The Effect of Damping on Radiative Corrections to Electron Scattering and the Problem of Infra-red Catastrophe

R. C. MAJUMDAR and A. N. MITRA

Department of Physics, University of Delhi, Delhi 8

(Received September 11, 1952)

The reaction of radiation on the scattering of electrons with radiative corrections is treated on the basis of covariant theory of radiation damping. It is shown that the ultra-violet divergences are removed by the process of usual mass and charge renormalisations; and the infra-red divergence, which still persists in the second order corrections to radiationless electron scattering is exactly compensated by the corresponding divergence occurring in the scattering with the emission of radiation. In § 1 the interaction Hamiltonian for the scattering processes are given after Feynman-Dyson formalism. In § 2 the covariant formulation of radiation damping and the scattering cross-sections are given. The § 3 deals with the infra-red divergences involved in the electron scattering.

Introduction

The problem of infra-red catastrophe in electron scattering and its interpretation by Bloch and Nordsieck, Pauli and Fierz, and Braunbek and Weinmann showed for the first time the significant part played by the virtual quanta accompanying the electrons and the inadequacy of the customary perturbation theory where use is made of expansion in powers of $\alpha = e^2/4\pi \hbar c$. By making use of a canonical transformation these authors were able to include the virtual quanta in defining the state of the electron in closed form, at least in non-relativistic approximation, without expansion of the series in powers of $\alpha$. It was thereby shown that the infra-red divergence associated with the scattering of electron with the emission of radiation was exactly compensated by the radiative corrections to electron scattering without radiation. It was further clarified by Pauli and Fierz that the compensation takes place only for the emission of quanta of low frequencies. But for the emission of high frequency virtual quanta involving energies higher than the electron’s kinetic energy we are faced with new divergences (ultra-violet) which are again logarithmic and remain uncompensated, there being no corresponding ultra-violet term in the radiative scattering probability. The origin of new divergent terms was investigated by Dancoff and Lewis from relativistic considerations. It was shown by Lewis that they are all attributed to divergent mass increase of the electron. The work of Bethe and Oppenheimer further revealed a new difficulty as regards the compensation of the infra-red divergent terms. They showed that the complete compensation does not take place when one takes account of the effect of the radiation damping on the electron scattering by the perturbation method of Heitler and Wilson. It has been, however, recently shown by...
one of us\(^9\) that the application of the theory of radiation damping as proposed by Heitler and formulated by Pauli\(^1\) when properly modified to take account of radiative corrections, removes completely the infra-red divergence difficulties pointed out by Bethe and Oppenheimer, at least up to the order of \(\alpha\). Recently a new formulation of a covariant electrodynamics has been developed by Tomonaga\(^13\) and Schwinger\(^19\) which gives a powerful and consistent method for investigating the radiative corrections successfully up to any order of approximations, and in which the ultra-violet divergences causing serious difficulties in previous calculations are removed unambiguously by the process of charge and mass renormalisations. It has been already shown by Schwinger that in the new quantum electrodynamics the infra-red divergence, which is left out after mass and charge renormalisations in the second order corrections to radiationless electron scattering, compensates the infra-red divergence involved in the scattering with the emission of radiation. Schwinger has, however, not considered the effect of the radiation damping on the electron scattering. In present paper we shall extend Schwinger’s calculations to study the effect or radiation damping on the problem of electron scattering by a time independent scalar field and show that the scattering in this case also becomes completely free from divergences, infra-red or ultra-violet, after the usual charge and mass renormalisations. Contrary to Bethe and Oppenheimer, the infra-red divergences involved in the scattering with radiation is still being cancelled by the corresponding term due to the scattering without the emission of radiation, the extra term needed for the compensation being supplied by the higher order radiative corrections. In the first article we shall briefly work out the interaction Hamiltonian for the scattering processes after Feynman-Dyson formalism. In the second article the theory of radiation damping and scattering cross-section will be given. In the third article we shall discuss the problem of the infra-red divergences involved in the radiative corrections to electron scattering when radiation damping is taken into account.

\section{1. Covariant formulation of interaction Hamiltonian}

We shall first derive briefly Schwinger results for the interaction of an electron with the electromagnetic field from the Feynman-Dyson formalism. Fig. 1, gives the lowest order elastic scattering of an electron by an external field and the corresponding matrix element is

\[ H(x_0) = -ie\bar{\psi}(x_0)\gamma_\mu\psi(x_0)A_\mu(x_0). \]  

The diagrams describing the second order radiative corrections to the electron scattering are obtained by introducing the vacuum polarisation, self-energy and the vertex parts to the Fig. 1 as shown in Fig. 2.

