ON THE WAVES DUE TO THE ROLLING OF A SHIP

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SUMMARY

A thin rigid flat plate is partially immersed in a fluid and is constrained to rotate about a horizontal axis in its plane. When the plate performs small oscillations about the vertical a wave motion is set up which is studied in this paper. It is supposed that the plate is so wide that the motion can be treated as two-dimensional. An explicit expression is given for the amplitude at infinity of the wave motion; this is applied to the wave damping of the rolling motion of a ship. It is considered, however, that a flat plate is not in general a good approximation for this purpose. Further developments will be given in a later paper.

If a ship floating partially immersed on the water surface is forced to execute simple harmonic oscillations about a horizontal axis lying in the longitudinal plane of symmetry, a stationary state is quickly established, in which waves travel away from the ship. The motion can therefore be maintained only if energy is constantly supplied to the system. It was suggested by W. Froude (1) that the damping of the rolling motion of a ship without bilge keels might be due to this wave motion rather than to losses caused by skin friction and eddy-making, but no detailed comparison based on theory or experiment has yet been made. It is not known what features of the ship contribute most strongly to the wave-making. T. H. Havelock (2) has calculated the waves due to the rolling motion of a small submerged elliptic cylinder and has found results of the same order of magnitude as those given by Froude, but as the motion of the fluid near the cylinder differs from that near a partially immersed ship, he does not claim to be able to do more. In the present paper and in subsequent work it will be shown how the wave motion due to the rolling of a long partially immersed ship can be calculated, and how the form of the section can be taken into account. The simplifying assumptions underlying this work, and also the work of Froude and Havelock, are:

(1) The ship is replaced by a cylinder of infinite length, that is, variations in velocity parallel to the longitudinal axis of the ship are ignored. The fluid motion is thus the same in all transverse sections.

(2) The angle of rolling is so small that its square can be neglected, and the resulting wave motion is so small that the linearized free surface condition holds.
(3) The effects of viscosity and compressibility are neglected, so that a velocity potential satisfying Laplace's equation exists.

In this paper a very simple case will be considered. It will be assumed that the effect of the beam can be neglected; that is, the ship is here replaced by an infinitely thin plate rolling about a horizontal axis so that its mean position is vertical.

**Formulation**

Take rectangular cartesian coordinates with origin in the mean surface, the y-axis being taken vertically downwards. Suppose that a thin plate extending from above the surface to depth \( a \) is hinged at \((0, b)\) and forced to perform small simple harmonic oscillations of amplitude \( \theta_0 \) and period \( 2\pi/\sigma \) about its vertical mean position. It is required to find the motion of the fluid. It can be shown (3) that if the motion is irrotational, the potential satisfies Laplace's equation in two dimensions

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.
\]

The condition of constant pressure at the free surface is

\[
K\phi + \frac{\partial \phi}{\partial y} = 0, \quad y = 0,
\]

where \( \sigma^2 = gK \).

The plate communicates its normal velocity to the adjacent water particles; to first order in the amplitude this condition may be written

\[
\frac{\partial \phi}{\partial x} = \sigma \theta_0 (b-y) \sin \sigma t, \quad x = 0 \quad (0 < y < a).
\]

At infinity the motion consists of regular wave trains moving away from the origin; furthermore, there is clearly a symmetry in the motion expressed by

\[
\phi(x, y; t) = \phi(-x, y; t+\pi/\sigma).
\]

The method of an earlier paper (3) will be followed. When the horizontal velocity at all points on the vertical plane is known, the wave motion on each side of the plane can be determined. By definition the wave motions on each side of the plane have the same value of \( \partial \phi/\partial x \) on the plane. From physical considerations it is necessary that the vertical velocity \( \partial \phi/\partial y \) should also be continuous across the plane \( x = 0 \) for values of \( y \) greater than \( a \). This condition provides an equation for the horizontal velocity. Just below the plate the horizontal velocity will become infinite, the total flow remaining finite; the infinity disappears when a rounded edge is substituted for the sharp lower edge.
Assume that
\[(\alpha \alpha \theta_\theta)^{-1} \frac{\partial \phi}{\partial x} = f(y) \cos \omega t + g(y) \sin \omega t, \quad x = 0 \quad (0 < y < \infty),\]
where
\[f(y) = 0 \quad (0 < y < a),\]
\[g(y) = (b-y)/a \quad (0 < y < a).\]

The potential can then be expanded in the form
\[(\alpha \alpha \theta_\theta)^{-1} \phi_+ = A e^{-K_y \cos (Kx - \omega t)} + B e^{-K_y \sin (Kx - \omega t)} +
\int_0^\infty \{S(k) \cos \omega t + C(k) \sin \omega t\} e^{-K_y (k \cos ky - K \sin ky)} dk
\]
in the region \(x > 0,\)
\[(\alpha \alpha \theta_\theta)^{-1} \phi_- = -A e^{-K_y \cos (Kx + \omega t)} + B e^{-K_y \sin (Kx + \omega t)} -
\int_0^\infty \{S(k) \cos \omega t + C(k) \sin \omega t\} e^{K_y (k \cos ky - K \sin ky)} dk
\]
in the region \(x < 0.\)

