Quantization of Gauge Theory with Scalar-Field-Dependent Metric

Shinichi DEGUCHI

Atomic Energy Research Institute
College of Science and Technology, Nihon University, Tokyo 101

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We investigate the BRS quantization of a gauge theory in which the Killing metric is replaced with a function of scalar fields. This type of gauge theory becomes important when we discuss internal gauge symmetries based on noncompact Lie groups. We point out that, in order to assure the BRS invariance, the Lagrangian density must contain fourth order terms of FP (anti-) ghost fields. It is also shown that a part of the gauge fields becomes massive and their propagator has the same high-energy behavior as that of massless gauge fields.

§ 1. Introduction

About ten years ago, Cahill formulated a gauge theory of internal symmetry based on the noncompact group $GL(n, R)(K=R, C)$. In this theory, the Lagrangian density of gauge fields is constructed through the use of the scalar fields which are called "internal metric tensor", and thus the appearance of ghosts with negative energy is avoided. Zee and Kim applied Cahill's theory to the grand unified theory.² Dell proposed a unified theory including the gravity whose non-gravitational part is identical with Cahill's theory.³,4

On the other hand, internal gauge symmetries based on noncompact Lie groups were also investigated by Julia and Luciani and by Hull et al.⁵,⁶ as an application of the nonlinear realization of Lie groups.⁷ They treated a general framework of the gauge theory in which the Lagrangian density of gauge fields is constructed by using a positive-definite scalar-field-dependent metric instead of the Killing metric. Cahill's theory is reduced to a particular case of Julia and Luciani's theory (see Appendix B).

The gauge theories mentioned above can be understood as an extension of the massive Yang-Mills field theory proposed by Kunimasa and Goto,⁸ and can be used as a framework for (grand) unified theories including both massive and massless gauge fields.

The purpose of the present paper is to investigate the quantum field theory of Julia and Luciani's gauge theory. We carry out the quantization by using the BRS formalism⁹,¹⁰ without imposing the unitary gauge condition.¹¹ Our discussion is valid for any (compact or noncompact) gauge group. The formulation in this paper can be regarded as an extension of the quantum theory of the massive Yang-Mills fields developed by Fukuda et al.¹²

In the next section, we briefly review the construction of the classical Lagrangian

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⁸ As for the $SL(2, C)$ gauge theory, the quantization was discussed by Popović and by Dell et al.⁶,¹⁰ They carried out the quantization of the system after imposing the unitary gauge condition on the Lagrangian (or on the Hamiltonian), and hence the propagator of the massive gauge fields becomes that of Proca fields.
density. In § 3, the Lagrangian density for quantum theory is set up by adding a gauge fixing and a Faddeev-Popov (FP) ghost terms to the classical Lagrangian density. We there point out that these gauge-fixing and FP ghost terms cannot be obtained by the usual path-integral method. Necessary field equations and Noether’s currents are also derived. We perform the canonical quantization of the system in § 4. It is verified that the BRS charge exactly generates the BRS transformation. In § 5, we introduce asymptotic fields and examine the properties of those fields. The unitarity of the physical S-matrix is discussed. Section 6 is devoted to a summary and some remaining discussion. We add a short comment on the anti-BRS symmetry. Appendix A contains some geometrical formula which are useful in this paper. In Appendix B, we show that Cahill’s Lagrangian density can be rewritten in the form obtained in § 2.

§ 2. Classical system

Let $G$ be a Lie group and $H$ its subgroup. The generators $\{T_a\}^* (A=1, 2, \cdots, r; \ r=\text{dim}_R G \ (\text{dim}_R \text{ denote real dimension})$ of $G$, which satisfy the commutation relation $[T_a, T_b]=i\epsilon_{abc}T_c$, then, are divided into two parts: The set $\{T_a\} (a=1, 2, \cdots, m; \ m=\text{dim}_R H)$ which generates the subgroup $H$, and the remainder $\{T_a\} (a=m+1, \cdots, r)$ which are associated with the $(r-m)$-dimensional coset space $G/H(\equiv \{g\}|g \in G)$. In this paper we shall only consider the “reductive” coset space, i.e., the case $f_{ab'}=0$.

We first consider the system consisting of gauge fields $A_\mu(x)=A_\mu^x(x)T_x \ (x=0, 1, 2, 3)$ and $r-m$ real scalar fields $\phi^a(x)$ which are coordinates in the coset space $G/H$.

To construct the Lagrangian density of $\phi^a$, we define

$$E_\mu(\phi)=\frac{1}{i\kappa} v(\phi)^{-1} D_\mu v(\phi) \quad (2\cdot1)$$

with

$$D_\mu v(\phi)=\partial_\mu v(\phi) + iq A_\mu v(\phi) \quad (2\cdot2)$$

Here $\kappa$ and $q$ are dimensionless constants and $v(\phi)(\subseteq G)$ is a coset representative, which transforms under the (left) action of $g(\subseteq G)$ according to

$$v(\phi) \mapsto v(\phi') = gv(\phi)h(\phi, g)^{-1} \quad (h \in H). \quad (2\cdot3)$$

The vector $E_\mu$ belongs to the Lie algebra of $G$, and hence can be expanded in terms of $\{T_a\}$ as $E_\mu=E_\mu^aT_a$. Taking into account the condition $f_{ab'}=0$, we can show that, under the gauge transformation

$$A_\mu \mapsto A'_\mu = gA_\mu g^{-1} + \frac{1}{iq} g\partial_\mu g^{-1} \quad (2\cdot4)$$

and (2·3), the components $E_\mu^{(a)}$ of $E_\mu$ transform, homogeneously, as

$$E_\mu^{(a)}(\phi) \mapsto E_\mu^{(a)}(\phi') = E_\mu^{(b)}(\phi)D_\mu^a(h), \quad (2\cdot5)$$

*\footnote{Any element $X$ in the Lie algebra of $G$ can be expressed as $X=X^aT_a$ with “real” coefficients $X^a$.}
where $D_b^a$ is a submatrix of the $D_b^A$ defined in (A·11). To make an invariant quantity from $E_{\mu}^{(a)}$, we introduce a constant metric $I_{AB}(=I_{BA})$ which is invariant under the actions of $H$ in the following sense:

$$I_{CD}A^c_{A}(h)D^b_{B}(h)=I_{AB} \quad \text{(for all } h(\in H)) \tag{2\cdot6}$$

and satisfies the condition $I_{aa}=0$. We, then, can set up the Lagrangian density of $\phi^a$ as

$$\mathcal{L}_\phi\equiv\frac{1}{2}\mu^2I_{ab}E_{\nu}^{(a)}(\phi)E_{\nu}^{(b)}(\phi), \tag{2\cdot7}$$

where $\mu$ is a constant with dimension of mass. By using Eqs. (A·8) and (A·10), the Lagrangian density (2·7) can be rewritten in the following form:

$$\mathcal{L}_\phi\equiv\frac{1}{2}\mu^2g_{ab}(\phi)D_\nu\phi^aD_\nu\phi^b, \tag{2\cdot8}$$

where $g_{ab}(\phi)$ is the metric tensor on $G/H$ defined in (A·18) and the covariant derivative of $\phi^a$ is defined by

$$D_\mu\phi^a=\partial_\mu\phi^a+qe^{-1}A^b_\mu L_b^a(\phi). \tag{2\cdot9}$$

The Lagrangian density $\mathcal{L}_\phi$ is nothing but that of the nonlinear $\sigma$-model coupled with gauge fields.