The matrix elements of various processes of Fig. 2 are given by

\[ H_{\text{quad}}^{(x_0)} = \frac{ie\alpha\pi}{2} \int dx_1 dx_2 \text{Spur}(\gamma_\mu S_\mu(x_0-x_1)\gamma_\nu S_\nu(x_1-x_0)) \]

\[ \times A_\mu(x_0) D_\mu(x_1-x_2) \bar{\psi}(x_2)\gamma_\nu\psi(x_2), \]  

(2)
The Effect of Damping on Radiative Corrections

\[ H^{(\psi)}(x_0) = -\frac{i\alpha}{2} \int dx_1 dx_2 A_\mu^a(x_0) [\bar{\nu}(x_0)\gamma_\mu S_\nu(x_1-x_0)\gamma_\nu \phi(x_2) \]
\[ \times D_\nu(x_2-x_1) + \bar{\phi}(x_2)\gamma_\nu S_\nu(x_2-x_0)\gamma_\nu D_\nu(x_1-x_0) S_\nu(x_0-x_1)\gamma_\nu \phi(x_0)] \]
\[ - \frac{e}{2\hbar} \delta mc \int dx_1 A_\mu^a(x_0) [\bar{\nu}(x_0)\gamma_\mu S_\nu(x_1-x_0)\gamma_\nu \phi(x_2) \]
\[ + \bar{\phi}(x_1) S_\nu(x_0-x_1)\gamma_\nu \phi(x_0)], \]
\[ H^{(\psi)}(x_0) = -\frac{i\alpha}{2} \int dx_1 dx_2 A_\mu^a(x_0) [\bar{\nu}(x_1)\gamma_\nu S_\nu(x_0-x_1)\gamma_\nu \phi(x_2) \]
\[ \times \gamma_\nu \phi(x_0)] D_\nu(x_1-x_2), \]

where \( a = \alpha^2 / 4\pi \hbar c \) and \( D_\nu(x) \) and \( S_\nu(x) \) are Feynman's functions for the electromagnetic field and electron respectively and given by

\[ D_\nu(x) = \frac{-2i}{(2\pi)^4} \int \frac{e^{-ipx}}{p^2 + \lambda^2} dp; \quad S_\nu(x) = \frac{-2i}{(2\pi)^4} \int e^{-ipx} dp \frac{i\gamma^\nu - x_0}{p^2 + x_0^2}. \]

\( \lambda \) is introduced in \( D_\nu(x) \) to avoid the anticipated infra-red divergences and will be made to \( \rightarrow 0 \) after all operations are performed. Similarly the lowest order matrix element for bremsstrahlung is obtained from the Fig. 3 and is given by

\[ H^{(\psi)}(x_0) = -\frac{i\alpha^2}{2\hbar c} \int dx_1 [\bar{\nu}(x_0)\gamma_\mu S_\nu(x_1-x_0)\gamma_\nu \phi(x_1) \]
\[ + \bar{\phi}(x_1)\gamma_\nu S_\nu(x_0-x_1)\gamma_\nu \phi(x_0)] A_\mu^a(x_0) A_\mu(x_1), \]

where \( A_\mu^a(x_0) \) is the external field and \( A_\mu(x_1) \) the electromagnetic field.

We now proceed to calculate the second order corrections to the charge-current. The charge-current due to the vacuum polarisation can be expressed as

\[ J_\nu^\phi(x_0) = i\alpha e \int dx_1 G_{\mu\nu}(x_0-x_1) \partial A_\nu(x_1), \]

where \( \partial A_\nu(x_1) = -e/2 \int dx_2 D_\nu(x_1-x_2) \bar{\phi}(x_2)\gamma_\nu \phi(x_2). \)
Now by making the Fourier integral representation of $S_F(x)$ and evaluating the spurs we can write

$$G_{\mu\nu}(x) = \text{Spur} \{ \gamma_\mu S_F(x) \gamma_\nu S_F(-x) \}. \quad (9)$$

It is easily seen that the requirements of charge conservation and gauge invariance are satisfied if

$$\int \frac{k'_\mu dk''}{k'^2 + x_0^2} = 0, \quad \int \frac{k''_\mu dk'_{\nu}}{k''^2 + x_0^2} = 0. \quad (11)$$

Introducing new variables $k'_\mu$ and $p_\mu$ defined by

$$k'_\mu = \frac{1}{2} k_\mu (1 - \nu) + p_\mu, \quad k''_\mu = \frac{1}{2} k_\mu (1 + \nu) - p_\mu, \quad (12)$$

and writing

$$\frac{1}{(k''^2 + x_0^2)} = \frac{1}{2} \int d\nu \left[ \frac{k'^2 + k''^2}{2} + \frac{\nu}{2} (k'^2 - k''^2) + x_0^2 \right]^{-2}, \quad (13)$$

we obtain, following Schwinger

$$G_{\mu\nu}(x) = \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial x_\nu} [\Box] \right) G_1(x), \quad (14)$$

where

$$G_1(x) = \frac{1}{(2\pi)^4} \int d\nu (1 - \nu^2) \int dk dp e^{ikx} \left( x_0^2 + p^2 + \frac{k^2}{4} (1 - \nu^2) \right)^{-2}. \quad (15)$$

by a partial integration with respect to $\nu$.

The expression (7) for the charge-current is thus reduced to

$$\partial j_\mu^a(x_0) = \left( \frac{L_\mu \log \frac{P^2}{(x_0^2 - x_1)} - 1}{8\pi^2 x_0^2} \right) \partial_\mu \left( x_0 - x_1 \right) F_1(x_0 - x_1) \quad (16)$$

where

$$F_1(x) = \frac{1}{(2\pi)^4} \int d\nu \frac{1}{1 + (\nu^2/4x_0^2)} e^{ikx} dk, \quad (18)$$

and

$$P_0 = \sqrt{P^2 + x_0^2}. \quad (19)$$

The diagram (b) corresponding the self-energy part gives rise only to mass and
The Effect of Damping on Radiative Corrections

charge renormalisations. The charge-current corresponding to this diagram is given by

\[
\partial j^\mu_+(x_o) = \frac{i\alpha e c}{2} \int dx_1 dx_2 [\bar{\psi}(x_2) \gamma_\mu S_\nu(x_1-x_2) K(x_2-x_1) \psi(x_2) \\
+ \bar{\psi}(x_2) K(x_1-x_2) S_\nu(x_0-x_1) \gamma_\mu \psi(x_0)] \\
+ \frac{e}{2\hbar} \partial m^2 \int dx_1 [\bar{\psi}(x_0) \gamma_\mu S_\nu(x_1-x_0) \psi(x_1) + \bar{\psi}(x_1) S_\nu(x_0-x_1) \gamma_\mu \psi(x_0)],
\]  

