Mass Relations for $1^-$ and $1^+$ Nonets
Based on Weinberg's Sum Rules

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A mass relation among isospin singlets of $1^+$ and $1^-$ nonets is obtained by making use of Weinberg's sum rules based on asymptotic $U(3) \otimes U(3)$. Possible existence of the ninth member of $1^+$-nonet is suggested and its mass is predicted from this relation.

Successful relations between masses of $P$ and $A_1$, and also of $K^*$ and $K_A$, have been obtained by making use of the spectral function sum rules based on chiral symmetry groups. In this note we wish to discuss possible mass relationship among other members of $1^-$ and $1^+$ nonets along the similar line.*

First we enumerate the sum rules we have used:

\[ \int dm^2 \{ m^2 \rho_{00}^V (m^2) + \rho_{00}^V (m^2) \} = 0, \]
\[ \int dm^2 \{ m^2 \rho_{00}^A (m^2) + \rho_{00}^A (m^2) \} = 0, \]
\[ \int dm^2 \{ \rho_{00}^V (m^2) \} = \int dm^2 \{ \rho_{00}^A (m^2) \}, \]
\[ \int dm^2 \{ \rho_{00}^V (m^2) \} = \int dm^2 \{ \rho_{00}^A (m^2) \}, \]
\[ \int dm^2 \{ \rho_{00}^V (m^2) \} = \int dm^2 \{ \rho_{00}^A (m^2) \}, \]

where $\rho_{\alpha \beta}^{V,A}(m^2)$, with $U(3)$ indices $\alpha$ and $\beta (=0, 1, 2, \ldots 8)$, are defined, with obvious notations, as

\[ \Delta_{\mu \nu}^{V,A} (q) = -i \int d^4x e^{-iqx} \langle 0 | T (J_{\mu \nu}^{V,A} (x) J_{\nu \mu}^{V,A} (0)) | 0 \rangle \]
\[ = \int dm^2 \rho_{\alpha \beta}^{V,A} (m^2) (\delta_{\mu \nu} + m^{-2} q_{\mu} q_{\nu}) / q^2 + m^2 - i \epsilon \]
\[ + \int dm^2 \rho_{\alpha \beta}^{V,A} (m^2) q_{\mu} q_{\nu} / q^2 + m^2 - i \epsilon + \text{Schwinger terms}. \]

As is well known, Eqs. (1) and (2) are Weinberg’s first sum rules extended to the asymptotic chiral $U(3) \otimes U(3)$. Equations (3), (4) and (5) are, on the other hand, the second sum rules obtained on the basis of the asymptotic chiral symmetry group $U_Y(1) \otimes U_Y(1) \otimes U_B(1) \otimes U_B(1)$. In effect, we have assumed Weinberg’s first sum rules for chiral $U(3) \otimes U(3)$ and the second sum rules for the subgroup $SU(2) \otimes SU(2) \otimes U_Y(1) \otimes U_Y(1) \otimes U_B(1) \otimes U_B(1)$ of $U(3) \otimes U(3)$.\(^*\)

When combined with an additional assumption that the spectral functions appearing in (1) \(\sim\) (5) are dominated by the vector, axial vector and pseudoscalar mesons, these sum rules imply the following mass relationship among isospin singlet vector and axial vector mesons,

$$
\frac{(m_Y^2(8) - m_Y^2(\omega)) (m_Y^2(1) - m_Y^2(\omega))}{m_Y^2(1) m_Y^2(8)} = \frac{(m_A^2(8) - m_A^2(\omega)) (m_A^2(1) - m_A^2(\omega))}{m_A^2(1) m_A^2(8)}, \quad (6)
$$

where $m_Y(1)$ and $m_Y(8)$ are the masses of vector mesons of unitary singlet and the $T=Y=0$ member of an octet respectively, while $m_Y(\omega)$ and $m_Y(\varphi)$ are physical masses of ordinary $\omega$ and $\varphi$ mesons. $m_A(1)$, $m_A(8)$, $m_A(\omega)$ and $m_A(\varphi)$ are defined as the $1^+$-counterparts of the corresponding members of $1^-$ nonet.

Below we shall briefly describe the derivation of the mass relation (6). For this purpose we first note that

$$
m_Y^2(\omega) \tan \theta_Y = m_Y^2(\varphi) \tan \theta_B \quad (7)
$$

follows from Eq. (1).\(^3\) Similarly from Eq. (2), we get**

$$
m_A^2(\omega) \tan \theta_A = m_A^2(\varphi) \tan \theta_B \quad (8)
$$

In each case, mixing angles are defined by the following matrix elements of hypercharge and baryon number vector and axial vector currents:

$$
\left( \frac{\sqrt{3}}{2} \right) \langle 0 | J_{\rho}^{\nu,A}(0) | \varphi_{Y,A} \rangle = \langle 0 | J_{\rho}^{\nu,A}(0) | \varphi_{Y,A} \rangle = \frac{m_{Y,A}^2(\varphi)}{f_{\pi Y,A}^2} \epsilon_\mu \cos \theta_{Y,A},
$$