Next, let us construct the Lagrangian density of the gauge fields $A_\mu$. The field strength of $A_\mu$ is defined by

$$F_{\mu\nu}\equiv F^{\mu\nu}_{\rho}T_\rho=\partial_\mu A_\nu-\partial_\nu A_\mu+i[q[A_\mu,A_\nu]], \tag{2\cdot10}$$

and it transforms as

$$F_{\mu\nu}\rightarrow F'_{\mu\nu}=gF_{\mu\nu}g^{-1}=F^{\mu\nu}_{\rho}D^a_\rho(g)T_B. \tag{2\cdot11}$$

If $G$ is a compact semi-simple Lie group, one can use, as a Lagrangian density of $A_\mu$, the usual form (i.e., $-\frac{1}{4}K_{AB}F_{\mu\nu}^{A}F^{\mu\nu}_{B}$) with the help of the Killing metric $K_{AB}$ of $G$. For noncompact Lie groups, however, we are not allowed to use such a form, since the Killing metric of noncompact groups is not positive-definite. In contrast, the $H$-invariant metric $I_{AB}$ can be positive-definite and we may, with the aid of $I_{AB}$, construct a Lagrangian density with positive-definite kinetic terms for $A_\mu$, even if the gauge group $G$ is noncompact. From $I_{AB}$ and $v(\phi)$, we define the scalar-field-dependent metric

$$G_{AB}(\phi)=I_{CD}A^c_{A}(v(\phi)^{-1})D^b_{B}(v(\phi)^{-1}), \tag{2\cdot12}$$

for which, by using (2·3) and (2·6), we can verify the transformation behavior

$$G_{AB}(\phi)\rightarrow G_{AB}(\phi')=G_{CD}(\phi')D^c_{A}(g^{-1})D^b_{B}(g^{-1}). \tag{2\cdot13}$$

Taking into account (2·11) and (2·13), we can set up a new type of Lagrangian density for $A_\mu$ as
\[ \mathcal{L}_A = -\frac{1}{4} G_{AB}(\phi) F_{\mu\nu}^A F^{\mu\nu B}. \]  
(2.14)

The total classical Lagrangian density of the system thus is given by
\[ \mathcal{L}_{cl} = \mathcal{L}_A + \mathcal{L}_\phi. \]  
(2.15)

This type of Lagrangian density was first given by Julia and Luciani,\(^9\) and has been treated by several authors.\(^6,13\)

§ 3. Lagrangian density for quantum theory

Let us introduce the Nakanishi-Lautrup fields \( B^A(x) \), the FP ghost fields \( C^A(x) \) and the FP anti-ghost fields \( \bar{C}^A(x) \) in order to quantize the system described by the Lagrangian density (2.15). Here we assume that \( C^A \) and \( \bar{C}^A \) are real Grassmann numbers with ghost numbers 1 and -1 respectively.

The BRS transformations, for \( A_\mu^B \) and \( \phi^a \) are determined respectively from (2.4) and (A.2) as follows:
\[ \delta A_\mu^B = \partial_\mu C^B - q C^D A_\mu^D f_{CD}^B, \]  
(3.1)

\[ \delta \phi^a = -q \kappa^{-1} C^A L_A^a(\phi), \]  
(3.2)

where \( L_A^a(\phi) \) are the Killing vectors on \( G/H \). We define the BRS transformation for a tensor field \( T_{a-\mu}^a(\phi) \) on \( G/H \), such as \( g_{ab}(\phi) \) and \( G_{ab}(\phi) \), as
\[ \delta T_{a-\mu}^a(\phi) = \delta \phi^c \partial_a T_{a-\mu}^c(\phi) = -q \kappa^{-1} C^A L_A^a(\phi), \]  
(3.3)

where \( \partial_a \equiv \partial/\partial \phi^a \) and \( L_A \equiv L_A^a(\phi) \partial_a \). To keep the relation \([\delta, \partial_a] = 0\) for \( T_{a-\mu}(\phi) \), we assume that \( \delta \) and \( \partial_a \) satisfy
\[ [\delta, \partial_a] = -\partial_b (\delta \phi^b)) \partial_a . \]  
(3.4)

This relation yields the transformation behavior (3.3) for \( \partial_a T_{a-\mu}(\phi) \). Similar to the usual Yang-Mills theory,\(^9,10\) we define the BRS transformations for \( C^A, \bar{C}^A \) and \( B^A \) as follows:
\[ \delta C^A = -\frac{q}{2} (C \times C)^A, \]  
(3.5)

\[ \delta \bar{C}^A = i B^A, \]  
(3.6)

\[ \delta B^A = 0, \]  
(3.7)

where \((U \times V)^A \equiv U^B V^C f_{CB}^A\). Then, the equalities \( \delta^2 A_\mu^A = \delta^2 C^A = \delta^2 \bar{C}^A = \delta^2 B^A = 0 \) are obvious. The equality \( \delta^2 \phi^a = 0 \) can be verified by using (3.3) for \( T^a(\phi) = L_A^a(\phi) \), (3.5) and (A.6). We can furthermore show \( \delta^2 T_{a-\mu}(\phi) = 0 \) with the help of (3.4). The equality \( \delta^2 g_{ab}(\phi) = 0 \) \( (\delta^2 G_{ab}(\phi) = 0) \) can also be checked from (A.17) and (A.6) ((A.24) and the Jacobi identity) in particular. The BRS transformation \( \delta \) therefore satisfies the nilpotency property: \( \delta^2 = 0 \).

Now, we construct the gauge-fixing term \( \mathcal{L}_{gf} \) and the FP ghost term \( \mathcal{L}_{fp} \) in such a way that \( \mathcal{L}_{gf} + \mathcal{L}_{fp} \) has the BRS invariance and the global gauge invariance. We
also note that in \( \mathcal{L}_{\text{gf}} \) and \( \mathcal{L}_{\text{fp}} \), the indices of the Lie algebra must be contracted with \( G_{\alpha B}(\phi) \) as in \( \mathcal{L}_A \). This will be important in order to determine the metric structure of the Fock space generated by asymptotic fields of the relevant fields. Taking these conditions and the usual Yang-Mills theory into account, we set the gauge-fixing and FP ghost terms in the following form:

\[
\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{fp}} = \delta \left[ i(1 - \omega) \partial^\alpha \bar{C} \star A_\mu - i \omega (\bar{C} \star \partial^\alpha A_\mu - \partial^\alpha (\bar{C} \star A_\mu)) - \frac{i}{2} a \bar{C} \star B \right],
\]

where \( a \) is a gauge parameter and the \(*\)-product is defined by \( U \star V \equiv G_{AB}(\phi) U^A V^B \). Here we note that the \(- \bar{C} \star \partial^\alpha A_\mu + \partial^\alpha (\bar{C} \star A_\mu) \) is not equivalent to the \( \partial^\alpha \bar{C} \star A_\mu \) unlike the case of the usual Yang-Mills theory, since \( G_{AB}(\phi) \) depends on \( (x^\mu) \) through \( \phi^a \). We hence distinguished these two terms with the parameter \( \omega \left( 0 \leq \omega \leq 1 \right) \). Carrying out the BRS transformation explicitly in (3.8) and using Eqs. (A.13), (A.20), (A.21) and (A.23)\textendash(A.25), \( \mathcal{L}_{\text{gf}} \) and \( \mathcal{L}_{\text{fp}} \) are respectively obtained as