(20)

where \( K(x) \) is given by:

\[
K(x) = -\frac{4}{(2\pi)^3} \int dk dp e^{-i(k-p) \cdot x_0} \bar{\psi}(p-k) \gamma_\nu \psi(p) \left[ \frac{i\gamma\cdot k}{(p-k)^2 + m^2} \right].
\]  

(21)

Now since

\[
\frac{1}{[(p-k)^2 + m^2][k^2 + l^2]} = \int_0^1 du \left[ (k-up)^2 + x_0^2 u^2 + l^2 (1-u) + u(1-u) (p^2 + x_0^2) \right]^{-2}
\]  

(22)

the transformation

\[
k_\mu \rightarrow k_\mu + u p_\mu
\]  

(23)

brings \( K(x) \) into the form

\[
K(x) = \frac{8}{(2\pi)^3} \int dk dp e^{-i(k-p)^2} \int_0^1 du \left[ \frac{i\gamma\cdot k}{(k^2 + A_0^2)} \frac{1}{(1-u)} \right] \cdot
\]  

(24)

Here the first derivative in the integrand represents the contributions from a surface term which must be added to take into account the effect of the displacement (23) at large values of \( |k| \) where the integrand does not tend to zero rapidly enough. Taking

\[
\int k_\mu f(k^2) dk = 0, \quad \int k_\mu k_\nu f(k^2) dk = \frac{1}{4} \partial_{\mu\nu} \int k^2 f(k^2) dk,
\]

and simplifying and rearranging terms as done by Karplus and Kroll, we obtain finally

\[
K(x) = \frac{8}{(2\pi)^3} \int dp e^{-i(k-p)^2} \int_0^1 du \left[ \left\{ \frac{1}{(k^2 + A_0^2)^2} - \frac{2i\pi^2 x_0}{4} \right\} \right]
\]  

(25)

Now substituting (25) in (20) and evaluating the integrals in the usual way, the first term which contributes to the self-energy cancels with the counter self-energy term as given by the last term of (20). We thus obtain finally, after replacing the renormalisation factor \( Z_2 \) by \( Z_2^{1/2} \), as shown by Karplus and Kroll,

\[
\partial j^\mu_+(x_o) = \frac{\alpha}{2\pi} \left( 2 \log \frac{x_0}{2k_{\text{min}}} - \log \frac{P+P_0}{x_0} + \frac{3}{4} \right) J_\mu(x_0),
\]  

(26)
where we have used the relation
\[ \epsilon \lambda = 2k_{\text{min}}. \]  
(26a)

This is proportional to the current operator and therefore can be removed by the procedure of charge renormalisation. It will, however, be shown that this cancels with the term from the vertex part when we add them together to obtain the total current.

We now calculate the contribution to the current arising from vertex part:
\[ \frac{\partial j^\mu}{\partial x^\nu}(x_0) = \frac{i\pi a e_c}{2} \int dx_1 dx_2 \bar{\psi}(x_1) K^\mu_\nu(x_0 - x_1, x_2 - x_0) \psi(x_2) \]  
(27)

where
\[ K^\mu_\nu(x_0 - x_1, x_2 - x_0) = D^\mu_\nu(x_1 - x_2) \gamma_\nu S^\mu_\nu(x_0 - x_1) \gamma_\mu S^\nu_\mu(x_0 - x_0) \gamma_\nu. \]  
(28)

By making the Fourier integral representation and proceeding in the usual way we can express (28) as
\[ K^\mu_\nu(x_0 - x_1, x_2 - x_0) = \left[ \frac{-2i}{(2\pi)^4} \right]^3 \int dx_1 dx_2 dk dk' dk'' e^{-ik(x_1-x_0)} e^{-ik'(x_2-x_0)} \times e^{-ik''(x_0-x_1)} \bar{\psi}(x_1) \gamma_\nu(i\gamma k' - x_0) \gamma_\mu(i\gamma k'' - x_0) \gamma_\nu \psi(x_2) \frac{(k^2 + k'^2)(k'^2 + x_0^2)(k''^2 + x_0^2)}{(k^2 + k^2)(k'^2 + x_0^2)(k''^2 + x_0^2)}. \]  
(29)

We now introduce the variables
\[ p^\mu_\nu = k^\mu - k^\mu, \quad p^\nu_\mu = k'^\mu - k^\mu, \]  
(30)

and note that \( \bar{\psi}(x_1) \) and \( \psi(x_2) \) occur on the left and right respectively and that the integrations are performed with respect to \( x_1 \) and \( x_2 \). This means that only values of \( p^\mu_\nu \) and \( p^\nu_\mu \) should occur for which \( p^\mu + x_0^2 = p'^\mu + x_0^2 = 0 \), and \( (i\gamma p + x_0) \) (on left) = 0 and \( (i\gamma p' + x_0) \) (on right) = 0.