$$
\left( \frac{\sqrt{3}}{2} \right) \langle 0 | J_{\omega}^{\nu,A}(0) | \omega_{Y,A} \rangle = \langle 0 | J_{\omega}^{\nu,A}(0) | \omega_{Y,A} \rangle = - \frac{m_{Y,A}^2(\omega)}{f_{\omega Y,A}^2} \epsilon_\mu \sin \theta_{Y,A},
$$

$$
\left( \frac{3}{2} \right)^{1/2} \langle 0 | J_{\rho}^{\nu,A}(0) | \varphi_{Y,A} \rangle = \langle 0 | J_{\rho}^{\nu,A}(0) | \varphi_{Y,A} \rangle = \frac{m_{Y,A}^2(\varphi)}{f_{\pi Y,A}^2} \epsilon_\mu \sin \theta_{Y,A},
$$

\(^*\) The motivation for this assumption comes from the fact that with the second sum rule for chiral $SU(2) \otimes SU(2)$, one can predict correct mass-ratio of $\rho$ and $A_1$. Our assumptions are consistent with the hypothesis of vector (axial vector) dominance for the spectral functions. See reference 3.

**\) Besides $1^+$, we expect possible contributions from $0^-$-mesons. However, due to the fact that $\eta(549)$ and $\eta'(958)$ are almost pure octet and singlet, respectively, they scarcely contribute to Eq. (2).
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\[
\left( \frac{3}{2} \right)^{1/2} \langle 0 | J^{V,A}_{\mu}(0) | \omega_{V,A} \rangle = \langle 0 | J^{V,A}_{\mu}(0) | \omega_{V,A} \rangle = \frac{m^2_{V,A}(\omega)}{f^{V,A}} e^{\mu} \cos \theta^{V,A}.
\]

Equations (3), (4) and (5) then yield

\[
\frac{1}{f^V f^0} \left\{ m^V(\phi) \cos \theta^V - m^V(\omega) \sin \theta^V \cos \theta^V \right\} = \frac{1}{f^V f^0} \left\{ m^A(\phi) \cos \theta^A - m^A(\omega) \sin \theta^A \cos \theta^A \right\},
\]

(9)

\[
\left( \frac{1}{f^V} \right)^2 \left\{ m^V(\phi) \cos^2 \theta^V + m^V(\omega) \sin^2 \theta^V \right\} = \left( \frac{1}{f^V} \right)^2 \left\{ m^A(\phi) \cos^2 \theta^A + m^A(\omega) \sin^2 \theta^A \right\},
\]

(10)

and

\[
\left( \frac{1}{f^0} \right)^2 \left\{ m^V(\phi) \sin^2 \theta^V + m^V(\omega) \cos^2 \theta^V \right\} = \left( \frac{1}{f^0} \right)^2 \left\{ m^A(\phi) \sin^2 \theta^A + m^A(\omega) \cos^2 \theta^A \right\},
\]

(11)

respectively. For later convenience, let us introduce angles $\theta^V$ and $\theta^A$ such that

\[
\tan \theta^V = \frac{m^V_{\omega}}{m^V(\phi)}, \quad \tan \theta^A = \frac{m^A_{\omega}}{m^A(\phi)}.
\]

(12)

We have not so far specified any $SU(3)$-breaking mechanism. We now consider two specific models.

The first model assumes, following Das, Mathur and Okubo,\textsuperscript{2} that $dm^2 \rho^{V,A}_{\alpha \beta}(m^2)$ satisfy the Gell-Mann-Okubo formula, i.e.

\[
\int dm^2 \rho^{V,A}_{\alpha \beta}(m^2) = S\delta_{\alpha \beta} + S' a_{a \beta} \quad (a, \beta = 1, 8).
\]

(13)

Combining with the first sum rule

\[
\int dm^2 \{ m^2 \rho^{V,A}_{\alpha \beta}(m^2) + \theta^{V,A}_{\alpha \beta}(m^2) \} = S\delta_{\alpha \beta},
\]

(14)

we obtain the following mass formulae for the vector and axial vector mesons,

\[
\frac{1}{2} (4 m_{\rho}^2 - m_{\omega}^2) = m^V(8),
\]

(15)

\[
\frac{1}{2} (4 m_{K}^2 - m_{\pi}^2) = m^A(8),
\]

(15')

with

\[
m^V_{\omega}(8) = m^V_{\rho}(\omega) \cos^2 \theta^{V,A} + m^V_{\omega}(\omega) \sin^2 \theta^{V,A}.
\]

(16)

Eliminating $f^V_{\rho,A}$ and $f^0_{\rho,A}$ from Eqs. (9), (10) and (11) and using Eqs. (12)
and (16) we finally obtain Eq. (6).