\[
\mathcal{L}_{\text{gf}} = - \partial^\alpha B \star A_\mu + \frac{1}{2} \alpha B \star B + \omega(k) \partial^\alpha \phi^a \left[ (\bar{e}_a \times B) \star A_\mu + (\bar{e}_a \times A_\mu) \star B \right],
\]

\[
\mathcal{L}_{\text{fp}} = - i \partial^\alpha \bar{C} \star \partial_\mu C + i \omega \left( (\partial^\alpha \bar{C} \star C) \star A_\mu + \omega (\partial^\alpha C \times \bar{C}) \star A_\mu + C A_\mu \right)
\]

\[
+ i \omega(k) \partial^\alpha \phi^a \left[ (\bar{e}_a \times \bar{C}) \star \partial_\mu C - (\bar{e}_a \times \partial_\mu C) \star \bar{C} - q (C \times \bar{C}) \star A_\mu \right]
\]

\[
+ (\bar{e}_a \times A_\mu) \star (C \times \bar{C}) - \frac{i}{2} a \bar{C} \star B - (C \times B) \star \bar{C} = 0.
\]

where \( \bar{e}_a \) are defined in (A.22). By adding \( \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{fp}} \) to the classical Lagrangian density (2.15), we get the total Lagrangian density for quantum gauge-field theory:

\[
\mathcal{L}_{\text{tot}} = \mathcal{L}_A + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{fp}}.
\]

It should be noted that the totally antisymmetric property of \( f_{ABC}(\phi) \equiv f_{AB} C_G(\phi) \) does not hold in general with respect to the indices \( A, B, C \) (see (A.21)). Hence, compared with the usual Yang-Mills theory, the gauge fixing term and the FP ghost term are much complicated and except for the Landau gauge \( \alpha = 0 \), the third order terms with respect to \( B^A, C^A \) and \( \bar{C}^A \) remain nonvanishing in \( \mathcal{L}_{\text{fp}} \). These third order terms, which are necessary for \( \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{fp}} \) to be BRS invariant, yield fourth order terms with respect to \( C^A \) and \( \bar{C}^A \) after the elimination of \( B^A \). Thus the gauge-fixing and FP ghost terms (3.8) cannot be derived from the usual path-integral formalism.\textsuperscript{13}

From the Lagrangian density (3.11), we derive, the Euler-Lagrange equations of motion for \( A_\mu^B \) and \( B^A \):

\[
\partial_\nu F^{\nu A}_B + q A^{C B} f_{C A} B^\mu - (\partial^\mu B)_A + i q \left( (\partial^\alpha \bar{C} \times C)_A + \omega (\partial^\alpha C \times \bar{C})_A - \omega \partial^\alpha C^B f_{B A} \bar{C} \right)
\]

\[
+ \omega(k) \partial^\alpha \phi^a \left[ (\bar{e}_a \times B)_A - \bar{e}_a^B f_{B A} C - i q ((\bar{e}_a \times (C \times \bar{C})))_A - \bar{e}_a^B f_{B A} (C \times \bar{C} C) \right]
\]

\[
+ \mu^2 q k^{-1} L_A^a g_{a b} D^\mu \phi^b = 0
\]

and

\[
(\partial_\mu A^\mu)_A + \alpha \left[ B_A - \frac{i}{2} q ((C \times \bar{C})_A - C^B f_{B A} \bar{C} C) \right] = 0.
\]
\[(1 - \omega) \kappa \partial^\mu \phi^a ((\bar{\epsilon}_a \times A_\nu)_A - \bar{\epsilon}_a \beta_B A_{\nu B} C A_{\mu C}) = 0, \quad (3.13)\]

where \( U_A \equiv G_{AB}(\phi) U_B \) and \((\partial_\mu U)_A \equiv G_{AB}(\phi) \partial_\mu U^B \).

Invariance of the Lagrangian density (3.11) under the BRS transformation yields, after the use of Eq. (3.12), the BRS current

\[
\mathbf{J}^{(b)}_{\mu} = B \star \partial^\mu C - \partial^\mu B \star C + q \left\{ (1 - \omega) (C \times B) \star A^\mu - \omega (C \times A^\mu) \star B \right. \\
+ iC \star (\partial^\mu \bar{C} \times C) - \frac{i}{2} \partial^\mu \bar{C} \star (C \times \bar{C}) + i\omega C \star (\partial^\mu C \times \bar{C}) - i\omega \partial^\mu C \star (C \times \bar{C}) \right. \\
+ \left. i\omega q^2 ((C \times \bar{C}) \star (C \times A^\mu) - (C \times (C \times A^\mu)) \star \bar{C}) + \omega \kappa \partial^\mu \phi^a \left( \bar{\epsilon}_a \times (B \\
- i\omega C \times \bar{C}) \right) \star C + (\bar{\epsilon}_a \times C) \star (B - i\omega C \times \bar{C}) + \frac{i}{2} q ((\bar{\epsilon}_a \times (C \times C)) \star \bar{C} \\
+ (\bar{\epsilon}_a \times \bar{C}) \star (C \times C)) \right\} + \partial_\nu (C \star F^{\nu \mu}). \quad (3.14)\]

The Lagrangian density (3.11) is also invariant under the scale transformation: \( \delta S C^a = C^a, \delta S \bar{C}^a = - \bar{C}^a \), and hence we have the corresponding current, that is, the FP ghost current

\[
\mathbf{J}^{(c)}_{\mu} = i \bar{C} \star \partial^\mu C - i \partial^\mu \bar{C} \star C - i\omega q^2 \left( (1 - \omega) (C \times \bar{C}) \star A^\mu - \omega (C \times A^\mu) \star \bar{C} \right) \\
+ i\omega \kappa \partial^\mu \phi^a \left( \bar{\epsilon}_a \times C \right) \star C - (\bar{\epsilon}_a \times C) \star \bar{C}. \quad (3.15)\]

For the currents (3.14) and (3.15), one can verify, with the help of the formulas given in Appendix A, the transformation behavior

\[
\delta \mathbf{J}^{(c)}_{\mu} = - \mathbf{J}^{(b)}_{\mu} + \partial_\nu (C \star F^{\nu \mu}) \quad (3.16)
\]

and

\[
\delta \mathbf{J}^{(b)}_{\mu} = \partial_\nu [\delta (C \star F^{\nu \mu})], \quad (3.17)
\]

which indicate the property of the BRS algebra.

