$$v/\Delta = K_1 + K_2 v,$$  (42)

which is an extension of Hubble's Law $v/\Delta = \text{const}$. As $v$ and $\Delta$ are observables, $K_1$ and $K_2$ are observable constants. They are given in terms of $R$ by

$$K_1 = R_0'/R_0^2, \quad K_2 = (4R_0'^2 - R_0R_0')/2R_0'R_0^2,$$  (43)

where dashes denote differentiations with respect to $t$, and the subscript denotes evaluation at the time of observation, i.e. at the origin. These equations therefore give $R_0'$ and $R_0''$ in terms of $R_0$.

In addition to $K_1$ and $K_2$, there are two other observable constants $K_3$, $K_4$, given by

$$K_3 = 2\pi\delta_0, \quad K_4 = 8\pi p_0,$$  (44)

where $\rho$ is the pressure and $\delta$ the proper density. It can be shown from (38) that

$$2\pi \delta = -\lambda + 3(kR + R^3)/2R^3, \quad 8\pi \rho = \lambda - (kR^2 - R'^2 + 2RR'')/R^4,$$  (45)

where $\lambda$ is the cosmical constant at present undetermined. Evaluating at the origin, these equations with (44) give two relations between $R_0$, $R_0'$, $R_0''$ and $\lambda$.

Eliminating $R_0'$, $R_0''$ and $\lambda$ from these relations and (43), we find

$$k/2R_0^2 = K_1(K_1 - K_2) + K_3 + K_4.$$  (46)

Hence, if

$$K_1(K_1 - K_2) + K_3 + K_4 > 0, \quad k = 1, \quad \sigma = -1.$$  (47)

Thus $k$ can be determined, and if $k = 0$, $R_0$, $R_0'$ and $R_0''$ can be found from (46) and (43). Substituting in either of (45), we get

$$\lambda = 9K_1^2 - 6K_1K_2 + 2K_3 + 3K_4.$$  (48)

If $k = 0$, $R_0$ cannot be determined, but we find that $\lambda$ is still given by (48). From (47), this can now be written

$$\lambda = 3K_1^2 - 4K_3 - 3K_4.$$  (49)

* From special relativity, we should say that the actual velocity of recession, $V$, is given by $(1 + V)/\sqrt{1 - V^2} = D = 1 + v$. Thus $v$ can take any values, but $V$ cannot exceed unity.
ON THE PERIODIC ORBITS IN THE NEIGHBOURHOOD OF THE TRIANGULAR EQUILIBRIUM POINTS IN THE RESTRICTED PROBLEM OF THREE BODIES.

Peder Pedersen.

(Communicated by E. Strömgren)

Introduction.—Since Gyldén * in 1884 began the study of the infinitesimal periodic orbits around the libration points in the restricted problem of three bodies, a number of investigations of the problem have appeared. Most of these investigations are based on the variational equations that correspond to the differential equations of the problem. As E. Strömgren † has shown, the variational equations are suitable as a basis for numerical integrations. In theoretical investigations, however, essential peculiarities of the periodic orbits are often lost when the variational equations are used.

The following example may illustrate the question: Using the variational equations one finds that the period of the periodic orbits around a libration point is a function of the two finite masses, while it is independent of the size of the orbit. Using the differential equations, however, one finds that the period must depend on the size of the orbit. Due to the fact that only terms of the first order are retained in forming the variational equations, the essential property of the orbits that the period varies with the size is lost. When terms of higher order are retained in the analysis the dependence of period on size reappears.

In the present investigation I will show that the period of the periodic orbits around the libration points $L_4$ and $L_5$ appear as a function of the size of the orbits, when terms of second and third order are retained in the series expressions, limiting, however, the investigation to the case that the masses have values in the neighbourhood of the critical values $\mu_0$ and $1 - \mu_0$. It will further be shown that the critical values of the masses depend on the

† Publikationer fra Københavns Observatorium.
size of the orbits in such a way that the critical value corresponding to the small mass increases with the size of the orbit. Finally, it will be shown that classes of periodic orbits with *infinitesimal limiting orbits* exist for values of the small mass greater than the critical value $\mu_0$.

I am indebted to Professor Elis Strömgren for some valuable suggestions.

§ 1. We shall begin with fixing the units of length, mass and time. As unit of length we choose the distance between the two finite masses, as unit of mass the sum of the two finite masses, and, finally, the unit of time is determined by putting the angular velocity of the two finite masses equal to $\tau$. With this choice of units the gravitational constant reduces to 1.

It should be mentioned that these units which are convenient in theoretical investigations are not identical with the units adopted in the so-called Copenhagen problem.

We shall call the values of the two finite masses $\mu$ and $1-\mu$, assuming $\mu \leq \frac{1}{2}$. The motion of the three masses ($\mu$, $1-\mu$ and the infinitesimal mass) will be referred to a rotating co-ordinate system, the origin of which is the centre of mass of the two finite masses; the $x$-axis passes through the finite masses, the positive direction being that from the origin to $\mu$. The positive direction of revolution is chosen as the direction of the absolute motion of the two finite masses. In this co-ordinate system the co-ordinates of $\mu$ and $1-\mu$ are $(\mu, 0)$ and $(-\mu, 0)$.

The differential equations of motion of the infinitesimal mass are:

\[
\dot{x} - 2\dot{y} - x = \frac{\partial \Omega}{\partial x}, \tag{1}
\]

\[
\dot{y} + 2\dot{x} - y = \frac{\partial \Omega}{\partial y},
\]

where

\[
\Omega = \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}, \tag{2}
\]

and $\rho_1$ and $\rho_2$ are determined by the equations

\[
\rho_1^2 = (x + \mu)^2 + y^2, \quad \rho_2^2 = (x + \mu - 1)^2 + y^2. \tag{3}
\]

Fig. 1 shows the positions of the libration points when $\mu$ has the critical value $\mu_0$ ($\mu_0 = 0.03852 \ldots$; $\mu_0$ is determined by the equation $\mu_0(1-\mu_0) = \frac{1}{27}$).

$L_4$ and $L_5$ have the co-ordinates ($\frac{1}{2} - \mu$, $\frac{1}{2} \sqrt{3}$) and ($\frac{1}{2} - \mu$, $-\frac{1}{2} \sqrt{3}$).

To investigate the motion of the infinitesimal mass in the neighbourhood of the libration point $L_4$ we introduce a new co-ordinate system with its origin in $L_4$ and with axis parallel to the axis of the former co-ordinate system. We then have

\[
x = \xi + \frac{1}{2} - \mu, \quad \tag{4}
\]

\[
y = \eta + \frac{1}{2} \sqrt{3},
\]
hence the differential equations of motion of the infinitesimal mass in the new co-ordinate system are:

\[ \dot{\xi} - 2\dot{\eta} - \xi - \frac{1}{2} + \mu = \frac{\partial \Omega}{\partial \xi}, \]

\[ \dot{\eta} + 2\dot{\xi} - \eta - \frac{1}{2} \sqrt{3} = \frac{\partial \Omega}{\partial \eta}, \]  

(5)

where

\[ \Omega = \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2}, \]  

(6)

and where \( \rho_1 \) and \( \rho_2 \) are determined by the equations

\[ \rho_1^2 = (\xi + \frac{1}{2})^2 + (\eta + \frac{1}{2} \sqrt{3})^2, \]

\[ \rho_2^2 = (\xi - \frac{1}{2})^2 + (\eta + \frac{1}{2} \sqrt{3})^2. \]  

(7)

The differential equations possess the Jacobian integral

\[ \xi^2 + \eta^2 = \xi^2 + \eta^2 + (1 - 2\mu)\xi + \sqrt{3} \eta + 2\Omega - C, \]  

where \( C \) is the Jacobian constant.

