Progress of Theoretical Physics, Vol. 70, No. 3, September 1983

New Factorized $S$-Matrix and Its Application to Exactly Solvable $q$-State Model. II

Kiyoshi Sogo, Yasuhiro Akutsu and Takayuki Abe

Institute of Physics, College of General Education
University of Tokyo, Komaba, Tokyo 153

(Received April 16, 1983)

The partition function of the $q$-state vertex model is derived by the use of the inversion method. The connection of the vertex model to the solid on solid model is established by the generalized Wu-Kadanoff-Wegner transformation. The critical behavior of the model is discussed in relation with the roughening transition.

§ 1. Introduction

In a previous paper, to be referred to as I hereafter, we gave the solution of the factorization equations for the $q$-state $S$-matrix with the CPT invariances and the charge conservation symmetry. It was also shown that this factorized $S$-matrix corresponds to a solvable $q$-state vertex model satisfying the Yang-Baxter relation.

In this paper II we consider the thermodynamic property of the $q$-state model. The partition function of the $q$-state vertex model is calculated by the inversion method (or Stroganov-Shankar-Baxter method). On the other hand, our $q$-state vertex model with charge conservation can be transformed into a physical system known as the solid on solid model (SOS model) by the generalized Wu-Kadanoff-Wegner transformation. The critical behavior of this SOS model is discussed.

The outline of this paper is the following. In § 2 the partition function of the $q$-state vertex model is obtained. To perform this calculation the inversion method is formulated for our model. In § 3 the connection of the $q$-state vertex model to the SOS model is established. The critical behavior of the model is argued by using the result of § 2. It is shown that our system exhibits an infinite order phase transition and other many transitions at the critical values of the parameter. This peculiar behavior is related to the roughening transition. Summary and discussion are given in § 4.

§ 2. Partition function of the $q$-state vertex model

In this section we use the inversion method to calculate the partition function of the $q$-state vertex model discussed in paper I. The inversion method may be summarized as follows. If the $S$-matrix elements or the Boltzmann weights satisfy the factorization equations, the CPT invariances and the crossing symmetry, the partition function per site can be computed by solving the functional equations which represent these symmetries.

For the simplicity of expressions, we change the notations in I as follows: $\gamma \rightarrow \lambda$, $\lambda \theta \rightarrow -u$. For example, the $S$-matrix elements for $q=3$ are given in this notation by

$$S_{ij}(u) = \text{sh}(\lambda - u)\text{sh}(2\lambda - u),$$
\[ S_{\ell\ell}^{(u)} = \text{sh} \ u \ \text{sh}(\lambda - u), \]
\[ S_{\ell\ell}^{(u)} = \text{sh}(2\lambda)\text{sh}(\lambda - u), \]
\[ S_{\ell\ell}^{(u)} = \text{sh}(2\lambda)\text{sh} u, \]
\[ S_{\ell\ell}^{(u)} = \lambda \text{sh}(2\lambda), \]
\[ S_{\ell\ell}^{(u)} = \text{sh} u \ \text{sh}(\lambda + u), \]
\[ S_{\ell\ell}^{(u)} = \text{sh} \lambda \ (2\lambda) - \text{sh} u \ \text{sh}(\lambda - u). \] (2.1)

And the crossing symmetry and the unitarity condition are expressed as
\[ S_{\ell\ell}^{(u)} = S_{\ell\ell}^{(\ell - u)}, \] (2.2)
\[ \sum_{\ell = -S}^{S} S_{\ell\ell}^{(u)} S_{\ell\ell}^{(-u)} = \delta_{\ell\ell} \delta_{\ell\ell} \prod_{\ell = 1}^{2S} [\text{sh}^2(\rho_\ell) - \text{sh}^2 u]. \] (2.3)

We have called Eq. (2.3) also the unitarity condition although the normalization factor is not included in S-matrix elements.