As a result of this transformation (29) becomes
\[ K^\mu_\nu(x_0 - x_1, x_2 - x_0) = \left[ \frac{-2i}{(2\pi)^4} \right]^3 \int dk dk' dk'' e^{i\theta^\mu_\nu(x_1-x_0)} e^{i\theta'^\mu_\nu(x_0-x_2)} \times \bar{\psi}(x_1) \gamma_\nu(i\gamma k' - x_0) \gamma_\mu(i\gamma k'' - x_0) \gamma_\nu \psi(x_2) \frac{(k^2 + 2k'k')}{(k^2 + 2k'k')(k^2 + k'^2)}. \]  
(31)

Writing now
\[ \frac{1}{(k^2 + 2k'k')(k^2 + 2k'k')(k^2 + k'^2)} = \int_0^1 du \int_{-1}^1 dv \left[ \frac{1}{k^2 + k^2(1-u)} + ku \left( p' + p'' + v(p'' - p') \right) \right]^{-3} \]  
(32)

and making the transformation
\[ k_{\mu} \rightarrow k_{\mu} - \frac{u}{2} \{ (p^\mu_\nu + p^\nu_\mu) + v(p^\nu_\mu - p^\nu_\mu) \}, \]  
(33)

we obtain
where

\[ K_\mu(x_0 - x_1, x_2 - x_3) = \left[ \frac{-2i}{(2\pi)^4} \right] \int dk \, dk' \, dp' \cdot \delta^{(4)}(x_2 - x_0) \cdot \frac{1}{(k^2 + \Lambda^2)^3} \int duu' \int dv' \]

\[ \times \gamma_v \left[ i\gamma \left\{ k + p' - u/2 \left( p' + p'' + v(p'' - p') \right) \right\} - x_0 \right] \times \gamma_w \left[ i\gamma \left\{ k + p' - u/2 \left( p' + p'' + v(p'' - p') \right) \right\} - x_0 \right] \gamma_v \]  

(34)

where

\[ \Lambda^2 = \Lambda_0^2 + \frac{\mu^2(1 - \nu^2)}{4} (p' - p'')^2, \quad \Lambda_0^2 = \nu^2 x_0^2 + \nu (1 - \nu). \]  

(35)

Here no surface term is necessary since the divergence is only logarithmic.

In evaluating the integral we note the symmetry of the denominator with respect to \( k \)-integration and discard terms linear in \( k \cdot k_0 \) which do not contribute, while replace \( k \cdot k_0 \) by \((1/4)\delta_{k_0} k^2 \). After these simplifications the factor involving \( \gamma \)'s in (34) becomes

\[ \gamma_v \left[ i\gamma \left\{ p' - \frac{u}{2} \left( p' + p'' - p' \right) \right\} - x_0 \right] \times \gamma_w \left[ i\gamma \left\{ p'' - \frac{u}{2} \left( p' + p'' - p' \right) \right\} - x_0 \right] \gamma_v - k^2 \gamma_v. \]  

(36)

This expression has already been obtained by Schwinger and following him we can write after performing the integrations

\[ K_\mu(x_0 - x_1, x_2 - x_3) = -\frac{1}{2\pi^2} \delta(x_0 - x_1) \gamma_v \left[ 2 \log \frac{\lambda}{\lambda^2} - \frac{5}{4} \right] \delta(x_1 - x_0) \]

\[ + \frac{2}{4\pi^2} \gamma_v \partial(x_1 - x_2) \left[ \log \frac{x_0}{2k_{\min}} \left( F_0(x_0 - x_1) + F_1(x_0 - x_1) \right) + \frac{1}{2} F_0(x_0 - x_1) \right] \]

\[ + F_1(x_0 - x_1) + \frac{1}{2} G(x_0 - x_1) \]  

\[ = \frac{2i}{4\pi^2} \partial(x_1 - x_2) x_0 \gamma_v \partial \partial x_0 \]  

(37)

Substituting (37) in (27) we obtain finally

\[ \delta j'_\mu(x_0) = -\frac{a}{2\pi} \left[ \frac{2\log \frac{x_0}{2k_{\min}} + \frac{3}{4} \log \frac{P_0}{x_0}}{2k_{\min}} \right] J_\mu(x_0) \]

\[ + \frac{a}{2\pi} \log \frac{x_0}{2k_{\min}} \sum \left[ \frac{1}{2} F_0(x_0 - x_1) + F_1(x_0 - x_1) \right] J_\mu(x_1) dx_1 \]

\[ + \frac{a}{2\pi} \sum \left[ \frac{1}{2} \partial F_0(x_0 - x_1) + F_1(x_0 - x_1) + \frac{1}{2} G(x_0 - x_1) \right] J_\mu(x_1) dx_1 \]

\[ + \frac{a}{2\pi} \partial \frac{\partial}{\partial x_0} F_0(x_0 - x_1) m_{\mu\nu}(x_1) dx_1, \]  

(38)

where

\[ m_{\mu\nu}(x) = \frac{e}{2x_0} \phi(x) \sigma_{\mu\nu} \phi(x), \quad \sigma_{\mu\nu} = \frac{1}{2i} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \]  