FIG. I.

§ 2. The above-mentioned critical value of the mass \( \mu (\mu_0 = 0.03852 \ldots) \) has the well-known property that infinitesimal periodic orbits exist around the libration points \( L_4 \) and \( L_5 \) when \( \mu \leq \mu_0 \), while no such orbits exist for \( \mu > \mu_0 \). Infinitesimal periodic orbits around the libration points \( L_1, L_2 \) and \( L_3 \) exist for all values of \( \mu \).

The critical value \( \mu_0 \) is, however, determined, not from the differential equations (1) or (5), but from variational equations corresponding to (5) and obtained from (5) by retaining terms of the first order only. The
critical value $\mu_0$ can therefore be said to correspond to the variational equations, and as the variational equations are only a first approximation to the differential equations, similarly $\mu_0$ is only a first approximation to the critical mass corresponding to (5).

In the following we shall show how the critical value of the mass can be determined as a function of the size of the periodic orbit by retaining terms up to the third order in the series expressions. Only in the limiting case that the dimension of the orbit is zero the critical mass has the value $\mu_0$.

§ 3. Now let us assume that the infinitesimal mass, the motion of which we want to examine, is situated in the immediate neighbourhood of the libration point $L_4$. We shall assume its co-ordinates $\xi$ and $\eta$ to be small quantities of the first order, and similarly we shall assume that $\dot{\xi}$ and $\dot{\eta}$ are small quantities of the first order. With the aid of the differential equations (5) we then see at once that $\dot{\xi}$ and $\dot{\eta}$ are also small quantities of the first order.

We now expand the two terms on the right-hand sides of the equations (5), $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$, in series and retain terms up to the third order. In the expansion of $\Omega$ it will then be necessary to retain terms up to the fourth order, as the order of the terms is diminished by 1 by the process of differentiation.

We start with expanding $\frac{1}{\rho_1}$ and $\frac{1}{\rho_2}$, then find the series expression for $\Omega$ and finally for $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$.

Equations (7) can be written in the following form:

\[ \rho_1^2 = 1 + \dot{\xi} + \sqrt{3} \eta + \xi^2 + \eta^2, \]
\[ \rho_2^2 = 1 - \dot{\xi} + \sqrt{3} \eta + \xi^2 + \eta^2. \]  

(9)

From these equations we get

\[ \frac{1}{\rho_1} = 1 - \frac{1}{2} \dot{\xi} - \frac{1}{2} \sqrt{3} \eta - \frac{1}{8} \xi^2 - \frac{1}{2} \sqrt{3} \xi \eta + \frac{1}{8} \eta^2 \]
\[ + \frac{3}{7} \xi^2 \cdot \eta - \frac{3}{8} \sqrt{3} \xi^2 \eta - \frac{3}{8} \xi \eta^2 - \frac{3}{8} \sqrt{3} \eta^3 \]
\[ - \frac{27}{32} \xi^2 + \frac{3}{8} \sqrt{3} \xi^2 \eta + \frac{1}{8} \xi \eta^2 + \frac{3}{8} \xi \eta^2 + \frac{1}{8} \xi^2 \eta^2 + \frac{1}{8} \xi \eta^2 + \frac{1}{8} \xi^2 \eta^2 + \frac{1}{8} \xi \eta^2. \]

(10)

In these equations we get

\[ \frac{1}{\rho_2} = 1 + \frac{1}{2} \dot{\xi} - \frac{1}{2} \sqrt{3} \eta - \frac{1}{8} \xi^2 - \frac{1}{2} \sqrt{3} \xi \eta + \frac{1}{8} \eta^2 \]
\[ - \frac{3}{7} \xi^2 - \frac{3}{8} \sqrt{3} \xi^2 \eta + \frac{3}{8} \xi \eta^2 - \frac{3}{8} \sqrt{3} \eta^3 \]
\[ - \frac{27}{32} \xi^2 + \frac{3}{8} \sqrt{3} \xi^2 \eta + \frac{1}{8} \xi \eta^2 + \frac{3}{8} \xi \eta^2 + \frac{1}{8} \xi^2 \eta^2 + \frac{1}{8} \xi \eta^2 + \frac{1}{8} \xi^2 \eta^2 + \frac{1}{8} \xi \eta^2. \]

Inserting in (6) we next find the expansion of $\Omega$:

\[ \Omega = 1 - \frac{1}{2}(1 - 2\mu)\dot{\xi} - \frac{1}{2} \sqrt{3} \eta \]
\[ - \frac{1}{8} \xi^2 + \frac{3}{8} \sqrt{3} (1 - 2\mu) \xi \eta + \frac{1}{8} \eta^2 \]
\[ + \frac{3}{7} (1 - 2\mu) \xi^2 - \frac{3}{8} \sqrt{3} \xi^2 \eta - \frac{3}{8} (1 - 2\mu) \xi \eta^2 - \frac{3}{8} \sqrt{3} \eta^3 \]
\[ - \frac{27}{32} \xi^2 + \frac{3}{8} \sqrt{3} (1 - 2\mu) \xi^2 \eta + \frac{1}{8} \xi \eta^2 + \frac{3}{8} \sqrt{3} (1 - 2\mu) \xi \eta^2 + \frac{1}{8} \xi \eta^2. \]

(11)
From (11) we finally obtain the expansions of $\frac{\partial \Omega}{\partial \xi}$ and $\frac{\partial \Omega}{\partial \eta}$. These are then inserted in (5) and the following differential equations, on which the following analysis is based, are obtained:

$$\begin{align*}
\xi' - 2\eta' - 3\xi - \frac{3}{4}\sqrt{3}(1 - 2\mu)\eta \\
- \frac{1}{3} \frac{1}{6} (1 - 2\mu) \xi^2 + \frac{3}{8} \sqrt{3} \xi \eta + \frac{3}{32} (1 - 2\mu) \eta^2 \\
+ \frac{3}{32} \xi \eta + \frac{3}{32} \sqrt{3} (1 - 2\mu) \xi^2 \eta - \frac{15}{32} \xi^2 \eta^2 - \frac{9}{32} \sqrt{3} (1 - 2\mu) \eta^3 = 0,
\end{align*}$$

$$\begin{align*}
\eta' + 2\xi' - \frac{3}{4} \sqrt{3} (1 - 2\mu) \xi' - \frac{9}{8} \eta \\
+ \frac{3}{32} \sqrt{3} \xi^2 + \frac{3}{32} (1 - 2\mu) \xi \eta + \frac{3}{32} \sqrt{3} \eta^2 \\
+ \frac{3}{32} \sqrt{3} (1 - 2\mu) \xi^2 \eta - \frac{15}{32} \xi^2 \eta^2 - \frac{9}{32} \sqrt{3} (1 - 2\mu) \xi \eta^3 - \frac{9}{32} \eta^3 = 0.
\end{align*}$$

The Jacobian integral corresponding to (12) is obtained from (8):

$$\begin{align*}
\xi^2 + \eta^2 = 2 + \frac{3}{8} \xi^2 + \frac{3}{8} \sqrt{3} (1 - 2\mu) \xi \eta + \frac{9}{8} \eta^2 \\
+ \frac{3}{8} (1 - 2\mu) \xi^3 - \frac{3}{8} \sqrt{3} \xi^2 \eta - \frac{3}{32} (1 - 2\mu) \xi^2 \eta^2 - \frac{3}{8} \sqrt{3} \eta^3 \\
- \frac{3}{8} \xi^4 - \frac{3}{8} \sqrt{3} (1 - 2\mu) \xi^2 \eta^2 - \frac{3}{32} \xi^2 \eta^3 - \frac{3}{32} \sqrt{3} (1 - 2\mu) \xi \eta^2 + \frac{3}{8} \eta^4 - C,
\end{align*}$$

where the Jacobian constant has the same value as in (8).