Following the usual procedure of the inversion method,\(^{21-40}\) we find that the partition function per site \(x(u)\) satisfies the functional equations,
\[ x(u) x(-u) = \rho_0 \prod_{\ell = 1}^{2S} \frac{\text{sh}(\rho_\ell u)}{\text{sh}^2(\rho_\ell)}, \] (2.4)
\[ x(u) = x(\lambda - u), \] (2.5)

where \(\rho_0 = x(0)\), the partition function at \(u = 0\). Equation (2.4) is essentially an unitary condition (2.3) (see also (2.24) of I). The function \(x(u)\) has the trivial translation symmetry of \(2\pi i\): \(x(u + 2\pi i) = x(u)\), because S-matrix elements has this symmetry. More precisely \(x(u)\) satisfies the equation
\[ x(u + \pi i) = x(u). \] (2.6)

Equation (2.6) can be shown from the equality
\[ PL_n(u) P^{-1} = (-1)^{2S} L_n(u + \pi i), \] (2.7)

where the matrix \(L_n(u)\) is introduced in (1.2.12) and \((2S + 1) \times (2S + 1)\) matrix \(P\) is defined by
\[ P = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^{2S} \end{pmatrix}. \] (2.8)

Now we compute \(x(u)\) from the functional equations (2.4), (2.5) and (2.6). In this paper we restrict the consideration to the region \(0 < u < \lambda\) (F model like region).

(i) \(x(u)\) for the real \(\lambda\)

First we consider the case of real \(\lambda\), which may correspond to the low temperature
region as for the case $q=2$. From (2.6) we can express $\ln x(u)$ in a Fourier series form,

$$\ln x(u) = \sum_{n=1}^{\infty} C_n e^{2nu}.$$  \hspace{1cm} (2.9)

The coefficients $C_n$ are determined from (2.4) and (2.5) as follows. From (2.4) we have

$$C_n + C_{-n} = \frac{2}{\pi} \sum_{\rho=1}^{2,s} (1/n) e^{-2\rho in},$$  \hspace{1cm} (2.10)

where we have used the formula

$$\ln \sinh(\lambda - u) \sinh(\lambda + u) = 2(\lambda - \ln 2) - \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\lambda n}(e^{2nu} + e^{-2nu}).$$  \hspace{1cm} (2.11)

From (2.5) we have

$$C_{-n} = e^{2\lambda n} C_n.$$  \hspace{1cm} (2.12)

Together with (2.10) and (2.12) we obtain

$$C_n = -\frac{1}{\pi} \sum_{\rho=1}^{2,s} \frac{e^{-2\rho in}}{n(1 + e^{2\lambda n})}.$$  \hspace{1cm} (2.13)

Thus the function $\ln x(u)$ is determined as

$$\ln x(u) = \ln \varphi_0 + \sum_{n=1}^{\infty} \frac{2(\sum_{\rho=1}^{2,s} e^{-2\rho in}) \sinh(nu) \sinh(n(\lambda - u))}{n \sinh(n\lambda)}.$$  \hspace{1cm} (2.14)

It is clearly seen that the expression (2.14) has the crossing symmetry (2.5).

(ii) $x(u)$ for the imaginary $\lambda$

Now we consider the case of pure imaginary $\lambda$, which may correspond to the high temperature region as for the case $q=2$. We make the variable transformations; $\lambda = i\mu$ and $u = iv$. Then Eqs. (2.4) and (2.5) become

$$x(v)x(-v) = \varphi_0 \prod_{\rho=1}^{2,s} \frac{\sin((\rho \mu - v)\sin((\rho \mu + v))}{\sin^2((\rho \mu)},$$  \hspace{1cm} (2.15)

$$x(v) = x(\mu - v).$$  \hspace{1cm} (2.16)