(39)

and
Thus the complete second order radiative correction to the Hamiltonian density for the elastic scattering of the electron is given by

\[ H^{II}(x_0) = -ie \int dx_1 \bar{\psi}(x_1) \Gamma'_\mu(x_0 - x_1) \psi(x_1) A^*_\mu(x_0), \]  

where

\[ \Gamma'_\mu(x) = \frac{\alpha}{4\pi} \gamma_\mu \log \frac{x_0}{2k_{\text{min}}} \left[ F_0(x) + F_1(x) \right] \]

\[ + \frac{\alpha}{4\pi} \gamma_\mu \frac{\Box^2}{x_0^2} \left( \frac{1}{2} F_0(x) + \frac{1}{3} F_2(x) + \frac{1}{2} G(x) \right) \]

\[ + \frac{\alpha}{2\pi} \left( -i \frac{1}{2x_0} \right) \sigma_{\mu\nu} \frac{\partial}{\partial x_\nu} F_0(x). \]  

We shall now restrict the external potential to be that of a time independent scalar field and accordingly write

\[ A^*_\mu(x_0) = iN(r). \]  

The matrix element of the integral Hamiltonian for the elastic scattering for transitions from a state \( p \) to a state \( q \) in a static field \( V(r) \) including the second order radiative corrections becomes

\[ \langle p | H^{II} | q \rangle = eV_{pq} u(p) [\gamma_4 + \Gamma_4(p-q)] u(q), \]  

where \( \Gamma'_\mu(p-q) \) is the Fourier transform of \( \Gamma'_\mu(x) \) defined by

\[ \Gamma'_\mu(p-q) = \int e^{-i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}} \Gamma'_\mu(x) dx, \]

\[ \Gamma_4(p-q) = -\frac{\alpha}{4\pi} \gamma_4 \left[ 4\mathbf{p}^2 A(\lambda) + i \frac{\mathbf{q}}{x_0} F_0(\lambda) \gamma \cdot (\mathbf{p}-\mathbf{q}) \right], \]

where \( A(\lambda), F_0(\lambda) \) have been already defined by S. III and

\[ V_{pq} = \int e^{-i(\mathbf{p}-\mathbf{q}) \cdot \mathbf{r}} V(r) d\mathbf{r}, \]

the integral being extended over the surface \( t=\text{constant} \).

We now proceed to evaluate the Hamiltonian density for the bremsstrahlung. Making use of the relation

\[ S_\phi(\mathbf{p}) = \int_{-\infty}^{\infty} dx e^{ipx} S_\phi(x) = \frac{-2i(\gamma \mathbf{p} - x_0)}{\mathbf{p}^2 + x_0^2}, \]  

and putting

\[ A_\phi(x) = A_\phi(k) e^{\pm i\kappa x}, \]  

where \( \pm \) signs correspond to the emission and absorption of a quantum respectively, and performing the integration in (6) over the surface \( t=\text{constant} \) the matrix element of the integral Hamiltonian reduces to
The Effect of Damping on Radiative Corrections

\[(\phi | H^{(0)}(q,k)) = 4\pi i u \sqrt{\frac{\hbar c}{2k}} V_{\nu, q+k} \bar{u} \phi \left[ \gamma_4 \frac{i \gamma(q+k)-x_0}{2qk} \gamma_\nu \right] u(q), \tag{50} \]

where we have put:

\[A_\nu(k) = \sqrt{\frac{\hbar c}{2k}} e_\nu, \quad \phi(x) = u(q) e^{ip_\nu}, \quad \bar{\phi}(x) = \bar{u}(p) e^{-ip_\nu}, \tag{51} \]

where \(e_\nu\) is the unit vector in the direction of the polarisation of the emitted photon and

\[V_{\nu, q+k} = \int V(r) e^{-i(p-q-k) \cdot r} \, dr. \tag{52} \]

For slowly moving particles, the expressions (44) and (52) reduce to:

\[(\phi | H^{(0)}(q)) = \epsilon V_{\nu} \bar{u} \phi \gamma_4 \left[ 1 - \frac{\epsilon\delta}{4\pi} \frac{(p-q)^2}{x_0^2} - \frac{i\epsilon}{4\pi x_0} \gamma_\nu (p-q) \right] u(q), \tag{53} \]

and

\[\epsilon \frac{V_{\nu} C_1}{\hbar^2} \bar{u} \phi \gamma_4 e \cdot (q-p) u(q), \tag{54} \]

where

\[\delta = \frac{4}{3} \log \frac{x_0}{2k_{\text{min}}} + \frac{17}{30}, \quad C_1 = \frac{e}{x_0 \sqrt{2\hbar c}}. \tag{55} \]

§ 2. Radiation damping and the scattering cross-section

The covariant theory of radiation damping has already been discussed by Fukuda and Miyazima. \(^{14}\) The \(S\)-matrix of Heisenberg describing the scattering processes is defined by

\[S = 1 + \sum_n \left( \frac{-i}{\hbar c} \right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n P(H^e(x_1), \ldots H^e(x_n)), \tag{56} \]

where \(P\) is the chronological ordering operator and \(H^e(x_0)\) is the Hamiltonian for the external field, including the radiative corrections. The \(S\)-matrix is further expressed as

\[S = 1 - \frac{i}{\hbar c} R, \tag{57} \]

where \(R\) is proportional to amplitude of the scattered wave. Making the Cayley transformation:

\[S = \frac{1 - (i/2\hbar c)K}{1 + (i/2\hbar c)K}, \tag{58} \]

where \(K\) is a Hermitian matrix, we obtain

\[R = K - \frac{i}{2\hbar c} KR, \tag{59} \]
which is the covariant generalisation of Heitler's equation for radiation damping. $K$ has 
the following expansion:
\[
K = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \left( -i \frac{1}{\hbar c} \right)^n \int_{-\infty}^{\infty} dx_1 \cdots dx_n \ H^x(x_1) \cdots H^x(x_n) \epsilon(\sigma_1, \sigma_2) \cdots \epsilon(\sigma_{n-1}, \sigma_n),
\]
(60)
where $\epsilon(\sigma, \sigma') = +1$ if $\sigma'$ antedates $\sigma$ and $-1$ if $\sigma'$ succeeds $\sigma$.