§ 4. Finding the periodic orbits around $L_4$ is equivalent to determining the coefficients $a$ and $b$ in the Fourier expansions of $\xi$ and $\eta$:

$$\begin{align*}
\xi = a_0 + a_1 \cos \omega t + a_{-1} \sin \omega t + a_2 \cos 2\omega t + a_{-2} \sin 2\omega t \\
+ a_3 \cos 3\omega t + a_{-3} \sin 3\omega t,
\end{align*}$$

$$\begin{align*}
\eta = b_0 + b_1 \cos \omega t + b_{-1} \sin \omega t + b_2 \cos 2\omega t + b_{-2} \sin 2\omega t \\
+ b_3 \cos 3\omega t + b_{-3} \sin 3\omega t.
\end{align*}$$

The roman numbers denote the order of the coefficients. An examination of the variational equations shows that $a_1$, $a_{-1}$, $b_1$ and $b_{-1}$ are of the first order, while the other coefficients are of higher order. We assume that the coefficients with the indices 0 and $\pm 2$ are of the second order, that the coefficients with indices $\pm 3$ are of the third order and that the coefficients with indices numerically greater than 3 are of higher order than the third. As only terms up to the third order are retained in the expansions for $\xi$ and $\eta$, we break off the Fourier expansion after the first seven terms. We then have to determine seven coefficients $a$ and seven coefficients $b$ by the condition that $\xi$ and $\eta$ given by expressions (14) shall satisfy (12).

The first step is forming the Fourier expansions of the quantities of the first order $\xi$, $\eta$, $\xi'$ and $\eta'$, of the quantities of the second order $\xi^2$, $\xi \eta$ and $\eta^2$, and of the quantities of the third order $\xi^3$, $\xi^2 \eta$, $\xi \eta^2$ and $\eta^3$; terms up to the third order are retained. For greater clearness the Fourier coefficients are arranged in schematic form.

The next step is the determination of the four coefficients $a_1$, $a_{-1}$, $b_1$
## Coefficient Scheme

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<td>$\omega a_1$</td>
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and $b_{-1}$, terms of the first order only being retained. In the differential equations

$$
\xi - 2\eta - \frac{3}{2} \xi - \frac{1}{2} \sqrt{3}(1 - 2\mu) \eta = 0,
$$

$$
\eta + 2\xi - \frac{1}{2} \sqrt{3}(1 - 2\mu) \xi - \frac{3}{2} \eta = 0,
$$

which are obtained from (12) when only terms of the first order are retained, we insert

$$
\xi = a_1 \cos \omega t + a_{-1} \sin \omega t,
$$

$$
\eta = b_1 \cos \omega t + b_{-1} \sin \omega t.
$$

Equating the coefficients of $\cos \omega t$ and $\sin \omega t$ in the two equations (15) to zero, we get the following system of four equations:

$$
\begin{align*}
(\omega^2 + \frac{3}{4}) \cdot a_1 + \frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot b_1 + 2\omega \cdot b_{-1} &= 0, \\
\frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot a_1 + (\omega^2 + \frac{3}{4}) \cdot b_1 - 2\omega \cdot a_{-1} &= 0, \\
-2\omega \cdot b_1 + (\omega^2 + \frac{3}{4}) \cdot a_{-1} + \frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot b_{-1} &= 0, \\
2\omega \cdot a_1 + \frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot a_{-1} + (\omega^2 + \frac{3}{4}) \cdot b_{-1} &= 0.
\end{align*}
$$

From the two first equations (17) we can express $a_{-1}$ and $b_{-1}$ in terms of $a_1$ and $b_1$:

$$
\begin{align*}
2\omega \cdot a_{-1} &= \frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot a_1 + (\omega^2 + \frac{3}{4}) \cdot b_1, \\
2\omega \cdot b_{-1} &= - (\omega^2 + \frac{3}{4}) \cdot a_1 - \frac{3}{2} \sqrt{3}(1 - 2\mu) \cdot b_1.
\end{align*}
$$

Inserting these expressions for $a_{-1}$ and $b_{-1}$ into the two last equations (17), we get two equations which determine $a_1$ and $b_1$:

$$
\begin{align*}
[\omega^4 - \omega^2 + \frac{3}{4} - \frac{3}{2} \sqrt{3}(1 - 2\mu)^2] \cdot a_1 &= 0, \\
[\omega^4 - \omega^2 + \frac{3}{4} - \frac{3}{2} \sqrt{3}(1 - 2\mu)^2] \cdot b_1 &= 0.
\end{align*}
$$

If now

$$
\omega^4 - \omega^2 + \frac{3}{4} = \frac{3}{2} \sqrt{3}(1 - 2\mu)^2 > 0,
$$

then we have

$$
a_1 = b_1 = a_{-1} = b_{-1} = 0.
$$

The mathematical condition for infinitesimal periodic orbits therefore is

$$
\omega^4 - \omega^2 + \frac{3}{4} = \frac{3}{2} \sqrt{3}(1 - 2\mu)^2 = 0.
$$

A further condition is that $\omega^2$ should be real and positive.

From (22) we get

$$
\omega^2 = \frac{1}{2} \pm \sqrt{1 - 27\mu(1 - \mu)}.
$$

The condition for $\omega^2$ being real is

$$
\mu(1 - \mu) \leq \frac{1}{27}
$$

or the product of the two finite masses should be less than or equal to $\frac{1}{27}$.

The value of the small mass that correspond to the mass-product $\frac{1}{27}$ has already been introduced and denoted by $\mu_0$. Hence we have

$$
\mu_0(1 - \mu_0) = \frac{1}{27}.
$$

We now assume that the values of the finite masses satisfy (24). From (22) we can then determine $\omega$ in such a way that equations (19) are satisfied.
for all values of $a_1$ and $b_1$, the last-mentioned quantities being therefore arbitrary constants of integration. Having chosen $a_1$ and $b_1$ we can find $a_{-1}$ and $b_{-1}$ from equations (18).