Let us next consider another model in which, instead of Eqs. (15) and (15'), we assume

\[ \frac{1}{2} (4 m_K^2 - m_\rho^2) = m_{\pi^0}^2 (8), \quad (17) \]
\[ \frac{1}{2} (4 m_{K_A}^2 - m_{\rho}^2) = m_{\omega}^2 (8), \quad (17') \]

with

\[ m_{\pi^0}^2 (8) = m_{\pi^0}^2 (\varphi) \cos^2 \theta_{V,A} + m_{\pi^0}^2 (\omega) \sin^2 \theta_{V,A}. \quad (18) \]

As was noted by Oakes and Sakurai, the mass formulae (17) and (17') can be derived by the following SU(3) symmetry breaking mechanism,

\[ \int dm^2 \{ m^{-1} \rho_{\pi^0} (m^2) \} = S_0 a + S' d_{a b}. \quad (19) \]

Proceeding in a similar manner, we again arrive at Eq. (6) with \( m^2 \) replaced by \( m^{-2} \) (6). Thus we have shown that the mass relation of the form Eq. (6) is valid in both models.

Equation (6) can be used to relate the masses of \( \omega_\pi, \omega_\rho \) and \( \omega_A, \omega_{A'} \). As an example we try to identify \( \omega_A = D (1285) \) and \( \varphi_A = E(1420) \), and find that Eq. (6) is not satisfied.* There is in fact some evidence that \( E \) is not 1+, but rather 0-. Assuming \( D (1285) \) as \( \varphi_A \), Eq. (6) then could be used to estimate the mass of the ninth member of the 1+ nonet \( \omega_A \). For this purpose, let us rewrite Eq. (6) as

\[ m_{\pi^0}^2 (8) = m_{\pi^0}^2 (\varphi) \cos^2 \theta_{V,A} + m_{\pi^0}^2 (\omega) \sin^2 \theta_{V,A}. \quad (18) \]

![Fig. 1. Mass relation between \( \varphi_A \) and \( \omega_A \) based on the formula (6).](image)

![Fig. 2. Possible mass splitting patterns of 1- and 1+ nonets based on asymptotic chiral U(3) \( \otimes \) U(3).](image)

\* In the first paper of the footnote on page 1531, possible assignment of \( \omega_A = D(1285) \) and \( \varphi_A = E(1420) \) has been suggested. However, if the 1+ assignment for \( E \)-meson were ruled out, we could still assign \( \varphi_A = D(1285) \) and expect \( m_A (\omega) \geq 1500 \text{ MeV} \). One of us (K. K.) would like to thank Professor W. W. Wada for helpful comments.
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(\(m_\lambda^2(8) - m_\lambda^2(\omega)\)) (\(m_\lambda^2(\phi) - m_\lambda^2(8)\))

\[
\frac{(m_\lambda^2(\phi) + m_\lambda^2(\omega) - m_\lambda^2(8)) m_\lambda^2(8)}{(m_\lambda^2(\phi) + m_\lambda^2(\omega) - m_\lambda^2(8)) m_\lambda^2(8)} = (m_\rho^2(8) - m_\rho^2(\omega)) (m_\rho^2(\phi) - m_\rho^2(8)) .
\]

The right-hand side can be estimated from mass levels of 1−-nonet. Identifying \(m_\lambda(\phi) = m_\rho(1285)\) and \(m_\lambda(8) = 1390\) MeV or 1440 MeV estimated either from (15′) or (17′), we obtain \(m_\lambda(\omega) = 1800\) MeV or 1620 MeV. The corresponding mixing angles are \(\theta_\lambda = 25.0°\) and 41.1° respectively. These situations are given graphically in Figs. 1 and 2.

Finally it should be emphasized that our arguments presented in this note are entirely based on the assumption that the asymptotic chirality is a better symmetry of strong interaction than \(SU(3)\) and therefore the second sum rules for the chirality preserving subgroup of \(U(3) \otimes U(3)\) may still be valid in broken \(SU(3)\). Only under such assumptions, Eq. (6) has been derived* and applied for predicting possible mass relationship among isospin singlet members of 1− and 1+ nonets.

References


*) In this connection, it is interesting to note that in the phenomenological Lagrangian theory, Eq. (6) can be derived from the following Lagrangian,

\[
-L = \frac{1}{2} \left( m_\rho^2(8) \phi^2(8) + m_\rho^2(1) \phi^2(1) \right) + m_\rho(8) m_\rho(1) A_\rho \phi(1) \phi(8)
+ \frac{1}{2} \left( m_\lambda^2(8) \phi^2(8) + m_\lambda^2(1) \phi^2(1) \right) + m_\lambda(1) m_\lambda(8) A_\lambda \phi(1) \phi(8)
\]

with a condition \(A_\rho = A_\lambda\).