The New Decay Mode in the Electric Dipole Transitions

--- Inversely Proportional to the Eighth Power of Time ---

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We investigate the part of the electric dipole transition, whose time dependence is inversely proportional to the 8th power of time in the leading order. To be specific, we examine the electric dipole transition \((2p\rightarrow 1s)\) of the hydrogen atom and the hydrogen-like atoms. The results are of academic interest only, since they are so small to propose any experimental test at the moment.

In previous papers, we have presented the general theory of quantum mechanical resonances\(^1\) and discovered a new decay mode (with dependence on a certain inverse power of time) of unstable particles,\(^2\) apart from the well-known and overwhelmingly dominant exponential decay mode. In this paper, we shall discuss explicitly such a decay mode associated with the electric dipole transition \((2p\rightarrow 1s)\) of the hydrogen atom and other hydrogen-like atoms (namely, systems consisting of one electron plus a nucleus). To this end, we shall begin with the brief summary of our argument developed in the previous paper,\(^2\) modified to the case of electric dipole transitions.

Let us take two channels, \(P\)- and \(D\)-channel to be

\[\text{\(P\)-channel: one \(H\)-atom with no photons,}\]
\[\text{\(D\)-channel: one \(H\)-atom with one photon.}\]

We are using the Fock space representation. Let us denote the (free) Hamiltonians in these two channels by \(H_P\) and \(H_D\).

Then we see that

\[H_P = H,\]
\[H_D = H + \omega a_{k\epsilon}a_{k\epsilon}^\dagger,\]

where \(H\) is the Hamiltonian for the hydrogen atom, \(a_{k\epsilon}^\dagger(a_{k\epsilon})\) is the creation (annihilation) operator of a photon with momentum \(k\) and polarization \(\epsilon\), and

\[\omega = |k|\]

The channel coupling \(H_w\) is the electric dipole operator:

\[H_w = ie\sqrt{4\pi|\omega|\sqrt{2\omega}}(er)a_{k\epsilon}^\dagger + \text{h.c.}\]

Now, we can write down the basic Schrödinger equation for our problem:
For the $2p \to 1s$ transition, we should take $Q$ to be

$$Q = -E(1s).$$

We denote the $2p$ wave function (wave function in the $P$-channel) by $\Phi_B$ and the "continuum" state wave function in the $D$-channel by $\phi_k$ (following the notation used in Ref. 2):

\begin{align*}
H_P \Phi_B &= E(2p) \Phi_B = -B \Phi_B, \\
H_D \phi_k &= \omega \phi_k = E_k \phi_k,
\end{align*}

or more explicitly

\begin{align*}
\Phi_B &= R_{21} Y_{1m}, \\
\phi_k(0) &= R_{10} Y_{00} a_k |0\rangle, \\
E(1s) &= -\frac{1}{2} \alpha^2 m_e, \quad E(2p) = -\frac{1}{8} \alpha^2 m_e.
\end{align*}

From these equations we can derive the survival probability amplitude of the $2p$-state (at rest) of the $H$ atom prepared at time $t=0$

\begin{equation}
A(t) = e^{-i(E_0-(i/2)\Gamma_0)t} \int_{-\infty}^{0} \frac{d^3 k}{(2\pi)^3} \left| \langle \Psi_k | H_W | \Phi_B \rangle \right|^2 e^{-iE_k t},
\end{equation}

and the survival probability:

\begin{equation}
P(t) = |A(t)|^2
= \left| e^{-i(E_0-(i/2)\Gamma_0)t} + \eta_1 \left( \frac{T_1}{t} \right)^n + \eta_2 \left( \frac{T_2}{t} \right)^{n+1} + \cdots \right|^2,
\end{equation}

where $E_0$ and $\Gamma_0$ are the resonance energy (i.e., $E_0 = E(2p) - E(1s) = \omega$) and its width, respectively. $n$ is a certain number which depends on the nature of final states ($n=4$ for dipole transitions, see Eq. (13)). The coefficients $T_1$, $T_2$, $\cdots$ stem from the Taylor expansion of $|\langle \Psi | H_W | \Phi_B \rangle|^2$ at $E=0$, i.e.,

\begin{align*}
\eta_1 \left( \frac{T_1}{t} \right)^n &= \langle \Psi_k | \Phi_B \rangle^2 \int_{-\infty}^{0} \frac{4\pi k^2}{(2\pi)^3} \frac{dk}{dE_k} dE_k e^{-iE_k t}, \\
\eta_2 \left( \frac{T_2}{t} \right)^{n+1} &= \frac{d}{dE_k} \langle \Psi_k | \Phi_B \rangle^2 \int_{-\infty}^{0} \frac{4\pi k^2}{(2\pi)^3} \frac{dk}{dE_k} dE_k e^{-iE_k t},
\end{align*}

where phase factors $\eta_1$, $\eta_2$, $\cdots$ ($|\eta_1| = |\eta_2| = \cdots = 1$) have been introduced to make $T_1$, $T_2$, $\cdots$ real (and positive) and

$E_k = |k| = \omega$.

$T_1$, $T_2$, $\cdots$ can be evaluated by making use of perturbation theory, e.g.,
\[
\eta(T_1/t) = \left(\frac{T_1}{t}\right)^n = \int_{-\infty}^{0} \frac{4\pi k^2}{(2\pi)^3} \frac{dk}{dE_k} dE_k \frac{\langle \phi_k | H_w | \Phi_B \rangle^2}{E - (Q - B)} \bigg|_{E=0} e^{-i\mathbf{k} \cdot \mathbf{r}}
\]

and so on. We shall retain only the leading term \((T_1/t)\) below because the higher order terms in Eq. (5) are very small as discussed in Ref. 2). Equations (4) and (5) are valid for "large" \(t(\tau \approx 1/\tau_0)\).