\section*{§ 4. Quantization}

The conjugate momenta \( \pi^{A_A}_A = \partial L_{\text{Tot}} / \partial \dot{\phi}_A \) for the relevant fields \( \phi_A = (A_\mu^A, \phi^a, B^4, C^A, \bar{C}^a) (A' = a \text{ for } \phi^a \text{ and } A' = A \text{ for the others}) \) are given as follows:

\[
\begin{align*}
\pi^{A_A}_A &= 0, \quad (4.1a) \\
\pi^{A_A}_A &= F_{BA}, \quad (i = 1, 2, 3) \quad (4.1b) \\
\pi^a &= \mu^2 g_{ab} D_b \phi^b + \omega \kappa ((\bar{\epsilon}_a \times (B - i\omega C \times \bar{C})) \star A_0 + (\bar{\epsilon}_a \times A_0) \star (B - i\omega C \times \bar{C}) \\
&+ i(\bar{\epsilon}_a \times \bar{C}) \star \bar{C} - i(\bar{\epsilon}_a \times \bar{C}) \star C), \quad (4.1c) \\
\pi^B_A &= - A_{0A}, \quad (4.1d)
\end{align*}
\]
\[ \pi^c_A = i(\partial)\phi - i\omega q(\partial^b f_{ba} c A_{0c} + A_b^b f_{ba} c C_c) - i\omega \phi^a((\bar{e}_a \times C)_A - \bar{e}_a^b f_{ba} c C_c), \]  
\[ \pi^{\bar{c}}_A = -i(\partial)\phi + i\omega q f_{ba} c A_{0c}, \]  
where \((U)_A = G_{AB}(\phi) U^B\). It should be noted that (4.1d) and (4.1a) give the constraints \(\chi_{1A} = \pi^{\phi_A} + A_{0A} = 0\) and \(\chi_{2A} = \pi^{A_a} A_a\) respectively.

We next consider the Poisson bracket, which is defined as
\[ \{f(\Phi_i(t, x), \pi^{\phi_i}(t, x)), g(\Phi_i'(t, y), \pi^{\phi_i'}(t, y))\}_P. \]
\[ = \sum_{T, T'} \left[ (-1)^{r_{\phi_i}} \frac{\partial f}{\partial \Phi_i'} \frac{\partial g}{\partial \pi^{\phi_i'}} - (-1)^{r_{\phi_i} + r_{\phi_i'}} \frac{\partial g}{\partial \Phi_i} \frac{\partial f}{\partial \pi^{\phi_i}} \right] \delta^3 \]  
\[ \delta^3 = \delta^3(x - y) \]
for any functions \(f, g\) of the canonical variables \(\Phi_i\) and \(\pi^{\phi_i}\). Here, in the exponents of \((-1)\), \(\Phi_i\) takes \(1(0)\) for \(\Phi_i = \mathcal{C}^A\), \(\bar{C}^A\) (the other fields) and \(f\) take \(1(0)\) if the function \(f\) is odd (even) order monomial with respect to the Grassmann numbers. The same rule also holds for \(g\). The Poisson brackets (4.2) for \(\chi_{0A}\) \((a = 1, 2)\) yield the following matrix:
\[ M_{0A, Tb} = (\chi_{0A}, \chi_{0B})_P = \begin{pmatrix} 0 & G_{AB} \\ -G_{AB} & 0 \end{pmatrix} \delta^3. \]  
In what follows, we assume that \(G_{AB}(\phi)\) has the inverse, \(G^{AB}(\phi)\); that is, all the gauge fields are propagated. Then \(\chi_{0A} = 0\) become second class constraints\(^{15}\) because \(M_{0A, Tb}\) has the inverse. To quantize systems with second class constraints, we have to use the Dirac bracket. In the present case, it is defined by
\[ \{f, g\}_D = \{f, g\}_P + \int d^3 z f^R G^{AB}(\phi(t, z)) \{f, \chi_{1A}(t, z)\}_P. \]
\[ -1(1 \rightarrow 2)), \]  
(4.4)
Then we find that
\[ (A_0^A, \pi^{A_a})_D = 0, \]  
(4.5a)
\[ (A_0^A, B^B)_D = G^{AB} \delta^3, \]  
(4.5b)
and
\[ (A_0^A, \pi^a)_D = -G^{AB} A_0^C \partial_0 G^{BC} \delta^3. \]  
(4.6)
The other Dirac brackets for the canonical variables are reduced to the Poisson brackets for the same canonical variables. We here notice that, unlike the usual Yang-Mills theory, the Dirac bracket (4.6) remains non-vanishing owing to the \(\phi^a\)-dependence of \(G_{AB}\).

The canonical quantization is carried out with the canonical (anti-)commutation relation
\[ [f, g](\equiv fg - (-1)^{r_f} rf) = i(f, g)_D, \]  
(4.7)
which is defined taking into account the property \((f, g)_0 = -(-1)^{f_0} (g, f)_0\) of the Dirac bracket \((4\cdot4)\). From \((4\cdot7)\), the equal-time (anti-)commutation relations 
\([\Phi^{A}\,(t, x), \Phi^{B}\,(t, y)]\) can be calculated. First, through the help of \((4\cdot1b), (4\cdot1c), (4\cdot1e)\) and \((4\cdot1f)\), we obtain the following (anti-)commutation relations:

\[
\begin{align*}
[A_{\alpha}^{A}, \dot{A}_{\beta}] &= iG^{AB}\delta_{\alpha}^{A}\delta^{3}, \\
[B_{\alpha}^{A}, \dot{A}_{\beta}^{B}] &= iG^{AC}(\delta_{\alpha}^{C}\delta_{\beta}^{B} + qA_{\alpha} f_{dc}^{B})\delta^{3}, \\
[B_{\alpha}^{A}, \dot{C}_{\beta}^{B}] &= -i\mu G^{BC}C^{B}\delta^{3}, \\
[C_{\alpha}^{A}, \dot{C}_{\beta}^{B}] &= -\{C_{\alpha}^{A}, \dot{C}_{\beta}^{B}\} = G^{AB}\delta^{3}, \\
[A_{0\alpha}^{A}, \mu^{\alpha}] &= -i(1-\omega)\mu^{-1}g^{ab}G^{AB}A_{0\alpha}^{A}\partial_{\beta}G_{Bc}\delta^{3}, \\
[\mu^{\alpha}, \mu^{\beta}] &= i\mu^{\alpha}[\mu^{\beta}], \\
[B_{\alpha}^{A}, \mu^{\alpha}] &= (i\mu \omega^{-1}G^{AB}L_{\alpha}^{0} - \omega^{-1}g^{ab}J^{ABC}\partial_{\beta}G_{BC})\delta^{3}, \\
[C_{\alpha}^{A}, \mu^{\alpha}] &= [\mu^{\alpha}, \dot{C}_{\alpha}^{A}] = -i\omega^{-1}g^{ab}G^{AC}\bar{C}_{\beta}^{C}\partial_{\beta}G_{Bc}\delta^{3}, \\
[A_{0\alpha}^{A}, \dot{C}_{\beta}^{B}] &= i(1-\omega)\omega^{-1}g^{ab}G^{AC}G^{BD}A_{0\alpha}^{E}\bar{C}_{\beta}^{C}\partial_{\beta}G_{cB}\delta^{3}, \\
[B_{\alpha}^{A}, \dot{C}_{\beta}^{B}] &= \omega\{\mu^{\alpha}, \dot{C}_{\beta}^{B}\} = [\mu^{\alpha}, \dot{C}_{\beta}^{B}] = G^{AC}\bar{C}_{\beta}^{C}C_{\beta}^{D}\partial_{\beta}G_{cB}\delta^{3}, \\
[C_{\alpha}^{A}, \dot{C}_{\beta}^{B}] &= i(\omega^{-1})g^{ab}G^{AC}G^{BD}\bar{C}_{\beta}^{C}C_{\beta}^{D}\partial_{\beta}G_{cB}\delta^{3},
\end{align*}
\]