Since we are concerned with the scattering under the influence of a time-independent 
classical field we shall now specialise our discussion to the surface $t = \text{constant}$. It is then 
easily verified that
\[
K = \int_{-\infty}^{\infty} dx \ H^x(x).
\]
(61)

Since the conservation of energy holds in the scattering process, we can conveniently express 
\[
\rho_{OA} = 2\pi \delta(\rho_{OA} - \rho_{OB}) V_{AB}, \quad K_{AB} = 2\pi \delta(\rho_{OA} - \rho_{OB}) H_{AB},
\]
(62)
where $\rho_{OA}$, $\rho_{OB}$ are the energies of the initial and final states $A$ and $B$ respectively 
and $H_{AB}$ is proportional to the integral Hamiltonian. From (59), (61) and (62) we im-
mmediately obtain the familiar equation of radiation damping:
\[
V_{AB} = H_{AB} - \frac{\hbar c}{2\pi} \rho_C \omega_{OB} V_{OB},
\]
(63)
where $\rho_C \omega_{OC}$ is the number of states $C$ lying in the energy range $E_C$ and $E_C + dE_C$.

It is to be emphasised that in the equation (63) the matrix elements include the 
radiative corrections up to any order of approximations, whereas in the corresponding equation 
of Heitler only the non-vanishing lowest orders of the matrix elements are retained. The 

differential cross-section is now given by
\[
d\sigma_{AB} = \frac{2\pi}{\hbar^2 \omega_{OA}} |V_{AB}|^2 \rho_R.
\]
(64)

§ 3. Radiative corrections to electron scattering

(A) Without damping (Schwinger): We shall now investigate the elastic scattering 
of an electron in Coulomb field. We write (43) in the form
\[
A_r(x_0) = \frac{ieZ}{4\pi|p|},
\]
(65)
whence we obtain from (44)
\[
(\rho | H^{(0)} | q) = \frac{e^2 Z}{|p - q|^2} \tilde{u}(\rho) \left[ \gamma_4 + V_4(p - q) \right] u(q).
\]
(66)

The cross-section for radiationless scattering is then given by
\[
d\sigma_{AB}^{(0)} = \frac{e^2}{\rho^2 |p_q|} \cdot \frac{1}{2} \sum_{\text{Spin}} |H_{AB}^{(0)}|^2 dQ_q
\]

Downloaded from https://academic.oup.com/ptp/article-abstract/8/4/479/1857564/The-Effect-of-Damping-on-Radiative-Corrections-to
by guest on 16 September 2017
The Effect of Damping on Radiative Corrections

\[ -2dQ_o \left( \frac{Zu}{(\beta - \beta')^2} \right)^2 \int \frac{1}{4} \text{Spur} \left\{ (i\gamma p - x_0)(\gamma_4 + \Gamma_4(q - p))(i\gamma q - x_0)(\gamma_4 + \Gamma_4(p - q)) \right\}, \]  

which is the expression (2.64) of S. III.

For the scattering with the emission of radiation the integral Hamiltonian follows immediately from (50)

\[ H_{\text{lab}}^{(0)} = 4\pi a \sqrt{\frac{\hbar c}{2k}} \cdot \frac{eZ}{|p - q - \vec{k}|} \cdot u(p) \left[ 2(e\vec{p}) - (e\vec{r})(\gamma \vec{k}) \right] \times \gamma_4 2(eq) + (\gamma \vec{k})(eq) + \frac{2(\gamma \vec{k})(eq)}{2qk} \]  

The cross-section for radiative scattering is thus given by

\[ d\sigma_{\text{lab}}^{(0)} = \frac{1}{2} \sum_{\text{spin}, \text{pol}} |H_{\text{lab}}^{(0)}|^2 \frac{2\pi}{k^2} \frac{\vec{q} \cdot p_0 q_0}{|\vec{p}|} \cdot \frac{d^3 \vec{k}}{(2\pi)^3} \cdot dQ_o \]

\[ = \frac{\pi^2}{\hbar c} \frac{\vec{q}}{|\vec{p}|} \cdot \frac{a^2}{(2\pi)^b} \cdot \frac{eZ^2}{|\vec{p} - \vec{q} - \vec{k}|} \cdot \frac{d^3 \vec{k}}{|\vec{k}|} \cdot dQ_o \times \text{Spur} \left\{ (i\gamma p - x_0) \left( \frac{2(e\vec{p}) - (e\vec{r})(\gamma \vec{k}) \gamma_4 + \gamma_4 2(eq) + (\gamma \vec{k})(eq)}{2qk} \right) \right\}. \]  

We have thereby utilised the relation

\[ \sum_{\text{spin}, \text{pol}} u_a(p) u_0(p) = -\frac{1}{2p_0} (i\gamma p - x_0)_{ab}. \]  

The expression (69) agrees with the lowest order cross-section for bremsstrahlung obtained by Heitler. It is to be observed that for the essentially elastic scattering in which a small fraction of the electron kinetic energy is radiated we obtain from (69)

\[ d\sigma_{\text{lab}}^{(0)} = \frac{a}{2\pi^2} \left( \frac{Zu}{2p_0^2 \beta^2} \right)^2 \left( 1 - \beta^2 \sin^2 \theta/2 \right) \frac{d^3 \vec{k}}{2|\vec{k}|} \cdot dQ_o \sum_{\text{pol}} \left( \frac{eq}{qk} - \frac{ep}{pk} \right)^2, \]  

which is equivalent to expression (2.80) of S. III. It has been shown by Schwinger (S. III) that in the non-relativistic approximation the infra-red divergence contained in (67) is cancelled by the similar one arising from (69).