It is easily seen, however, that one of the arbitrary constants is of no importance when the form of the periodic orbit only is of interest. The two equations:

$$
\xi = a_1 \cos \omega (t - t_0) + a_{-1} \sin \omega (t - t_0), \\
\eta = b_1 \cos \omega (t - t_0) + b_{-1} \sin \omega (t - t_0),
$$

(26)

correspond to the same motion as the two equations (16), but rearranging the expressions on the right-hand sides of (26) we immediately see that other coefficients of $\cos \omega t$ and $\sin \omega t$ are obtained. The arbitrary constant $t_0$, which may vary without changing the orbital motion, we shall call the phase-constant. It is possible to choose the phase-constant in such a way that the coefficient of $\cos \omega t$ in the equation

$$
\eta = b_1 \cos \omega (t - t_0) + b_{-1} \sin \omega (t - t_0)
$$

(27)

vanishes; $t_0$ then has to satisfy the equation

$$
b_1 \cos \omega t_0 - b_{-1} \sin \omega t_0 = 0.
$$

(28)

In other words, we can choose the phase-constant in such a way that no terms with $\cos \omega t$ appear in the expansion of $\eta$.

We now return to the two equations (16) in which we have considered $a_1$ and $b_1$ to be arbitrary constants. Let us assume that the phase-constant has been so fixed that

$$
b_1 = 0.
$$

(29)

In the equations there remains the arbitrary constant $a_1$, which we shall call the orbit-constant. The quantities $a_{-1}$ and $b_{-1}$ can be found expressed by $a_1$ with the aid of (18). As $\xi$ and $\eta$ have been assumed to be small quantities of the first order, $a_1$, too, must be of the first order. In the following we shall denote $a_1$ by $\epsilon$:

$$
a_1 = \epsilon,
$$

(30)

where $\epsilon$ is a small quantity of the first order.

Introducing (29) and (30) in (18) we get two equations for determining $a_{-1}$ and $b_{-1}$:

$$
2\omega \cdot a_{-1} = \frac{3}{4} \sqrt{3} (1 - 2\mu) \epsilon, \\
2\omega \cdot b_{-1} = - (\omega^2 + \frac{3}{4}) \epsilon.
$$

(31)

We can summarize the above discussion as follows: When

$$
\mu (1 - \mu) \leq \sqrt{3},
$$

infinitesimal periodic orbits around $L_4$ exist, corresponding to the following equations:

$$
\xi = \epsilon \cos \omega t + \frac{3}{8\omega} \sqrt{3} (1 - 2\mu) \epsilon \sin \omega t, \\
\eta = - \left( \frac{1}{8\omega} + \frac{3}{8\omega} \right) \epsilon \sin \omega t.
$$

(32)
§ 5. We shall now give a short account of the method by which the other Fourier coefficients in the Fourier expressions for $\xi$ and $\eta$ are determined.

Using the values of $a_1$, $a_{-1}$, $b_1$ and $b_{-1}$ just found:

\begin{align*}
a_1 &= \epsilon, \quad a_{-1} = \frac{3}{8\omega} \sqrt{3(1 - 2\mu)} \epsilon, \\
b_1 &= 0, \quad b_{-1} = -\left(\frac{1}{2\omega} + \frac{3}{8\omega}\right) \epsilon,
\end{align*}

we can compute the coefficients of the second order in the first column of the coefficient scheme corresponding to $\xi^2$, $\xi \eta$ and $\eta^2$. Introducing now the values found, together with $a_0$ and $b_0$ in the differential equations (12), we can find the constant terms in the Fourier expressions (12a) and (12b). Putting these terms equal to zero two equations of the first order are found for the determination of $a_0$ and $b_0$. As the determinant corresponding to these equations does not vanish, $a_0$ and $b_0$ are determined as functions of $\epsilon$.

In a similar way it is possible to compute the coefficients of the second order in the coefficient scheme corresponding to $\cos 2\omega t$ and $\sin 2\omega t$ and $\xi^2$, $\xi \eta$ and $\eta^2$. Introducing the values thus found, together with $a_2$, $a_{-2}$, $b_2$ and $b_{-2}$ in the differential equations (12), we can determine the coefficients of $\cos 2\omega t$ and $\sin 2\omega t$ in the Fourier expansions of (12a) and (12b).

Equating these four coefficients to zero, we obtain four equations of the first order for the determination of $a_2$, $a_{-2}$, $b_2$ and $b_{-2}$. As the determinant corresponding to these four equations does not vanish, $a_2$, $a_{-2}$, $b_2$ and $b_{-2}$ can be found as functions of $\epsilon$.

With the values of $a_0$, $b_0$, $a_2$, $a_{-2}$, $b_2$ and $b_{-2}$ thus found we can compute the coefficients of the third order in the coefficient scheme corresponding to $\cos \omega t$ and $\sin \omega t$ and $\xi^3$, $\xi \eta$, $\eta^3$, $\xi^2 \eta$, $\xi \eta^2$ and $\eta^3$. Introducing in the differential equations the values found together with $a_1$, $a_{-1}$, $b_1$ and $b_{-1}$ corresponding to $\xi$, $\eta$, $\xi$, $\eta$, $\xi$ and $\eta$ and considered as unknown, we can determine the coefficients of $\cos \omega t$ and $\sin \omega t$ in the Fourier expressions of (12a) and (12b). Equating these four coefficients to zero, we get for the more exact determination of $a_1$, $a_{-1}$, $b_1$ and $b_{-1}$ four equations of the form:

\begin{align*}
-\left(\omega^2 + \frac{3}{4}\right) \cdot a_1 - \frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot b_1 - 2\omega \cdot b_{-1} + q_1 & = 0, \\
-\frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot a_{-1} - \left(\omega^2 + \frac{3}{4}\right) \cdot b_{-1} + 2\omega \cdot a_1 + q_2 & = 0, \\
2\omega \cdot b_{-1} - \left(\omega^2 + \frac{3}{4}\right) \cdot a_{-1} - \frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot b_{-1} + q_3 & = 0, \\
-2\omega \cdot a_1 - \frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot a_{-1} - \left(\omega^2 + \frac{3}{4}\right) \cdot b_{-1} + q_4 & = 0,
\end{align*}

where $q_1$, $q_2$, $q_3$ and $q_4$ are known functions of $\epsilon$, which are small quantities of the second order.

From the first two equations (34) we find $a_{-1}$ and $b_{-1}$ in terms of $a_1$ and $b_1$:

\begin{align*}
2\omega \cdot a_{-1} & = \frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot a_1 + \left(\omega^2 + \frac{3}{4}\right) \cdot b_1 - q_2 \cdot \epsilon, \\
2\omega \cdot b_{-1} & = \left(\omega^2 + \frac{3}{4}\right) \cdot a_1 - \frac{3}{4} \sqrt{3(1 - 2\mu)} \cdot b_1 + q_1 \cdot \epsilon.
\end{align*}
Inserting these values of $a_{-1}$ and $b_{-1}$ in the last two equations (34), we get the two equations:

\[
\begin{align*}
\omega^4 - \omega^2 + \frac{1}{4} \beta - \frac{1}{4} \beta (1 - 2\mu)^2 & = a_1 = \left[ (\omega^2 + \frac{1}{4}) q_1 - 2\omega q_4 - \frac{3}{4} \sqrt{3} (1 - 2\mu) q_2 \right], \\
\omega^4 - \omega^2 + \frac{1}{4} \beta - \frac{1}{4} \beta (1 - 2\mu)^2 & = b_1 = \left[ (\omega^2 + \frac{1}{4}) q_2 + 2\omega q_3 - \frac{3}{4} \sqrt{3} (1 - 2\mu) q_1 \right].
\end{align*}
\]  