where \(J^{ABC} \equiv iG^{AB}(B^{C} - i\omega \partial_{C}G_{Bc}) - qG^{DB}C^{C}C_{\beta}^{D}\). The other equal-time (anti-)commutation relations involving \(A^{A}, \dot{A}^{A}, C^{A}\) and \(\dot{C}^{A}\) vanish. Use of Eq. \((3\cdot12)\) yields the commutation relations involving \(B^{A}\):

\[
\begin{align*}
[A_{0\alpha}^{A}, \dot{B}_{\beta}^{B}] &= \{i(1-\omega)\omega^{-1}qG^{BD}(A_{0\alpha}^{E}\dot{f}_{bc}^{A} + G^{AB}\dot{f}_{bc}^{A})
+ (1-\omega)\omega^{-1}g^{ab}J^{BCF}G^{AH}A_{0\alpha}^{D}\partial_{\beta}G_{cB} - i\omega^{-1}g^{ab}G^{AC}G^{BD}\partial_{\beta}G_{cB}\}\delta^{3}, \\
[B_{\alpha}^{A}, \dot{B}_{\beta}^{B}] &= \omega\{\mu^{\alpha}, \dot{B}_{\beta}^{B}\} = [\mu^{\alpha}, \dot{B}_{\beta}^{B}] = G^{AC}\bar{C}_{\beta}^{C}C_{\beta}^{D}\partial_{\beta}G_{cB}\delta^{3}, \\
[C_{\alpha}^{A}, \dot{B}_{\beta}^{B}] &= \omega\{\mu^{\alpha}, \dot{B}_{\beta}^{B}\} = [\mu^{\alpha}, \dot{B}_{\beta}^{B}] = G^{AC}\bar{C}_{\beta}^{C}C_{\beta}^{D}\partial_{\beta}G_{cB}\delta^{3}, \\
\end{align*}
\]

and the others which are identical with the commutation relations obtained from \((4\cdot8b), (4\cdot8c), (4\cdot8g)\) and \((4\cdot8j)\) through the Leibniz rule: 
\([\Phi^{A}, \Phi^{B}] = \partial_{\alpha}[\Phi^{A}, \Phi^{B}] + [\Phi^{A}, \Phi^{B}], \] 
Furthermore, using Eq. \((3\cdot13)\), the commutation relations involving \(A_{\alpha}^{A}\) are obtained consistently with the Leibniz rule. We here give only the necessary commutation relation

\[
[A_{0\alpha}^{A}, \dot{A}_{\beta}^{A}] = \{i(1-\omega)\mu^{-2}g^{ab}G^{AC}G^{BD}A_{0\alpha}^{E}\partial_{\beta}G_{cB}\delta^{3} - i\omega G^{AB}\}\delta^{3}.
\]

By making use of the (anti-)commutation relations \((4\cdot8)\)~\((4\cdot10)\), one can show that the BRS charge \(Q_{(B)} = \int d^{3}x J_{(B)}^{0}\) defined from the BRS current \((3\cdot14)\) generates the BRS transformation as
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\[ [Q_{(b)}, \Phi^A] = -i \delta \Phi^A. \quad (4.11) \]

Similarly, the FP ghost charge \( Q_{(c)} = \int d^3 x j_{(c)} \) defined from the FP ghost current \((3.15)\) generates the scale transformation. We find

\[ [Q_{(c)}, \Phi^A] = -i \delta_3 \Phi^A, \quad (4.12) \]

where \( \delta_3 \Phi^A = (0, 0, 0, C^A, -\bar{C}^A) \). Furthermore, using \((3.17)\) and \((3.18)\), we can verify the BRS algebra:

\[ [Q_{(b)}, Q_{(c)}] = i Q_{(b)}, \quad (4.13) \]

\[ Q_{(b)}^2 = 0. \quad (4.14) \]

\section{5. Asymptotic fields}

Let us assume the existence of asymptotic fields and the asymptotic completeness. If we set up appropriate asymptotic conditions, asymptotic in-fields \( \Phi^{(\text{in})} \) (or similarly asymptotic out-fields \( \Phi^{(\text{out})} \)) of the relevant Heisenberg fields \( \Phi^A \) will be described by the quadratic part of the Lagrangian density \((3.11)\) in which the bare constants are replaced with the corresponding physical ones. We can put the condition \( v(0) = 1 \) (1 : the unit element of \( G \)) to the coset representative \( v(\phi) \) without loss of generality. Then, noticing that \( v(\phi) \) can be expanded around the unit element as \( v(\phi) = 1 + i \kappa \phi^a T_a + O((\kappa \phi)^3) \), we get the quadratic part of \((3.11)\):

\[ L_{\text{as}} = -\frac{1}{4} I_{ab}(\partial_\nu A^a - \partial_\nu A^b)(\partial^\nu A^{ab} - \partial^\nu A^{ba}) \]

\[ + \frac{1}{2} J_{ab}(\mu \partial_\nu \phi^a + mA^a)(\mu \partial^\nu \phi^b + mA^b) \]

\[ - I_{ab} \delta^a B^a A^b + \frac{1}{2} \alpha I_{ab} B^a B^b - i I_{ab} \bar{C}^a \partial_\mu C^b. \quad (5.1) \]

We regard, hereafter, the fields in \((5.1)\) as the asymptotic fields \( \Phi^{(\text{in})} \) (or \( \Phi^{(\text{out})} \)) and regard the constants \( \alpha \) and \( m \equiv (q/\kappa) \mu \) as physical ones. The field equations of the asymptotic fields are obtained as follows:

\[ \Box A^a_\nu - \partial_\nu \partial^\mu A^a_\mu - \partial_\nu B^a + m(\mu \partial_\nu \phi^a + mA^a) \delta^a_\nu = 0, \quad (5.2) \]

\[ \partial^\mu A^a_\mu + a B^a = 0, \quad (5.3) \]

\[ \mu \Box \phi^a + mA^a = 0, \quad \Box C^a = \Box \bar{C}^a = \Box B^a = 0. \quad (5.4) \]