This expression for the potential satisfies Laplace’s equation, the free surface condition, and the condition of symmetry. It has been shown \((3)\) that
\[A = 2 \int_0^\infty g(y) e^{-K_y} dy,\]
\[B = 2 \int_0^\infty f(y) e^{-K_y} dy,\]
\[-\frac{1}{2} \pi (K^2 + k^2) k C(k) = \int_0^\infty g(y)(k \cos ky - K \sin ky) dy,\]
\[-\frac{1}{2} \pi (K^2 + k^2) k S(k) = \int_0^\infty f(y)(k \cos ky - K \sin ky) dy.\]

By the definition of \(\phi_+\) and \(\phi_-\)
\[\frac{\partial \phi_+}{\partial x} = \frac{\partial \phi_-}{\partial x}\]
identically on \(x = 0.\)

Physical considerations of continuity require that
\[\frac{\partial \phi_+}{\partial y} = \frac{\partial \phi_-}{\partial y}, \quad x = 0 \quad (a < y < \infty),\]
or since both \(\phi_+\) and \(\phi_-\) tend to zero as \(y\) tends to infinity
\[\phi_+ = \phi_-, \quad x = 0 \quad (a < y < \infty),\]
whence
\[ Ae^{-Ky} = - \int_0^\infty S(k)(k \cos ky - K \sin ky) \, dk \quad (a < y < \infty), \]
\[ Be^{-Ky} = + \int_0^\infty C(k)(k \cos ky - K \sin ky) \, dk \quad (a < y < \infty). \]

Substituting for \( S(k) \) and \( C(k) \)
\[ \int_0^\infty f(u) \, du \left[ \log \frac{y+u}{y-u} - 2e^{-K(y+u)} \right]_{-\infty}^\infty \frac{e^v}{v} \, dv = 2\pi e^{-Ky} \int_0^\infty g(u)e^{-Ku} \, du, \]
\[ \int_0^\infty g(u) \, du \left[ \log \frac{y+u}{y-u} - 2e^{-K(y+u)} \right]_{-\infty}^\infty \frac{e^v}{v} \, dv = -2\pi e^{-Ky} \int_0^\infty f(u)e^{-Ku} \, du. \]

This is a pair of simultaneous integral equations for \( f(y) \) and \( g(y) \). These can be simplified by applying the differential operator
\[ K + \frac{d}{dy} \]
to both equations, which then reduce to
\[ \int_0^\infty \frac{KF(u)+f(u)}{y^2-u^2} \, du = 0 \quad (a < y < \infty), \]
\[ \int_0^\infty \frac{KG(u)+g(u)}{y^2-u^2} \, du = 0 \quad (a < y < \infty) \]
in one variable each, where
\[ F(y) = \int_a^y f(u) \, du, \quad G(y) = \int_a^y g(u) \, du; \]
\( f(y) \) and \( g(y) \) are prescribed in the range \( 0 < y < a \) and the integral equation for \( g(y) \) becomes, on putting \( \mu(y) = KG(y)+g(y) \),
\[ \int_a^\infty \frac{\mu(u) \, du}{y^2-u^2} = \frac{1}{2a} (1-Ka) \log \frac{y^2}{y^2-a^2} - (1-Ka) \frac{1}{2y} \log \frac{y+a}{y-a} + \]
\[ + \frac{1}{2K} \left( \frac{1}{2a} \log \frac{y+a}{y-a} - 1 \right) + (1-b/a) \left[ (1-Ka) \frac{1}{2y} \log \frac{y+a}{y-a} + \frac{1}{2K} \log \frac{y^2}{y^2-a^2} \right]. \]
This type of integral equation was studied in (3). The explicit solution is

\[
\mu(y) = -\frac{2}{\pi} \left( 1 - Ka \right) \frac{y}{a} \sin^{-1} \left( \frac{a}{y} \right) + (1 - b/a) - \frac{1}{2} Ka y \sqrt{y^2 - a^2}^{-1} - (1 - b/a) \left( 1 - Ka + 2 Ka \frac{y}{a} \sin^{-1} \left( \frac{a}{y} \right) \right) + py \sqrt{y^2 - a^2}^{-1},
\]

where \( p \) is a constant to be determined.