(36)

Introducing now in the equations (36) the values already chosen for the orbit-constant $a_1$ and the phase-constant $b_1$:

\[
a_1 = \epsilon, \quad b_1 = \phi,
\]

(37)

we get the two equations:

\[
\begin{align*}
\omega^4 - \omega^2 + \frac{1}{4} \beta - \frac{1}{4} \beta (1 - 2\mu)^2 & = (\omega^2 + \frac{1}{4}) q_1 + \frac{3}{4} \sqrt{3} (1 - 2\mu) q_2 + 2\omega q_4 = 0, \\
(\omega^2 + \frac{1}{4}) q_2 + 2\omega q_3 - \frac{3}{4} \sqrt{3} (1 - 2\mu) q_1 & = 0,
\end{align*}
\]

(38) \hspace{1cm} (39)

where (38) corresponds to (22). In what follows we shall consider equation (38), reverting to equation (39) later on.

We now limit the analysis by only considering values of $\mu$ in the neighbourhood of the critical value $\mu_0$, the smaller root of the equation:

\[
\mu_0 (1 - \mu_0) = \frac{1}{2} \gamma.
\]

(40)

The corresponding value of $\omega_0$, determined with the aid of the variational equations, is found from (23):

\[
\omega_0^2 = \frac{1}{2}, \quad \omega_0 = \frac{1}{2} \sqrt{2}.
\]

(41)

In the computations that follow we can exclude the negative value of $\omega_0$.

From (40) we find

\[
\mu_0 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{5}{3}}.
\]

(42)

The difference between the great mass $1 - \mu_0$ and the small mass $\mu_0$ is

\[
1 - 2\mu_0 = \sqrt{\frac{5}{3}}.
\]

(43)

For brevity we introduce the quantity $k$:

\[
k = \sqrt{\frac{5}{3}}.
\]

(44)

As we have assumed that $\mu$ lies in the neighbourhood of $\mu_0$, or that the mass-product lies in the neighbourhood of $\frac{1}{2} \gamma$, we can put

\[
\mu (1 - \mu) = \frac{1}{2} \gamma + \delta,
\]

(45)

where $\delta$ is infinitely small.

We now revert to equation (38). Introducing (45) in (38), we get

\[
(\omega^2 - \frac{1}{2})^2 + \frac{1}{4} \gamma \delta - (\omega^2 + \frac{1}{4}) q_1 + \frac{3}{4} \sqrt{3} (1 - 2\mu) q_2 + 2\omega q_4 = 0.
\]

(46)

The last three terms in equation (46) are small quantities of the second order. As only terms of the second order are retained in this equation, we can substitute $\omega_0$ and $\mu_0$ for $\omega$ and $\mu$ in these terms. We therefore introduce in the last terms of (46)

\[
\omega = \omega_0 = \frac{1}{2} \sqrt{2}, \quad 1 - 2\mu = 1 - 2\mu_0 = k,
\]

(47)
and we get the new equation
\[(\omega^2 - \frac{1}{3})^2 + \frac{2\pi}{4} \delta - \frac{11}{4} q_1 + \frac{3}{8} \sqrt{3kq_2} + \sqrt{2q_4} = 0. \tag{48}\]

The condition for real values of \(\omega\) and therefore the condition for
infinitesimal periodic orbits around \(L_4\) is
\[27\delta \leq 11q_1 - 3\sqrt{3kq_2} - 4\sqrt{2q_4}, \tag{49}\]
and we have thus found the upper limit of the mass-product
\[\frac{1}{11}[1 + 11q_1 - 3\sqrt{3kq_2} - 4\sqrt{2q_4}]. \tag{50}\]

§ 6. We now turn to the computation of the quantities \(q_1, q_2, q_3\) and
\(q_4\), which are functions of \(\mu\) and \(\epsilon\) of the following form:
\[q_n = f_n(\mu)e^2, \quad (n = 1, 2, 3, 4). \tag{51}\]

As in this equation only terms of the second order are retained, we can
in the computation of \(f_n(\mu)\) substitute \(\mu_0\) and \(\omega_0\) for \(\mu\) and \(\omega\). The
computations are based on equations (33), which with \(\mu = \mu_0\) and \(\omega = \omega_0\) become
\[
\begin{align*}
a_1 &= 2e, \\
b_1 &= 0, \\
a_{-1} &= \frac{8}{3} \sqrt{6k\epsilon}, \\
b_{-1} &= -\frac{8}{3} \sqrt{2\epsilon}.
\end{align*} \tag{52}
\]

With the values given by (52) we compute, as already explained, the
coefficients of the second order in the first column of the coefficient scheme
corresponding to \(\xi^2\), \(\xi\eta\) and \(\eta^2\). The results are as follows:
\[
\begin{array}{|c|c|}
\hline
\xi^2 & \frac{5}{9} \xi^2 \\
\xi\eta & -\frac{5}{6} \sqrt{3k\epsilon^2} \\
\eta^2 & \frac{8}{3} \xi^2 \\
\hline
\end{array} \tag{53}
\]

Introducing these values in the differential equations (12), we get for the
determination of \(a_0\) and \(b_0\) the two equations:
\[
\begin{align*}
\frac{3}{4} a_0 + \frac{3}{4} \sqrt{3k} b_0 &= -\frac{7}{3} \frac{5}{9} k \epsilon^2, \\
\frac{3}{4} \sqrt{3k} a_0 + \frac{3}{4} b_0 &= -\frac{5}{9} \frac{8}{3} \sqrt{3} \epsilon^2,
\end{align*} \tag{54}
\]
which have the following roots:
\[
\begin{align*}
a_0 &= -\frac{1}{3} \frac{8}{3} \frac{5}{9} k \epsilon^2, \\
b_0 &= \frac{8}{3} \frac{8}{3} \frac{5}{9} \sqrt{3} \epsilon^2. \tag{55}
\end{align*}
\]

Next we compute the coefficients of the second order in the coefficient
scheme corresponding to \(\cos 2\omega t\) and \(\sin 2\omega t\) and \(\xi^2, \xi\eta\) and \(\eta^2\).
The results are shown below:
\[
\begin{array}{|c|c|c|}
\hline
\xi^2 & \cos 2\omega t & \sin 2\omega t \\
\xi\eta & \frac{8}{3} \xi^2 & \frac{3}{9} \sqrt{6k} \epsilon^2 \\
\eta^2 & \frac{1}{6} \sqrt{3k\epsilon^2} & -\frac{8}{3} \sqrt{2\epsilon^2} \\
\hline
\end{array} \tag{56}
\]
Introducing these values in the differential equations (12), we get for the determination of \(a_2, a_{-2}, b_2\) and \(b_{-2}\) the four equations:

\[
\begin{align*}
\frac{11}{4}a_2 - \frac{3}{4}\sqrt{3}k b_2 + 2\sqrt{2}b_{-2} &= -\frac{9}{16}ke^2, \\
\frac{3}{4}\sqrt{3}ka_2 + \frac{1}{4}b_2 - 2\sqrt{2}a_{-2} &= \frac{1}{8}\sqrt{3}e^2, \\
-2\sqrt{2}b_2 + \frac{11}{4}a_{-2} + \frac{3}{4}\sqrt{3}k b_{-2} &= -\frac{10}{11}e^2, \\
2\sqrt{2}a_2 + \frac{3}{4}\sqrt{3}ka_{-2} + \frac{1}{4}b_{-2} &= -\frac{9}{8}\sqrt{2}ke^2,
\end{align*}
\]  