The equal-time (anti-)commutation relations \([\Phi^{(\text{in})}, \Phi^{(\text{in})}]\) (or similarly \([\Phi^{(\text{out})}, \Phi^{(\text{out})}]\)) obtained from \((5.1)\) have, except for \([B^a, \mu \phi^a] = i m I^{ab} \delta^a_3\), the identical forms as \((4.8)\)~\((4.10)\) with the condition \( \kappa = q = 0 \). Then, the 4-dimensional (anti-)commutation relations for the asymptotic fields can be calculated in the usual way. The results are
\[ [A^a_\mu(x), A^b_\nu(y)] = i I^{ab} \left\{ - \left( \eta_{\mu\nu} + \frac{1}{m^2} \partial^\tau \partial^\nu \right) \Delta(x-y;m^2) 
+ \partial^\tau \partial^\nu \left( \frac{1}{m^2} D(x-y) - a E(x-y) \right) \right\}, \] 
\[ (5.5a)^* \]
\[ [A^a_\mu(x), A^b_\nu(y)] = i I^{ab} \left\{ - \eta_{\mu\nu} D(x-y) + (1-\alpha) \partial^\tau \partial^\nu E(x-y) \right\}, \] 
\[ (5.5b) \]
\[ [A^a_\mu(x), B^b_\nu(y)] = - i I^{AB} \partial^\tau \partial^\nu D(x-y), \] 
\[ (5.5c) \]
\[ [A^a_\mu(x), \mu \phi^b(y)] = - i m I^{ab} \partial^\tau \partial^\nu E(x-y), \] 
\[ (5.5d) \]
\[ [B^a_\mu(x), \mu \phi^b(y)] = i m I^{ab} D(x-y), \] 
\[ (5.5e) \]
\[ [\mu \phi^a(x), \mu \phi^b(y)] = i I^{ab} (D(x-y) + a m^2 E(x-y)), \] 
\[ (5.5f) \]
\[ (C^A_\mu(x), \bar{C}^B_\nu(y)) = - I^{AB} D(x-y), \] 
\[ (5.5g) \]

and the other (anti-)commutation relations vanish. Here \( I^{AB} \) is the inverse of the \( H \)-invariant metric \( I_{AB} \). The commutation relation (5.5a) indicates that, in addition to the massive vector modes, the \( A^a_\mu \) contain the massless scalar modes. We hence separate the massive vector modes, \( U^a_\mu \), contained in \( A^a_\mu \) as

\[ U^a_\mu = A^a_\mu - \frac{1}{m^2} \partial^\nu (B^a_\mu - m \mu \phi^a), \] 
\[ (5.6) \]

which indeed satisfy the 4-dimensional commutation relation of the Proca fields:

\[ [U^a_\mu(x), U^b_\nu(y)] = - i I^{ab} \left( \eta_{\mu\nu} + \frac{1}{m^2} \partial^\tau \partial^\nu \right) \Delta(x-y;m^2). \] 
\[ (5.7) \]

The 4-dimensional commutation relation between \( U^a_\mu \) and any of the fields \( A^a_\mu \), \( \phi^a \), \( B^A \), \( C^A \) and \( \bar{C}^A \) vanishes. From (5.5) and (5.7), we obtain the following (anti-)commutation relations for the creation and annihilation operators of the respective asymptotic fields.\(^3\)

\[
\begin{array}{c|cccc|cc}
 & U_i^a & A_i^a & \mu \phi_i^a & A_{i\lambda}^a & B_i^a & C_i^a & \bar{C}_i^a \\
\hline
U_i^a & \delta_{ij} I^{ab} & 0 & 0 & 0 & 0 & 0 & 0 \\
A_i^a & 0 & \delta_{AI} I^{AB} & 0 & 0 & 0 & 0 & 0 \\
\mu \phi_i^a & 0 & I^{ab} & 0 & m I^{ab} & 0 & 0 & \times \delta_{kl} \\
A_{i\lambda}^a & 0 & 0 & 0 & 0 & - I^{ab} & 0 & \times \delta_{kl} \\
B_i^a & 0 & m I^{ab} & - I^{AB} & 0 & 0 & 0 & \times \delta_{kl} \\
C_i^a & 0 & 0 & 0 & 0 & 0 & i I^{AB} & \times \delta_{kl} \\
\bar{C}_i^a & 0 & 0 & 0 & 0 & - i I^{AB} & 0 & 0
\end{array}
\]

\[ (i, j=1, 2, 3; \ \sigma, \tau=1, 2; \ k, l=1, 2, 3, \cdots) \]

from which the metric structure of the total Fock space of the asymptotic fields is

\[^3\text{diag} \eta_{\mu\nu} = (1, -1, -1, -1).\]
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Next, we verify the BRS transformations of the relevant asymptotic fields $\psi_i^{A'} = (U_i^a, A_i^a, \mu^A, B_i^A, C_i^A, \bar{C}_i^A)$. In terms of the asymptotic fields, the renormalized BRS charge, $Q_{(B)}$, is expressed as

$$Q_{(B)} = \int d^3 x I_{\mathcal{AB}} (B_i^A \bar{C}_i^B - \bar{B}_i^A C_i^B),$$  \hspace{1cm} (5\cdot9)

which generates the BRS transformation of the asymptotic fields:

$$[Q_{(B)}, \psi_i^{A'}] = -i \delta \psi_i^{A'},$$  \hspace{1cm} (5\cdot10a)

where

$$\delta \psi_i^{A'} = (0, \partial_i C_i^a, -m C_i^a, 0, 0, iB_i^A).$$  \hspace{1cm} (5\cdot10b)

The transformation property (5\cdot10) shows that the transverse and longitudinal modes $U_i^a, A_i^a$ and the transverse modes $A_i^a$ belong to BRS-singlet representations, while the longitudinal mode $A_i^a$ and the scalar modes $\mu^A, B_i^A, C_i^A$ and $\bar{C}_i^A$ form a BRS-quartet.

By using (5\cdot8), (5\cdot10) together with hermiticity $Q_{(B)}^\dagger = Q_{(B)}$, we can prove the Kugo-Ojima theorem. \(^9\) $\langle f | f \rangle = \langle f | P^{(0)} | f \rangle$ holds for any state vector satisfying the subsidiary condition

$$Q_{(B)} f = 0,$$  \hspace{1cm} (5\cdot11)

where $P^{(0)}$ is the projection operator onto the Hilbert subspace $\mathcal{H}$ which is spanned solely by the BRS-singlet modes: $U_i^a$ and $A_i^a$. We can also put the condition $Q_{(C)} f = 0$, if necessary. From (5\cdot8), we see that even if $G$ is noncompact, the Hilbert subspace $\mathcal{H}$ has the positive-definite metric as long as the $H$-invariant metric $I^{AB}$ is positive-definite. In such a positive-definite case, all of the BRS-singlet modes can be regarded as the genuine physical particles, and the Kugo-Ojima theorem guarantees that any state vector satisfying the condition (5\cdot11) has positive-norm or zero-norm. Consequently, we get the unitary "physical" $S$-matrix, as usual. \(^9\)

§ 6. Summary and discussion

The results obtained in this paper are summarized as follows: First, we have constructed a new type of Lagrangian density which can give positive-definite kinetic terms for gauge fields, even if the gauge group is noncompact.

To quantize the classical system, we have introduced the gauge-fixing term and the FP ghost term in a BRS invariant manner. These terms are much complicated in comparison with those of the usual Yang-Mills theory, because the totally antisymmetric property of $f_{ABC}(=f_{AB}^\mu G_{DC}(\phi))$ does not hold in general. One will not be able to derive such complicated terms from the Faddeev-Popov path-integral formalism. Complexity of the total Lagrangian density is furthermore caused by the non-polynomial structure of the scalar-field-dependent metric $G_{AB}(\phi)$.

The canonical quantization of the system has been carried out by using Dirac's procedure. We have calculated the equal-time (anti-)commutation relations and
have found that the BRS transformation and the BRS algebra are exactly generated at the quantum level by the BRS charge.