(B) With damping (Bethe and Oppenheimer): We are now in a position to evaluate the effect of the radiation damping on the scattering of the electron. Following Bethe and Oppenheimer we shall assume that the matrix elements of the scattering potential energy \( V_e = eV_{pq} \) do not depend on the directions of the vectors \( \vec{p} \) and \( \vec{q} \) but make no restrictions on its magnitude. We shall further neglect the higher order effects such as the emission of two quanta or the scattering of quanta.

The equations of radiation damping for transitions without radiation and for scattering with emission of radiation are obtained from (63) and given respectively by
\[ (q|U|p) = (q|H|p) - \frac{i\pi p^2}{c\hbar(2\pi)^3} \int dQ' \left[ (q|H|p') (p'|U|p) \right. \\
+ \frac{d^3 k}{(2\pi)^3} \left( q|H|p'k \right) \left( p'k|U|p \right), \tag{72} \]

\[ (p'k|U|p) = (p'k|H|p) - \frac{i\pi}{c\hbar} \cdot \frac{p^2}{(2\pi)^3} \int dQ'' \left[ (p'k|H|p''k) (p''k|U|p) \right. \\
+ \left. (p'k|H|p'') (p''|U|p) \right]. \tag{73} \]

\( \vec{p}, \vec{q}, \vec{p}', \vec{p}'' \) denote the momenta of the electron, \( \vec{p} \) is the absolute value of the momentum which is the same for all states, the integral \( dQ' \) is over all directions of \( \vec{p}' \) and the integral \( d^3 \vec{k} \) is over magnitude, direction and polarisation of the light quantum.

To solve the equations \(72\) and \(73\) we assume

\[ (q|U|p) = \bar{u}(q) \gamma_4 [A_1 + (\vec{r} \cdot \vec{q}) A_2 + (\vec{r} \cdot \vec{p}) A_3 + (\vec{r} \cdot \vec{q}) (\vec{r} \cdot \vec{p}) A_4 + (\vec{r} \cdot \vec{q}) A_5] u(p), \tag{74} \]

\[ (qk|U|p) = -\frac{1}{k^{1/2}} \bar{u}(q) \gamma_4 [B_1 (\vec{r} \cdot \vec{q}) + B_2 (\vec{r} \cdot \vec{p}) + (\vec{r} \cdot \vec{r}) B_3 + (\vec{r} \cdot \vec{r}) (\vec{r} \cdot \vec{p}) B_4 + (\vec{r} \cdot \vec{q}) (\vec{r} \cdot \vec{p}) B_5] u(p). \tag{75} \]

It is to be noted that the coefficients \( A \)'s and \( B \)'s are matrices which involve \( \gamma_4 \) only and not the other \( \gamma \)'s.

We now substitute \(74\) and \(75\) in \(73\) and sum over the spins in the intermediate states and integrate over the angles. We thereby remember the relations \(70\) and note that in the terms containing an odd number of factors involving \( \vec{p}' \) vanish on integration and that

\[ \int (\vec{a} \cdot \vec{p}) (\vec{b} \cdot \vec{p}) dQ' = \frac{4\pi}{3} (\vec{a} \cdot \vec{b}) \vec{p}^2. \tag{76} \]

Equating the coefficients of \( (\vec{r} \cdot \vec{q}), (\vec{r} \cdot \vec{p}) \) etc. on both sides of the resulting equation we obtain

\[
\begin{align*}
B_1 &= V_1 C_1 - iW_1 C_1 \left( (p_0 + \chi_0 \gamma_4) A_1 + i\gamma^5 A_3 \right), \\
B_2 &= -V_1 C_1 - iW_1 \left( (p_0 + \chi_0 \gamma_4) \left( B_2 - \frac{1}{3} \rho^5 C_1 A_3 \right) \right), \\
B_3 &= -iW_1 \left( (p_0 - \chi_0 \gamma_4) \left( B_3 - \frac{1}{3} \rho^5 A_3 \gamma_4 \right) \right) - \frac{W_1 \rho^5}{3} \gamma_4 (B_1 - C_1 A_1), \\
B_4 &= -iW_1 \left( (p_0 + \chi_0 \gamma_4) \left( B_4 - \frac{1}{3} \gamma_4 A_3 \right) \right) + \frac{W_1 \rho^5}{3} \gamma_4 (B_3 - C_1 A_3), \\
B_5 &= -iW_1 \left( (p_0 - \chi_0 \gamma_4) A_3 - W_1 C_1 \rho^5 \gamma_4 \left( A_4 + \frac{1}{3} A_5 \right) \right),
\end{align*}
\]

where

\[ V_1 = eV_p, \quad W_1 = \frac{V_1 p}{4\pi \hbar c}. \tag{77a} \]
Similarly to solve (72) we substitute (74) and (75) in it and note that the photons \( \vec{k} \) emitted are virtual photons and the integration is over \( \alpha^2 \vec{k}^2 = k^2 dk dQ_k \). Performing the integrations and using the abbreviations

\[
\eta = \frac{\alpha}{4\pi} \delta^{\rho^2} x_0^2, \quad R = \frac{C_i^2 \rho^2}{3\pi^2} \log \frac{\rho^2}{2x_0 k_{\text{min}}},
\]