(57)

which have the roots:

\[
\begin{align*}
a_2 &= -\frac{11}{32}ke^2, & a_{-2} &= -\frac{11}{16}\sqrt{6}e^2, \\
b_2 &= -\frac{3}{16}\sqrt{3}e^2, & b_{-2} &= -\frac{3}{8}\sqrt{2}ke^2.
\end{align*}
\]  

(58)

With these values of \(a_0, b_0, a_2, a_{-2}, b_2\) and \(b_{-2}\) we can compute the coefficients of the third order in the coefficient scheme corresponding to \(\cos\omega t\) and \(\sin\omega t\) and \(\xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2\) and \(\eta^3\). The result is shown in the following scheme:

<table>
<thead>
<tr>
<th></th>
<th>(\cos\omega t)</th>
<th>(\sin\omega t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi^2)</td>
<td>(-\frac{9}{16}\sqrt{2}ke^2)</td>
<td>(-\frac{9}{16}\sqrt{6}ke^2)</td>
</tr>
<tr>
<td>(\xi\eta)</td>
<td>(\frac{3}{4}\sqrt{3}ke^2)</td>
<td>(\frac{3}{4}\sqrt{2}ke^2)</td>
</tr>
<tr>
<td>(\eta^3)</td>
<td>(\frac{3}{8}\sqrt{3}ke^2)</td>
<td>(-\frac{3}{8}\sqrt{6}ke^2)</td>
</tr>
<tr>
<td>(\xi^3)</td>
<td>(\frac{3}{4}\sqrt{3}ke^2)</td>
<td>(\frac{3}{4}\sqrt{6}ke^2)</td>
</tr>
<tr>
<td>(\xi^2\eta)</td>
<td>(\frac{3}{8}\sqrt{2}ke^2)</td>
<td>(\frac{3}{8}\sqrt{6}ke^2)</td>
</tr>
<tr>
<td>(\xi\eta^2)</td>
<td>(\frac{3}{8}\sqrt{3}ke^2)</td>
<td>(-\frac{3}{8}\sqrt{6}ke^2)</td>
</tr>
<tr>
<td>(\eta^3)</td>
<td>(\frac{3}{8}\sqrt{3}ke^2)</td>
<td>(-\frac{3}{8}\sqrt{6}ke^2)</td>
</tr>
</tbody>
</table>

(59)

Introducing these values in the differential equations (12), we get for the more exact determination of \(a_1, a_{-1}, b_1\) and \(b_{-1}\) a system of four equations corresponding to (34). Comparing the two systems of equations we can determine the values of \(q_1, q_2, q_3\) and \(q_4\):

\[
\begin{align*}
q_1 &= \frac{4}{15}ke^2, & q_3 &= \frac{8}{15}\sqrt{6}ke^2, \\
q_2 &= \frac{7}{10}\sqrt{3}ke^2, & q_4 &= \frac{3}{16}\sqrt{2}ke^2.
\end{align*}
\]  

(60)

§ 7. With the values of \(q_1, q_2, q_3\) and \(q_4\) thus found we can show that equation (39) is satisfied. As only terms of the second order are retained in the equation, we can put

\[
\mu = \mu_0, \quad \omega = \frac{1}{2}\sqrt{2},
\]  

(61)

and the equation takes the form

\[
5q_2 + 4\sqrt{2}q_3 - 3\sqrt{3}q_1 = 0.
\]  

(62)

Inserting the values of \(q_1, q_2\) and \(q_3\) from (60), we find that (62) and consequently (39) is satisfied.
The second equation of condition, equation (38), is satisfied when \((\omega^2 - \frac{1}{2})^2\) has the value indicated by (48). Inserting the values of \(q_1\), \(q_3\) and \(q_4\) given in (60) we get the equation
\[
(\omega^2 - \frac{1}{2})^2 + \frac{2}{5} \delta - \frac{3}{8} \frac{5}{4} \epsilon^2 = 0.
\]
(63)

The condition for real values of \(\omega\) and hence for infinitesimal periodic orbits around \(L_4\) is
\[
27\delta \leq \frac{9}{8} \frac{5}{4} \epsilon^2,
\]
and the upper limit of the mass product is
\[
\sqrt[3]{1 + \frac{9}{8} \frac{5}{4} \epsilon^2}.
\]
(64)
(65)

It appears from (65) that periodic orbits around \(L_4\) exist for values of the mass-product greater than \(\frac{1}{3}\).

Now let us assume
\[
\delta > 0.
\]
(66)

We can then determine an orbit-constant \(\epsilon_0\) by the equation
\[
\frac{9}{8} \frac{5}{4} \epsilon_0^2 = 27\delta,
\]
(67)

which, inserting in (63), gives
\[
(\omega^2 - \frac{1}{2})^2 - \frac{3}{8} \frac{5}{4} (\epsilon^2 - \epsilon_0^2) = 0.
\]
(68)

The condition for real values of \(\omega\) is
\[
\epsilon \geq \epsilon_0.
\]
(69)

For \(\epsilon = \epsilon_0\) there is only one value of \(\omega^2(\omega^2 - \frac{1}{2})\), corresponding to one periodic orbit (the limiting orbit). For \(\epsilon > \epsilon_0\) there are two values of \(\omega^2\), corresponding to two periodic orbits (short- and long-periodic). For \(\epsilon < \epsilon_0\) there exist no periodic orbits.

If, however, we assume
\[
\delta = 0,
\]
we get from (63)
\[
(\omega^2 - \frac{1}{2})^2 - \frac{3}{8} \frac{5}{4} \epsilon^2 = 0.
\]
(70)
(71)

For \(\epsilon = 0\) there is only one value of \(\omega^2(\omega^2 - \frac{1}{2})\), corresponding to the libration point \(L_4\). For \(\epsilon > 0\) there are two values of \(\omega^2\), corresponding to two periodic orbits (short- and long-periodic).

Finally, let us assume
\[
\delta < 0.
\]
(72)

We can then determine \(\epsilon_1\) by the equation
\[
\frac{9}{8} \frac{5}{4} \epsilon_1^2 = -27\delta,
\]
(73)

which, inserting in (63), gives
\[
(\omega^2 - \frac{1}{2})^2 - \frac{3}{8} \frac{5}{4} (\epsilon^2 + \epsilon_1^2) = 0.
\]
(74)

For \(\epsilon = 0\) there are two values of \(\omega^2\), corresponding to the libration point \(L_4\). For \(\epsilon > 0\) there are two values of \(\omega^2\), corresponding to two periodic orbits (short- and long-periodic).
§ 8. We shall now illustrate the previous results with the aid of a few figures.