The properties of the asymptotic fields have been investigated from the quadratic part of the total Lagrangian density. We have found that the gauge fields \( A_\mu^a \) associated with the coset space \( G/H \) become massive fields, while the gauge fields \( A_\mu^a \) for the subgroup \( H \) still remain massless. The Feynman propagator of \( A_\mu^a \) is obtained from the commutation relation (5.5a) as

\[
-\frac{i}{(2\pi)^4} I^{ab} \left[ \frac{1}{k^2 - m^2} \left( \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + a \left( \frac{k_\mu k_\nu}{k^2} \right) \right],
\]

which has, unlike the propagator of the Proca fields, the same high-energy behavior as that of massless gauge fields and satisfies a condition necessary for the renormalizability. This desirable result is owing to the freedom of the unphysical scalar fields \( \phi^a \) and \( B^a \). The complete proof of the renormalizability, however, is much more difficult owing to the non-polynomial structure of interaction terms involving \( \phi^a \). As mentioned already, the unitarity of the physical S-matrix is guaranteed by the Kugo-Ojima theorem as long as the metric \( I_{ab} \) is positive definite.

We finally comment on the anti-BRS transformation: \(^{10,16,17}\)

\[
\delta A_\mu^a = \partial_\mu \tilde{C}^a - \eta C^d A_\mu^c f_{cd}^b, \quad \delta \phi^a = - q \alpha^{-1} \tilde{C}^A L_A^a, \]

\[
\delta C^A = - \frac{q}{2} (\tilde{C} \times \tilde{C})^A, \quad \delta A^A = - i B^A - q (C \times \tilde{C})^A, \quad \delta B^A = - q (\tilde{C} \times B)^A. \tag{6.2}
\]

Unlike the case of the usual Yang-Mills theory, \(^{16}\) the gauge-fixing and FP ghost terms (3.8) are not invariant under the anti-BRS transformation (6.2) except for the \( \partial^\mu \tilde{C} \ast A_\mu \), which can be written as \( \delta (A_\mu \ast A^a) / 2 \). Thus, if we require both of the BRS and anti-BRS invariance on (3.8), the parameters \( \alpha \) and \( \omega \) must be zero. The non-Landau gauge term with the BRS and anti-BRS invariance is given by \( \delta \delta (C \ast \tilde{C}) \) instead of \( - i \delta (\tilde{C} \ast B) \). Carrying out the BRS and anti-BRS transformations, we find that \( \delta \delta (C \ast \tilde{C}) \) contains the third and fourth order terms with respect to \( B^A, C^A \), and \( \tilde{C}^A \). These higher order terms are necessary for the BRS and anti-BRS invariance, since \( \delta (B \ast B) = 2 (C \times B) \ast B \neq 0 \).

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Appendix A

In this appendix, we derive a number of formulas used in the text from the geometrical properties of the coset space \( G/H \). \(^{18}\)

To get geometrical formulas, let us first consider the infinitesimal gauge transfor-
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\[ g = 1 - iq \varepsilon^A T_A, \]  
\[ (A\cdot1) \]

where \( \varepsilon^A \) are infinitesimal parameters depending on \( (x^a) \). Through the transformation (2·3), the infinitesimal transformation (A·1) gives rise to the infinitesimal coordinate transformation in \( G/H \)

\[ \phi^a \rightarrow \phi^a = \phi^a - q \kappa^{-1} \varepsilon^A L_A^a(\phi) \]  
\[ (A\cdot2) \]

and the infinitesimal rotation in the tangent space of \( G/H \)

\[ h(\phi, g) = 1 - iq \varepsilon^A \Omega_a^a(\phi) T_a, \]  
\[ (A\cdot3) \]

where the vectors \( L_A^a(\phi) \) are Killing vectors and \( \Omega_a^a(\phi) \) are called \( H \)-compensators. Substituting \( (A\cdot1) \sim (A\cdot3) \) into (2·3), one can show that

\[ L_A v(\phi) = i \kappa (T_A v(\phi) - v(\phi) \Omega^a_a(\phi) T_a), \]  
\[ (A\cdot4) \]

where \( \partial_a = \partial / \partial \phi^a \) and \( L_A = L_A^a(\phi) \partial_a \). Further, the associativity \( (g_1 g_2) v(\phi) = g_1(g_2 v(\phi)) \) gives

\[ L_A \Omega_b^b(\phi) - L_B \Omega_a^a(\phi) = \kappa (f_{AB}^C \Omega_c^c(\phi) + \Omega_a^a(\phi) \Omega_b^b(\phi) f_{ab}^c) \]  
\[ (A\cdot5) \]

and

\[ [L_A, L_B] = \kappa f_{AB}^C L_C, \]  
\[ (A\cdot6) \]

where \( f_{AB}^C \) are structure constants in the commutation relation \([T_A, T_B] = i f_{AB}^C T_C\).

We next define the following 1-form on \( G/H \):

\[ e(\phi) = d \phi^a e_a(\phi) = \frac{1}{i \kappa} v(\phi)^{-1} d v(\phi), \quad (d = d \phi^a \partial_a = d x^a \partial_a) \]  
\[ (A\cdot7) \]

which belongs to the Lie algebra of \( G \). Hence, it can be expanded in terms of \( \{T_a\} \) as

\[ e^A_a(\phi) T_A = e^a_a(\phi) T_a + e^a_{(a)}(\phi) T_a = \frac{1}{i \kappa} v(\phi)^{-1} \partial_a v(\phi). \]  
\[ (A\cdot8) \]

The components \( e^a_{(a)}(\phi) \) constitute vielbeins on \( G/H \) and the components \( e^a_a(\phi) \) are called \( H \)-connections.\(^*)\) Multiplying (A·8) by \( L^a_a(\phi) \) and using Eq. (A·4), we obtain

\[ L^a_a(\phi) e^a_a(\phi) = D^b_b(\psi(\phi)^{-1}) - \Omega_a^a(\phi), \]  
\[ (A\cdot9) \]

\[ L^a_a(\phi) e^a_{(b)}(\phi) = D^b_b(\psi(\phi)^{-1}), \]  
\[ (A\cdot10) \]

where the matrix \( D^b_b \) is the adjoint representation of \( G \) defined by

\[ g T_A g^{-1} = D^b_b(g) T_B. \quad (g \in G) \]  
\[ (A\cdot11) \]

Substituting (A·3) into (A·11), we obtain the adjoint representation of (A·3):

\[ D^b_b(h) = \delta^b_b + q e^c_c \Omega^c(\phi) f_{AB}^c. \]  
\[ (A\cdot12) \]