(78)

and introducing

\[
\zeta = \eta - R = \frac{2\alpha \rho^2}{3\pi^2} \left[ \frac{17}{40} \log \frac{\rho^2}{x_0^2} \right],
\]

(79)

which is finite and free from \( k_{\text{min}} \), we can easily determine the coefficients \( A \)'s by comparing their coefficients on both sides of the equations. By retaining only the terms up to the order of \( \alpha \) and \( \rho^2 \) in matrix elements (74) and (75) we obtain finally the following solution for the amplitudes:

\[
A_1 = \frac{V_1(1-\eta)}{1+iW_1(\rho_0+x_0\gamma_4)} + \frac{V_1}{[1+iW_1(\rho_0+x_0\gamma_4)]^2} \left\{ iW_1\zeta(\rho_0+x_0\gamma_4) - \frac{iW_1\rho^2\gamma_4}{2x_0\pi} \right\},
\]

\[
A_2 = \frac{-iV_1\alpha/(4\pi x_0)}{1+iW_1(\rho_0+x_0\gamma_4)}, \quad A_3 = \frac{iV_1\alpha}{4\pi x_0} \times \frac{1+iW_1(\rho_0+x_0\gamma_4)}{1+2iW_1\rho_0},
\]

\[
A_4 \approx 0, \quad A_5 = \frac{V_1}{\rho^3} \cdot \frac{\gamma + iW_1\zeta(\rho_0+x_0\gamma_4)}{1+iW_1(\rho_0+x_0\gamma_4)},
\]

\[
B_1 = -B_2 = \frac{V_1C_1}{1+iW_1(\rho_0+x_0\gamma_4)}, \quad B_3 = B_4 = B_5 = 0.
\]

(80)

Substituting these expressions in (74) and (75) we get the matrix elements \( \langle g | U | \rho \rangle \) and \( \langle q k | U | \rho \rangle \) up to the order of \( \alpha \); the cross-section for radiationless scattering is then given by

\[
\frac{d\sigma_0}{dQ_q} = \frac{\rho^2}{4\pi^2 \hbar^2 c^2} \frac{1}{2} \sum_{\text{spin}} | \langle \rho | U | g \rangle |^2.
\]

(81)

The cross-section for scattering with the emission of radiation is given by

\[
\frac{d\sigma_1}{dQ_q} = \frac{\rho^2}{c^2 \hbar^2} \int_{k_{\text{min}}} \kappa^2 dk \frac{1}{2} \sum_{\text{spin, pol}} | \langle \rho | U | q k \rangle |^2 \frac{dQ_k}{(2\pi)^b}.
\]

(82)

Using formula (70) for the annihilation operator and evaluating the spurs, we get

\[
d\sigma_0 = \hbar^2 W^2 \left\{ \frac{1}{1+W^2} \left[ \frac{1}{2x_0^2} \frac{\eta}{1+\rho^2} - \frac{\rho^2}{2x_0^2} \right] - (1-\rho^2/2x_0^2) \left( \frac{2\eta - \rho^2}{4x_0^2} \right) \right\} dQ_q,
\]

(83)

and

\[
d\sigma_1 = \frac{2\hbar^2 W^2 R}{1+W^2} \left( 1 - \frac{\rho^2}{\rho^2} \right) dQ_q,
\]

(84)

where following Bethe and Oppenheimer, we have put
Adding the two cross sections (83) and (84) and noting the relation (79) we get finally,

\[ d\sigma = W^2 \kappa^2 \left[ \frac{1 + (\vec{p} \cdot \vec{q} + \vec{p}^2) / 2x_0^2}{1 + W^2} + \frac{(1 - \vec{p} \cdot \vec{q} / \vec{p}^2)(\vec{p}^2u/2\pi x_0^2 - 2\zeta)}{1 + W^2} \right. \\
\left. + \frac{2W^2}{(1 + W^2)^2} \left( \zeta - \frac{p^2u}{4\pi x_0^2} - \frac{p^2}{4x_0^2} \right) \right] d\Omega_q. \quad (86) \]

The expression (86) is thus entirely free from the infra-red divergences. It is to be observed that the infra-red catastrophe involved in the scattering with emission of radiation given by (84) is now exactly compensated by the corresponding term due to the second order radiative corrections, occurring in the radiationless scattering (83).

References

1) F. Bloch and A. Nordsieck, Phys. Rev. 52 (1937), 54.
3) W. Braunbek and E. Weinmann, ZS. f. Phys. 110 (1938), 360.
6) S. T. Epstein, Phys. Rev. 73 (1948), 177.
W. Heitler and S. T. Ma, Phil. Mag. 40 (1949), 651.
12) J. Schwinger, Phys. Rev. 74 (1948), 1439, I; Phys. Rev. 75 (1949), 651, II; Phys. Rev. 76 (1949), 790, III. This paper is referred in the Text as S. III.