Fig. 2 corresponds to equation (67), showing $\epsilon_0$ as a function of $\delta$. The curve is one branch of a parabola. A point to the left of the curve corresponds to two periodic orbits with the same orbit-constant, but with different periods. A point on the curve corresponds to one periodic orbit. For a point to the right of the curve there is no corresponding periodic orbit.

![Graph showing $\epsilon$ as a function of $\delta$.]

Fig. 3 corresponds to the equations (68), (71) and (74). The abscissa is $\epsilon$, the ordinate is $\omega^2 - \frac{1}{3}$. Seven curves corresponding to different values of $\epsilon_0$ and $\epsilon_1$ are shown; the values of $\epsilon_0$ and $\epsilon_1$ are shown below:

<table>
<thead>
<tr>
<th>No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_0$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>0.00</td>
<td>0.02</td>
<td>0.04</td>
<td>0.06</td>
</tr>
<tr>
<td>$\epsilon_1$</td>
<td>0.06</td>
<td>0.04</td>
<td>0.02</td>
<td>0.00</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$10^6 \cdot \delta$</td>
<td>-182</td>
<td>-81</td>
<td>-20</td>
<td>0</td>
<td>+20</td>
<td>+81</td>
<td>+182</td>
</tr>
</tbody>
</table>

The curves are branches of hyperbolae; the asymptotes form the curve corresponding to $\delta = 0$, i.e. to the mass-product $\frac{1}{3\tau}$. The curves that cut the ordinate axis (the $\epsilon_1$ curves) correspond to mass-products smaller than $\frac{1}{3\tau}$, while the curves that cut the abscissa axis (the $\epsilon_0$ curves) correspond to mass-products greater than $\frac{1}{3\tau}$. That part of the curves which lies above the abscissa axis corresponds to short-period orbits, while the part below the abscissa axis corresponds to long-period orbits. The run of the $\epsilon_0$ curves shows that there is a continuous transition from short-period to long-period orbits. As the orbit-constant $\epsilon$ of the short-period orbit decreases the period increases, and the orbit tends to the limiting orbit with orbit-constant $\epsilon_0$ and period $2\sqrt{2\pi}$. When the limiting orbit has been
reached both orbit-constant and period increase. The limiting orbit separates short-period and long-period orbits. Taken together these orbits and their continuations in both directions form a class.

The orbits of the class that corresponds to $\epsilon_0 = 0$ behave in a similar way, only the limiting orbit is here reduced to a point (the libration point).

Conditions are somewhat different regarding the $\epsilon_1$ curves, as these form two separate branches. Starting with the branch above the abscissa axis that corresponds to short-period orbits, we find as before that the period increases as the orbit-constant decreases. In the libration point the period reaches a relative maximum $2\sqrt{2}\pi(1 - \gamma \sqrt{1770}\epsilon_1)$. Continuing now along the other branch of the curve we start at the libration point where the period has the relative minimum $2\sqrt{2}\pi(1 + \gamma \sqrt{1770}\epsilon_1)$. Then follow orbits with increasing period as the orbit-constant increases. No orbits correspond to values of the period between $2\sqrt{2}\pi(1 - \gamma \sqrt{1770}\epsilon_1)$ and $2\sqrt{2}\pi(1 + \gamma \sqrt{1770}\epsilon_1)$. It is possible, but not certain, that in this case also the short- and long-period orbits belong to the same class.
§ 9. After this investigation of the equation of condition (63) we return to the problem of determining the Fourier coefficients of $\xi$ and $\eta$.

The final determination of the coefficients of the first order $a_{-1}$ and $b_{-1}$ is carried out with the aid of equations (35), inserting $a_1 = \varepsilon$ and $b_1 = 0$ and further inserting the values of $q_1$ and $q_2$ given by (60). In this way we get the following values:

\[
\begin{align*}
    a_{1} &= \varepsilon, \\
    b_{1} &= 0, \\
    2\omega \cdot a_{-1} &= -\frac{3}{4}\sqrt{3}(1 - 2\mu)\varepsilon - \frac{3}{4} \frac{8}{9} \sqrt{3}ke^3, \\
    2\omega \cdot b_{-1} &= -\left(\omega^2 + \frac{3}{4}\right)\varepsilon + \frac{3}{4} \frac{8}{9} \frac{8}{9} \frac{8}{9} \frac{8}{9} \frac{8}{9} e^3.
\end{align*}
\] (76)

It should be emphasized that $a_{-1}$ and $b_{-1}$ on account of the presence of $\omega$ in equations (76) contain terms of the second order also. This circumstance is of importance in the determination of the coefficients of the second order $a_0, b_0, a_2, a_{-2}, b_2$ and $b_{-2}$.

In the preliminary determination of these coefficients we computed the quantities of the second order in the coefficient scheme with the aid of the values of $a_1, a_{-1}, b_1$ and $b_{-1}$ given in (52). The results were given in the schemes (53) and (56). If, however, in computing these schemes we use the values of $a_{-1}$ and $b_{-1}$ given in (76), terms of the third order will appear in the coefficient scheme, and the expressions for the unknown coefficients $a_0, b_0, a_2, a_{-2}, b_2$ and $b_{-2}$ will therefore contain terms of the third order.

As, however, the introduction of the expressions (76) in the coefficient scheme would give rather complicated expressions, we shall not carry out the computation of the terms of the third order in $a_0, b_0, a_2, a_{-2}, b_2$ and $b_{-2}$.

The coefficients $a_3, a_{-3}, b_3$ and $b_{-3}$, which are of the third order, are computed from the columns corresponding to $\cos 3\omega t$ and $\sin 3\omega t$ in the coefficient scheme. We first compute the coefficients of the third order in the coefficient scheme corresponding to $\cos 3\omega t$ and $\sin 3\omega t$ and $\xi^3, \xi \eta, \eta^3, \xi^2 \eta, \xi^2 \eta^2$ and $\eta^3$, inserting the values of $a_1, a_{-1}, b_1$ and $b_{-1}$ given in (52). The result is shown in the scheme below:

<table>
<thead>
<tr>
<th></th>
<th>$\cos 3\omega t$</th>
<th>$\sin 3\omega t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^3$</td>
<td>$\frac{4}{9} \frac{7}{3} \frac{1}{9} ke^3$</td>
<td>$-\frac{4}{9} \frac{8}{3} \frac{8}{3} \frac{8}{3} \sqrt{3} e^3$</td>
</tr>
<tr>
<td>$\xi \eta$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{7}{3} \frac{7}{3} \sqrt{3} e^3$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{1}{3} \frac{1}{3} \sqrt{3} e^3$</td>
</tr>
<tr>
<td>$\eta^3$</td>
<td>$-\frac{2}{9} \frac{1}{3} e^3$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{1}{3} \frac{1}{3} \sqrt{3} e^3$</td>
</tr>
<tr>
<td>$\xi^2 \eta$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{1}{3} \frac{1}{3} \sqrt{3} e^3$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{1}{3} \sqrt{3} e^3$</td>
</tr>
<tr>
<td>$\xi^2 \eta^2$</td>
<td>$-\frac{2}{9} \frac{1}{3} \frac{1}{3} e^3$</td>
<td>$-\frac{2}{9} \frac{1}{3} \sqrt{3} e^3$</td>
</tr>
<tr>
<td>$\eta^3$</td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
</tbody>
</table>