If we express $\ln x(v)$ as

$$\frac{d^2}{dv^2} \ln x(v) = \int_{-\infty}^{\infty} dt e^{2\mu t} C(t),$$  \hspace{1cm} (2.17)

the function $C(t)$ can be determined from (2.15) and (2.16). For this purpose we use the formula

$$\frac{1}{\sin^2(\mu \pm v)} = \int_{-\infty}^{\infty} dt e^{2\mu t} \cdot 2 \int_{-\infty}^{\infty} \frac{e^{2(\pi - 2\mu)\tau \pi)}{\sin(\pi t)}},$$  \hspace{1cm} (2.18)

for $n\pi < \mu < (n+1)\pi$ ($n =$ integer). Hereafter we use the symbol $\tilde{a}$ as

$$\tilde{a} = a - n\pi \quad \text{for} \quad n\pi < a < (n+1)\pi$$  \hspace{1cm} (2.19)

for the simplicity of notation. From (2.15) with the use of (2.18) we have
\[ C(t) + C(-t) = -\sum_{\rho=1}^{2S} 4t \cdot \frac{\text{ch}(\pi - 2\rho \mu)t}{\text{sh}(\pi t)}. \quad (2.20) \]

From (2.16) we have
\[ C(-t) = e^{\rho \mu} C(t). \quad (2.21) \]

From (2.20) and (2.21) we obtain
\[ C(t) = -\sum_{\rho=1}^{2S} 4t \cdot \frac{\text{ch}(\pi - 2\rho \mu)t}{\text{sh}(\pi t)(1 + e^{2\rho \mu})}. \quad (2.22) \]

Substituting (2.22) into (2.17) and integrating (2.18) twice with respect to \( v \), we obtain
\[ \ln \chi(v) = \ln \rho_s + \sum_{\rho=1}^{2S} \int_{-\infty}^{\infty} dt \frac{\text{ch}(\pi - 2\rho \mu)t}{t \text{sh}(\pi t) \text{sh}(\mu v)} \cdot (2.23) \]

It is clearly seen that the expression (2.23) has the crossing symmetry (2.16).

§ 3. Solid on solid model as a realization of \( q \)-state model

In this section, we relate our newly obtained \( q \)-state model to the so-called solid on solid model (SOS model) of the roughening transition. This relationship is a generalization of the one which was found by Beijeren\(^7\) for \( q = 2 \) case (six vertex model).

Let us consider a solid-vapor interface. The SOS condition says that there should be no vapor beneath a solid atom, and the models with this condition are called SOS model. With this condition, the shape of interface can be described by \( \langle h_i \rangle \): The height of the interface at site \( i \) of the square lattice, and \( h_i \)'s take integer or half integer values.

For each configuration \( \{ h_i \} \), we assign a Boltzmann weight. Ordinary form of assignment is \( e^{-E} \), where \( E \) takes the form \( E = J \sum_i |h_i - h_j|^p \), with \( p \) being some positive number.\(^8\) Instead of using these pair-interaction form, in this paper we propose the assignment of the weight for each plaquette configuration. One such plaquette is depicted in Fig. 1. We denote the Boltzmann weight of this configuration as \( W(h_a, h_b, h_c, h_d). \)

![Fig. 1. A plaquette configuration in the SOS model.](image1)

![Fig. 2. The correspondence between the plaquette configuration and the vertex configuration.](image2)
The total weight of an interface configuration is given by

$$\text{Total weight} = \prod_{\text{all plaquettes}} W(\text{plaquette}). \quad (3.1)$$

It is quite natural to assume

$$W(h_a + n, h_b + n, h_c + n, h_d + n) = W(h_a, h_b, h_c, h_d), \quad (3.2)$$

where \(n\) is an arbitrary integer or half integer. In other words, the weight depends only on relative heights on the sites of plaquette. Then we can set any one of \(h_a = h_d\) equal to zero, when we discuss the configuration of only one plaquette.