It is interesting to calculate the critical time \(t^*\), at which

\[
e^{-(t^*/\tau_0)} = \left(\frac{T_1}{t^*}\right)^2
\]

holds, where \(\tau_0\) denotes the mean life \((\tau_0 = 1/\Gamma_0)\).

First, we write down the transition probability per unit time:

\[
\Gamma_0 = 2\pi \int |\langle f | H_w | i \rangle|^2 \frac{d^3 k}{(2\pi)^3} \delta(E_f - E_i),
\]

where \(|\langle f | H_w | i \rangle|^2\) is square of the transition matrix element between the initial state \(i\) and the final state \(f\), and \(|\langle f | H_w | i \rangle|^2 = 2J_{\text{vac}}|\langle i | \mathbf{r} | f \rangle|^2\),

\[
|\langle f | H_w | i \rangle|^2 = 2\pi \alpha \omega \langle i | \mathbf{e} \cdot \mathbf{r} | f \rangle^2,
\]

where

- \(\mathbf{e}\): polarization vector of emitted photon,
- \(E_i\): initial energy of hydrogen atom,
- \(E_f\): final energy of hydrogen atom,
- \(\omega = E_f - E_i\),

and \(\alpha = e^2 = 1/137(\hbar = c = 1)\). The transition probability per unit time of the \(2p \rightarrow 1s\) transition of the hydrogen atom is given by

\[
\Gamma = \frac{2^{11}}{3^9} \alpha^3 \omega
\]

\[
= 1.60 \times 10^{-19} \text{ sec}.
\]

In this case, we used the initial and final state as follows:

\[
|i\rangle = R_{21} Y_{10},
\]

\[
|f\rangle = R_{10} Y_{00}.
\]

Next, using Eq. (8),

\[
- \int_{-\infty}^{0} \frac{4\pi k^2}{(2\pi)^3} \frac{dk}{dE_k} dE_k \frac{\langle \phi_k | H_w | \Phi_B \rangle^2}{E - (Q - B)} \bigg|_{E_k=0} e^{-i\mathbf{k} \cdot \mathbf{r}}
\]
we find
\[ T_1^4 = \frac{2^{17}}{3^{11}} \frac{1}{\pi \alpha^e} \frac{1}{m_e^4}, \] (14)

namely \( n = 4 \) and \( \eta_1 = -1 \). Note that the emitted photon in the final state has "zero" energy \( \omega \approx 0 \).

The survival probability \( P(t) \) for macroscopic time is
\[ P(t) = |A(t)|^2 = e^{-t/\tau_0} + \left( \frac{T_1}{t} \right)^8 + \cdots. \] (15)

The leading term of inverse power part is given by \( (T_1/t)^8 \), i.e., inversely proportional to the 8th power of \( t \). The critical time \( t^* \), defined by Eq. (9) can be found from
\[ \Gamma_0 t^* - 8 \ln \Gamma_0 t^* = -8 \ln \Gamma_0 T_1. \] (16)

\( t^* \) for the \( 2p \rightarrow 1s \) transition of the hydrogen atom is
\[ t^* \approx 220 \tau_0. \] (17)

Finally, we can extend our result to the electric dipole transition of any hydrogen-like atom (one electron + nucleus of atomic number \( Z \)), whose \( \Gamma_0(Z) \) and \( T_1(Z) \) are related to those \( (\Gamma_0, T_1) \) of the hydrogen atom by the following scaling laws:
\[ \Gamma_0(Z) = Z^4 \Gamma_0, \]
\[ (T_1(Z))^4 = \frac{1}{Z^6} T_1^4, \]
so that
\[ \frac{\Gamma_0(Z)}{\Gamma_0 T_1} = Z^{8/2}. \]

Consequently, Eq. (16) will be modified to
\[ \Gamma_0(Z) t^* - 8 \ln \Gamma_0(Z) t^* = -8 \ln \Gamma_0(Z) T_1(Z) - 20 \ln Z . \] (18)

The numerical results for cases \( Z = 1, 10, 30, 50 \) and \( Z = 92 \) are depicted in Fig. 1.

Fig. 1. The figure depicts the critical time \( t^* \) defined by Eq. (18), \( \Gamma_0(Z) t^* - 8 \ln \Gamma_0(Z) t^* = -8 \ln \Gamma_0 T_1 - 20 \ln Z \) for the hydrogen-like atoms in the electric dipole transition \( 2p \rightarrow 1s \), whose transition probability per unit time is \( \Gamma_0(\tau_0 = 1/\tau_1). \)

The solid line shows the quantity \( \Gamma_0(Z) t^* - 8 \ln \Gamma_0(Z) t^* \) vs \( \Gamma_0(Z) t^* \). We marked here for several choices of atomic numbers \( Z = 1, 10, 30, 50 \) and \( Z = 92 \).
In general, the index $n$ of the inverse power mode of survival amplitude for the electric or magnetic $L$th multipole transition is given by $2L+2$, since the transition matrix element squared $|<f|H_w|i>|^2$ is proportional to $(L-1/2)$th power of energy. This means that the survival probability contains the leading inverse power term $(1/t)^{2L+4}$, so the critical time $t^*$ increases with $L$.

It is also easy to derive the part which depends on the inverse power of time for the radiative transition in atomic nuclei. In the case of both atomic and nuclear radiative transitions, the critical time $t^*$ is too large to verify the existence of the part with inverse power law.