\(^*)\) When we need to distinguish the indices of the local frame basis from the indices of the coordinate basis, we shall write the former with parentheses.
Then, (2.6) with \( D_A^a(h) \) given by (A.12) yields the (partial) antisymmetric property
\[
f_{ab}^c I_{dc} + f_{ac}^d I_{db} = 0. \tag{A.13}
\]
Now, from (2.3) and (A.7), we have
\[
\hbar v(\phi)^{-1} \left( \frac{\partial \phi^a}{\partial \phi^b} \left( v(\phi) h^{-1} \right) - \frac{\partial v(\phi)}{\partial \phi^a} \right) = i\kappa (e_a(\phi') - e_a(\phi)). \tag{A.14}
\]
After the substitution of (A.2) and (A.3), Eq. (A.14) is decomposed into two pieces:
\[
L_A e_a^b(\phi) + (\partial_a L_A^c(\phi)) e_c^b(\phi) = \kappa \Omega_A^a(\phi) e_a e^c(\phi) f_{cb} - \partial_a \Omega_A^c(\phi) \tag{A.15}
\]
and
\[
L_A e_a^{(b)}(\phi) + (\partial_a L_A^c(\phi)) e_c^{(b)}(\phi) = \kappa \Omega_A^a(\phi) e_a e^{(c)}(\phi) f_{ca}^b. \tag{A.16}
\]
We notice that the antisymmetric property (A.13) and Eq. (A.16) lead to the Killing-vector condition
\[
L_A g_{ab}(\phi) + (\partial_a L_A^c(\phi)) g_{cb}(\phi) + (\partial_b L_A^c(\phi)) g_{ca}(\phi) = 0, \tag{A.17}
\]
where \( g_{ab}(\phi) \) is the metric tensor on \( G/H \) defined by
\[
g_{ab}(\phi) = I_{cb} e_a^{(c)}(\phi) e_b^{(a)}(\phi). \tag{A.18}
\]
Equation (A.17) shows that \( L_A^a(\phi) \) in (A.2) are indeed Killing vectors on \( G/H \).

Let us consider the metric \( G_{AB}(\phi) \) defined in (2.12). By taking the total derivative of (A.11) for \( g = v(\phi)^{-1} \) and using (A.7), we obtain
\[
dD_A^a(v(\phi)^{-1}) = \kappa D_A^{(c)(v(\phi)^{-1}) e^a(\phi) f_{dc}^b}. \tag{A.19}
\]
This relation and the invariance of the structure constant:
\[
D_A^a(g) D_B^b(g) f_{de} = f_{AB} D_c^f(g) \tag{A.20}
\]
yield
\[
dG_{AB}(\phi) = \kappa \bar{\epsilon}^c(\phi) (f_{ca}^p G_{db}(\phi) + f_{cb}^p G_{da}(\phi)). \tag{A.21}
\]
Here \( \bar{\epsilon}^a(\phi) \) are components of the 1-form
\[
\bar{\epsilon}(\phi) = d\phi^a \bar{\epsilon}_a^A(\phi) T_A = \frac{1}{i\kappa} (dv(\phi)) v(\phi)^{-1} = e^a(\phi) D_a^A(v(\phi)) T_A \tag{A.22}
\]
and satisfy
\[
L_A^a(\phi) \bar{\epsilon}_a^B(\phi) = \delta_A^B - \Omega_A^a(\phi) D_a^B(v(\phi)) \tag{A.23}
\]
which follows from Eqs. (A.9) and (A.10). Multiplying both the sides of Eq. (A.21) by \( L_A^a(\phi) \) and using Eqs. (A.13), (A.20) and (A.23), Eq. (A.21) can be rewritten as
\[
L_A G_{AB}(\phi) = \kappa (f_{ca}^p G_{db}(\phi) + f_{cb}^p G_{da}(\phi)). \tag{A.24}
\]
Taking \( f_{ab}^c = f_{ab}^* = 0 \) into account, we can further obtain
\[
L_A \bar{\epsilon}_a^B(\phi) + (\partial_a L_A^b(\phi)) \bar{\epsilon}_b^B(\phi) = \kappa \bar{\epsilon}_a^C(\phi) f_{ca}^B - (\partial_a \Omega_A^c(\phi)) D_a^B(v(\phi)) \tag{A.25}
\]
from Eqs. (A.15), (A.16) and (A.19).

Appendix B

— The $SL(n, C)$ Gauge Theory —

In the $SL(n, C)$ gauge theory,\footnote{The $GL(n, C)$ theory is reduced to the $SL(n, C)$ theory by separating the $GL(n, C)/SL(n, C)$ part.} it is necessary to introduce the internal metric tensor $s(\phi)$; a positive hermitian matrix defined from the coset representative $v(\phi)$ of the coset space $SL(n, C)/SU(n)$ as

$$s(\phi) = v(\phi) v(\phi)^*.$$  \hspace{1cm} (B.1)

Using (2.3) and taking into account $h^* = h^{-1} (h \in SU(n))$, one can verify

$$s(\phi) \rightarrow s(\phi') = g s(\phi) g^*.$$ \hspace{1cm} (B.2)

From (2.2), the covariant derivative of the internal metric $s(\phi)$ is given by

$$D_\mu s = \partial_\mu s + iq A_\mu s - iqs A_\mu^*,$$ \hspace{1cm} (B.3)

which transforms in the same manner as $s(\phi)$.

The Lagrangian density of the $SL(n, C)$ gauge theory is constructed from $s(\phi)$ and the field strength $F_{\mu \nu}$ defined in (2.10). The original form introduced by Cahill is

$$\mathcal{L}_c = \mathcal{L}_A + \mathcal{L}_\phi$$ \hspace{1cm} (B.4a)

with

$$\mathcal{L}_A = -\frac{1}{4} \text{tr}(F_{\mu \nu} F^{\mu \nu} s^{-1})$$ \hspace{1cm} (B.4b)

and

$$\mathcal{L}_\phi = \frac{\mu^2}{8 \kappa^2} \text{tr}[(s^{-1} D_\mu s)(s^{-1} D^\mu s)],$$ \hspace{1cm} (B.4c)

where $\kappa$ is a dimensionless constant and $\mu$ is a constant with dimension of mass.

We can rewrite $\mathcal{L}_\phi$ in the following form:

$$\mathcal{L}_\phi = -\frac{1}{8} \mu^2 \text{tr}[(E_\mu - E_\mu^*)(E_\nu - E_\nu^*)],$$ \hspace{1cm} (B.5)

where $E_\mu$ is the vector defined in (2.1) and can be expanded as $E_\mu = E_\mu^A T_A$ with real coefficients $E_\mu^A$. The generators $\{T_A\}$ of $SL(n, C)$ consist of two parts: The hermite generators $\{T_a\}$ which generate the subgroup $SU(n)$, and the antihermite generators $\{T_a\}$. Hence, $\mathcal{L}_\phi$ can furthermore be rewritten as (2.7) whose metric is the $G/H$ part, $I_{ab}$, of the metric

$$I_{ab} = \frac{1}{2} \{\text{tr}(T_a T_b^*) + \text{tr}(T_b T_a^*)\}.$$ \hspace{1cm} (B.6)

For the $I_{ab}$ in (B.6), the invariance (2.6) can be checked directly. Also, substituting
$F_{\mu\nu}=F_{\mu\nu}^AT_A$ into (B·4b), $\hat{L}_A$ can be expressed in the form of (2·14) with the metric

$$G_{AB}(\phi) = \frac{1}{2} \left\{ \text{tr}(T_A s T_B s^{-1}) + \text{tr}(T_B s T_A s^{-1}) \right\}.$$  \hspace{1cm} (B·7)

We can verify directly that the $G_{AB}(\phi)$ in (B·7) is related to the $I_{AB}$ in (B·6) by (2·12). Therefore Cahill's Lagrangian density (B·4) can be rewritten in the form obtained in § 2.

References

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