The correspondence between the plaquette configuration and a vertex of our \(q\)-state model is shown in Fig. 2. The vertex model is defined on the dual lattice of that of the interface model. With this correspondence, the height difference between the nearest neighbor site \(\Delta h\) is allowed to take values \(S, S-1, \ldots, -S\), where we put \(q = 2S + 1\). In Beijeren's case, \(\Delta h\) is restricted to values \(\pm 1/2\). Note here that when \(q\) is even, we allow \(h_i\) to take half integer values (i.e., the solid forms body center cubic lattice, just like Beijeren's case). The correspondence stated here can be applied to more general IRF (Interaction Round a Face) models named by Baxter.\(^{9}\) In this sense we call this correspondence the generalized Wu-Kadanoff-Wegner transformation (original Wu-Kadanoff-Wegner transformation\(^{3,8}\) relates the eight vertex model to Ising model with four spin interaction). With the \(S\)-matrix elements, the above correspondence can be expressed as

$$S^\pm_{\ell, \ell^\prime}(-\theta) = W(\alpha, \beta, \gamma, \delta). \quad (3.3)$$

In this way, if we take the factorized \(S\)-matrix obtained in I as the l.h.s. of (3.3), we are to have a solvable SOS model according to the results of § 2.

Here we discuss the meaning of the symmetries of the \(S\)-matrix for an interface model. As was mentioned in I, our \(S\)-matrix has the CPT invariances and crossing symmetry. The CPT invariances are interpreted to dip-bump symmetry and reflection symmetries of the SOS model,

$$W(\alpha, \beta, \gamma, \delta) = W(-\alpha, -\beta, -\gamma, -\delta), \quad (3.4)$$

$$W(\alpha, \beta, \gamma, \delta) = W(\alpha, \delta, \gamma, \beta), \quad (3.5)$$

$$W(\alpha, \beta, \gamma, \delta) = W(\gamma, \beta, \alpha, \delta). \quad (3.6)$$

The crossing symmetry is related to the isotropy of the interface. For an isotropic SOS model the weights must be invariant under 90° rotation of configurations:

$$W(\alpha, \beta, \gamma, \delta) = W(\beta, \gamma, \delta, \alpha). \quad (3.7)$$

This yields a condition

$$S^\ell_{\ell^\prime}(\theta) = S_{\ell^\prime \ell}^{-\ell}(\theta), \quad (3.8)$$

which gives \(\theta = i\pi/2\) combined with the crossing symmetry (1.2.5). Therefore in the notation of § 2 must take

$$u = \lambda/2, \quad (3.9)$$

when we consider the isotropic SOS model.
Next we discuss the thermodynamic property of this SOS model. Hereafter we consider only the isotropic case. The free energy is given by substituting \( u = \lambda / 2 \) in Eqs. (2.14) and (2.23):

\[
\ln \chi = \ln \rho_0 + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{2 e^{-2pn\pi}}{n} \frac{\text{sh}^2(n\lambda / 2)}{\text{ch}(n\lambda)}, \quad (\lambda > 0)
\]

or

\[
\ln \chi = \ln \rho_0 + \sum_{p=1}^{\infty} \int_{-\infty}^{\infty} dt \frac{\text{ch}[\pi - 2p(t\mu)]\text{sh}^2(t\mu / 2)}{t \text{sh}(\pi t)\text{ch}(t\mu)} . \quad (\mu > 0)
\]

Both expressions have a singularity at \( \lambda = 0 \) (or \( \mu = 0 \)). Near \( \lambda = 0 \) (or \( \mu = 0 \)), the singular part of (3.10a) (or (3.10b)) behaves like

\[
(\ln \chi)_{\text{sing}} \sim e^{-\text{const} / \lambda}, \quad \text{(or } e^{-\text{const} / \mu})
\]

This critical behavior is typical of the roughening transition.\(^{7,8} \) In the case \( q = 2 \) (Beijeren's case), the calculation of height-height correlation function was performed\(^{10} \) and this explicitly shows that the point \( \lambda = 0 \) is really the roughening point. So we identify the point \( \lambda = 0 \) in our SOS model as the roughening point. The best way to confirm this identification is the investigation of the height-height correlation function. For this purpose, the quantum Gelfand-Levitan method which is now developing\(^{11,12} \) will be useful. But this exceeds the scope of this article and is the subject of future study.