\( g(y) \) is now uniquely determined, so that \( A \) and \( C(k) \) can be calculated, using the identities

\[
2 \int_a^\infty g(y) e^{-Kv} dy = \int_a^\infty \mu(y) e^{-Kv} dy,
\]

\[
\int_a^\infty g(y) (k \cos ky - K \sin ky) dy = k \int_a^\infty \{ \mu(y) - \mu(\infty) \} \cos ky dy - \mu(\infty) \sin ka.
\]

The calculation involves the Bessel functions \( J_n(x), I_n(x), K_n(x) \) defined by Watson (4). Thus

\[
A = -\frac{1 - Ka}{K^2 a} \left[ 1 + \frac{2}{\pi} \int_0^{Ka} uK_1(u) du \right] + \frac{aK_2(Ka)}{K^2a} - (1 - b/a) \frac{1}{K} \left[ 1 + \frac{2}{\pi} \int_0^{Ka} uK_1(u) du \right] + paK_1(Ka),
\]

\[
-\frac{1}{\pi} (K^2 + k^2) k C(k) = \frac{Ka}{4} J_2(ka) + \frac{1 - Ka}{ka} \int_0^{ka} uJ_1(u) du + (1 - b/a) \frac{K}{k} \int_0^{ka} uJ_1(u) du - \frac{1}{\pi} pKka J_1(ka).
\]

Since \( Be^{-Kv} = \int_0^\infty C(k)(k \cos ky - K \sin ky) dk \) \( (a < y < \infty) \), \( B \) can be evaluated by contour integration.

\[
B = \frac{2}{\pi} \left[ \frac{Ka}{4} I_2(Ka) + \frac{1 - Ka}{Ka} \int_0^{Ka} uI_1(u) du + (1 - b/a) \int_0^{Ka} uI_1(u) du \right] - \frac{1}{\pi} pKka I_1(Ka).
\]

These values of \( A \) and \( B \) have been deduced from \( g(y) \), the cosine component of the horizontal velocity. For the sine component \( f(y) \) the calculation is rather simpler:

\[
B = qaK_1(Ka),
\]

\[
A = qa\pi I_1(Ka),
\]

where \( q \) is another constant.
It is now possible to eliminate $p$ and $q$ from the four equations involving $A$ and $B$.

Using the identities

$$I_1(Ka)K_2(Ka)+K_1(Ka)I_2(Ka) = \frac{1}{Ka},$$

$$I_1(Ka)\int_0^{Ka} uK_1(u) du - K_1(Ka)\int_0^{Ka} uI_1(u) du = \frac{1}{2}\pi L_1(Ka),$$

where

$$L_1(x) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}x)^{2m+2}}{(m+\frac{1}{2})!(m+\frac{3}{2})!},$$

$$A = \frac{\pi a I_1(Ka)}{\pi^2 I_1^2(Ka)+K_1^2(Ka)} \left[ \frac{\pi}{2Ka} - \frac{\pi(1-Kb)}{K^2a^2} \left\{ I_1(Ka)+L_1(Ka) \right\} \right],$$

$$B = \frac{\alpha K_1(Ka)}{\pi^2 I_1^2(Ka)+K_1^2(Ka)} \left[ \frac{\pi}{2Ka} - \frac{\pi(1-Kb)}{K^2a^2} \left\{ I_1(Ka)+L_1(Ka) \right\} \right].$$

The amplitude at infinity is

$$Ka\theta_0 \sqrt{(A^2+B^2)} = \frac{\pi Ka^2\theta_0}{\sqrt{\pi^2 I_1^2(Ka)+K_1^2(Ka)}} \left\{ \frac{1}{2} - \frac{1-Kb}{Ka} \left\{ I_1(Ka)+L_1(Ka) \right\} \right\}.$$ 

**Application to the rolling motion of a ship**

In the example cited by Havelock $Ka$ has the value 0.24 so that it appears justified to assume $Ka$ small; also in many ships the roll axis is near the surface ($b = 0$). Under these conditions the amplitude reduces to

$$\frac{2}{3}K^2a^3\theta_0,$$

about 1.7 times the amplitude found by Havelock. Havelock estimated from Baker’s measurements that the waves would account for all the damping if their height (twice the amplitude) were 2.65 inches, whereas Havelock’s formula only gave 1.58 inches. The discrepancy disappears when the foregoing more accurate formula is used. Nevertheless, it is considered that the agreement in this case is fortuitous, as Baker’s experiments were made on ship-forms lacking a parallel middle body. For these Froude’s assumptions do not apply; moreover, the effect of the finite cross-section has been ignored.

**The wave-making properties of a hinged plate**

A plate hinged at its lower end is frequently used as a wave-maker in experimental tanks. If the effect of finite depth is ignored, the wave amplitude for small $Ka$ is

$$(\frac{1}{2}\pi-\frac{2}{3})K^2a^3\theta_0 = 0.9K^2a^3\theta_0.$$
On the other hand, if the space between the hinged and undisturbed water at a great depth is filled in, the amplitude is easily found to be

\[ \frac{2\theta_0}{K} (e^{-K\alpha} - 1 + K\alpha) \]

which, for small \( K\alpha \), is \( K\alpha^2\theta_0 \), and is much larger than before. This shows that there is a return flow underneath the plate which destroys the efficiency of a wave-maker of this type, unless the flow is prevented by means of a baffle.

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REFERENCES

1. W. Froude, Naval Science, 1 (1872), 411 and 3 (1874), 312.
2. T. H. Havelock, Phil. Mag. 29 (1940), 407.