Introducing these values in the differential equations (12), we get four equations for the determination of $a_3, a_{-3}, b_3$ and $b_{-3}$:
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\[
\begin{align*}
\frac{1}{4}a_3 + \frac{3}{4}\sqrt{3}k b_3 + 3\sqrt{2}b_2 - 3\sqrt{2}a_2 &= \frac{11}{8}\frac{3}{2}e^3, \\
\frac{3}{4}\sqrt{3}k a_3 + \frac{3}{4}b_3 - 3\sqrt{2}a_2 &= -\frac{9}{8}\frac{3}{2}\sqrt{3}ke^3, \\
3\sqrt{2}a_3 + \frac{3}{4}\sqrt{3}k a_3 &= \frac{7}{9}\frac{3}{2}\sqrt{2}e^3,
\end{align*}
\]

which have the roots:

\[
\begin{align*}
a_3 &= \frac{7}{8}\frac{3}{2}\sqrt{3}ke^3, & a_{-3} &= \frac{14}{9}\frac{3}{2}\sqrt{6}e^3, \\
b_3 &= -\frac{1}{8}\frac{3}{2}\sqrt{3}ke^3, & b_{-3} &= \frac{10}{9}\frac{3}{2}\sqrt{2}e^3.
\end{align*}
\]

§ 10. Owing to the complication of the expressions we did not carry out the computation of the terms of the third order in the coefficients of second order \(a_0, b_0, a_2, a_{-2}, b_2\), and \(b_{-2}\).

In the case, however, of the limiting orbits the terms of the second order in \(a_{-1}\) and \(b_{-1}\) disappear, and consequently the terms of the third order in \(a_0, b_0, a_2, a_{-2}, b_2\), and \(b_{-2}\) will also disappear. We therefore carry out the determination of the Fourier coefficients of \(\xi\) and \(\eta\) for these limiting orbits.

For the limiting orbits the equation of condition (63) yields only one value of \(\omega^2(\omega^2 = \frac{1}{2})\). The orbit-constant of the limiting orbit is given by equation (67):

\[
\frac{2}{3}e_0^2 = 27\delta,
\]

while \(\delta\) is given by equation (45):

\[
\mu(1 - \mu) = \frac{1}{2} + \delta.
\]

From (80) and (81) we obtain an equation which shows the connection between \(e_0\) and \(\mu\):

\[
27\mu(1 - \mu) = 1 + \frac{2}{9}\frac{3}{2}e_0^2,
\]

and from this we find, retaining terms up to the second order only,

\[
1 - 2\mu = k[1 - \frac{2}{9}\frac{3}{2}e_0^2].
\]

Introducing now \(\omega = \frac{1}{2}\sqrt{2}\) and the value of \(1 - 2\mu\) given by (83) in equations (76) we get the following values of the coefficients of the first order for the limiting orbits:

\[
\begin{align*}
a_1 &= e, & a_{-1} &= \frac{3}{8}\sqrt{6}k[e_0 - \frac{2}{9}\frac{3}{2}e_0^2], \\
b_1 &= 0, & b_{-1} &= -\frac{3}{8}\sqrt{2}[e_0 - \frac{2}{9}\frac{3}{2}e_0^2].
\end{align*}
\]

It appears from (84) that \(a_{-1}\) and \(b_{-1}\) contain no terms of the second order, and it is therefore possible to find the coefficients of the second order by substituting \(e_0\) for \(e\) in (55) and (58):

\[
\begin{align*}
a_0 &= -\frac{2}{9}\frac{3}{2}k_e^2, & b_0 &= \frac{3}{8}\frac{3}{2}\sqrt{3}e_0^2, \\
a_2 &= -\frac{2}{9}\frac{3}{2}k_e^2, & a_{-2} &= -\frac{1}{9}\frac{3}{2}\sqrt{6}e_0^2, \\
b_2 &= \frac{3}{8}\frac{3}{2}\sqrt{3}e_0^2, & b_{-2} &= -\frac{1}{9}\frac{3}{2}\sqrt{2}e_0^2.
\end{align*}
\]

In a similar way the coefficients of the third order are found from (79):

\[
\begin{align*}
a_3 &= \frac{7}{8}\frac{3}{2}\sqrt{3}e_0^3, & a_{-3} &= \frac{14}{9}\frac{3}{2}\sqrt{6}e_0^3, \\
b_3 &= -\frac{1}{8}\frac{3}{2}\sqrt{3}ke_0^3, & b_{-3} &= \frac{10}{9}\frac{3}{2}\sqrt{2}e_0^3.
\end{align*}
\]
Introducing now (84), (85) and (86) in (14) we get the Fourier expressions for the limiting orbits:

\[
\begin{align*}
\xi &= -\frac{11}{15}k\varepsilon_0^2 + \varepsilon_0 \cos \omega_0 t + \frac{3}{8} \sqrt{5}k[\varepsilon_0 - \frac{5}{12} \frac{1}{2} \varepsilon_0^2] \sin \omega_0 t \\
&\quad - \frac{1}{12} k\varepsilon_0^3 \cos 2\omega_0 t - \frac{1}{6} \sqrt{2} \varepsilon_0^2 \sin 2\omega_0 t \\
&\quad + \frac{71}{900} \varepsilon_0^2 \cos 3\omega_0 t + \frac{131}{600} \sqrt{3} k\varepsilon_0^3 \sin 3\omega_0 t, \\
\eta &= \frac{65}{32} \sqrt{3} k\varepsilon_0^2 - \frac{5}{8} \sqrt{2}[\varepsilon_0 - \frac{5}{12} \frac{1}{2} \varepsilon_0^2] \sin \omega_0 t \\
&\quad + \frac{71}{320} \sqrt{3} k\varepsilon_0^2 \cos 2\omega_0 t - \frac{1}{8} \sqrt{2} k\varepsilon_0^3 \sin 2\omega_0 t \\
&\quad - \frac{109}{32} \sqrt{3} k\varepsilon_0^3 \cos 3\omega_0 t + \frac{109}{32} \sqrt{3} k\varepsilon_0^3 \sin 3\omega_0 t,
\end{align*}
\]

(87)

where \(\omega_0 = \frac{1}{8} \sqrt{2}\).

The Jacobian constant for the limiting orbits is computed with the aid of column 1 of the coefficient scheme. The limiting orbits, all having the same period, have also the same Jacobian constant, viz.

\[
C = 2.
\]

(88)

§ 11. In order to illustrate equations (87) we introduce the values 0.1 and 0.2 for \(\varepsilon_0\). With \(\varepsilon_0 = 0.1\) we get from (87) the Fourier expressions:

\[
\begin{align*}
10^5 \xi &= -1190 + 10000 \cos \omega_0 t + 8291 \sin \omega_0 t \\
&\quad - 79 \cos 2\omega_0 t - 360 \sin 2\omega_0 t \\
&\quad + 14 \cos 3\omega_0 t + 9 \sin 3\omega_0 t, \\
10^5 \eta &= 293 - 8610 \sin \omega_0 t \\
&\quad + 40 \cos 2\omega_0 t - 177 \sin 2\omega_0 t \\
&\quad - 9 \cos 3\omega_0 t + 14 \sin 3\omega_0 t,
\end{align*}
\]

(89)