In (3.10b) there are additional singularities at \( \mu = \pi / p \) \((1 \leq p \leq 2S, 1 \leq I \leq p)\). The critical behavior at these points is the same as the KDP transition\(^{13} \) (first order phase transition).

Now if we consider the limit \( S \to \infty \), these singularities distribute densely in the region \([0, \pi]\). Such formation of "line of critical points" is very suggestive when we remind ourselves of the correspondence\(^{14} \) between the specific roughening model and classical XY model (or its equivalents: Coulomb gas model etc.) in two dimension. The latter shows the well-known Kosterlitz-Thouless\(^{15} \) (KT) transition. This correspondence tells the high (low) temperature behavior of the SOS model is just the low (high) temperature of the corresponding XY model. In the theory of KT transition, it is generally believed that there is a point \( T = T_{KT} \) below which the system is always at the criticality, and the essential singularity\(^{16} \) appears at \( T = T_{KT} \). In our SOS model with \( S \to \infty \) limit, the whole story of the KT transition is embodied exactly, when we permit the parameter \( \lambda \) to play the role of temperature.

But there arises one problem. In our SOS model some of the vertex weights (i.e., the Boltzmann weights) become negative in a certain range of

---

Fig. 3. Singularities of \( \chi \) in the complex \( \lambda \) plane.

The singular points are depicted by \( \bigcirc \). The solid lines represent the physical region, which ends at the points depicted by \( \times \). These points are \( i \pi / 0.491 \) \((q = 3)\) and \( i \pi / 0.291 \) \((q = 4)\) respectively.
$\mu \in [0, \pi]$. In other words, the system enters into unphysical region. For $q = 3$ and 4, we show the physical region of $\lambda$ in Fig. 3. Note that the starting point of the unphysical region (depicted by $\times$ in Fig. 3) is not necessarily equal to the critical point of the free energy. Therefore it may be allowed to say that our results on critical behavior remains to have physical meanings. It is not clear whether the positivity of the Boltzmann weights may be restored in the limit $S \to \infty$.

§ 4. Summary and discussion

In our series of papers (I and II), we have concentrated our study on the exactly solvable $q$-state model. In I, new $q$-state factorized $S$-matrix with CPT invariances and charge conservation symmetry has been obtained. And this factorized $S$-matrix has been identified with the $q$-state vertex model which satisfies the Yang-Baxter relation. The obtained $q$-state model possesses some good features; first it is new, second it is solvable and third it has a physical application. In paper II, the last two features are investigated. In § 2 the partition function of our $q$-state vertex model is computed exactly by the use of the inversion method. In § 3 the relationship between our vertex model and the solid on solid model of crystal surface is established. And the critical behavior of this solid on solid model, which is equivalent to a special case of the $q$-state vertex model, is analyzed and it is found that this solid on solid model exhibits a property similar to the two-dimensional classical XY model. The partition function obtained in § 2 and the relationship between the vertex model and the solid on solid model are new results.

Let us consider some related problems in the rest of this section. As is seen from I and II, the inductive method and the inversion method are powerful in producing new factorized $S$-matrix and in computing the partition function. These methods are applicable to the $q$-state factorized $S$-matrix with $Z_2$ symmetry. The details will be reported in a forthcoming paper.

It may be instructive to consider our result in a wide scope of general $q$-state model. Our solution makes one trajectory in the multi-dimensional parameter space. This trajectory lies in both physical and unphysical region, and crosses many sheets of critical surfaces as mentioned in § 3. In this sense it is desirable to examine other $q$-state models with different symmetries, to obtain the complete phase diagram.

Acknowledgements

The authors dedicate their sincere thanks to Professor Takeo Izuyama and Miki Wadati for valuable discussions and continuous encouragement.

References

1) K. Sogo, Y. Akutsu and T. Abe, preceding paper.
K. Sogo, Y. Akutsu and T. Abe