Application of the Extended Siegert Theorem to Muon Capture Reactions

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On the basis of the conserved vector current theory the extended Siegert theorem is given and applied to the muon capture reactions. As a consequence a relation among the nuclear matrix elements is found in each forbiddenness.

§ 1. Introduction

The partial rate of muon capture in complex nuclei gives us an important information on the nuclear structure as well as on the meson cloud effect of nucleons in the nucleus. Experiments along this line have been carried out for the reactions $^6C^12 \rightarrow ^7B^12$, $^3He^3 \rightarrow ^4H^4$, and $^16O^16 \rightarrow ^16N^16$. In particular, recent Columbia and Berkeley experiments on $^16O$ revealed the usefulness of “forbiddenness” in muon capture reactions, in which three first-forbidden and no third-forbidden partial transitions were observed. The forbiddenness introduced by Morita and Fujii is the same as that of beta decay. However, the number of independent nuclear matrix elements in each forbidden transition is much larger than that of beta decay because of the larger momentum transfer. The theory will be much simpler if certain relations among the nuclear matrix elements are found to exist independent of the nuclear model.

We shall point out here that such relations certainly exist, if we accept the hypothesis of conserved vector current (c.v.c.) which received experimental supports already in nuclear beta decay and in pion beta decay. The c.v.c. hypothesis is an analogue of charge conservation in the electromagnetic interaction. The Siegert theorem is a consequence of charge conservation and effectively applied to determine the form of the electromagnetic interaction of the nucleus. On the basis of the c.v.c. theory the Siegert theorem was...
extended to the vector interaction of the beta decay by Fujita,\(^{24}\)\(^{-27}\) and in combination with the Ahrens-Feenberg approximation\(^{28}\) it gives a simple relationship for nuclear matrix elements.

The purpose of this paper is to push further the parallelism of the vector interaction in muon capture and beta decay and apply the extended Siegert theorem to find the relation among the nuclear matrix elements in muon capture reactions.

\section{2. Extended Siegert theorem}

In this section we recapitulate the essence of the extended Siegert theorem in beta decay. The formalism presented here seems to have an immediate application to the finite nuclear size effect and the finite de Broglie wavelength effect in RaE spectrum.\(^{29}\) By a trivial modification in lepton current the argument goes quite parallel for the muon capture reactions, as is shown in \S\ 3.

Strictly speaking, the interaction for nuclear beta decay cannot be described by the sum of free nucleon operators because of the exchange forces among the nucleons. However, in electric multipole interactions of vector such an ambiguity is completely removed by the Siegert theorem. This is also true in beta decay interaction of vector type by the conserved vector current hypothesis.

The transition amplitude due to the weak vector current is written as

\[ T^{(v)} = \langle f | \int d^3 x J^I_\mu (x) \cdot L_\mu (x) | i \rangle \]

for \( \beta^\pm \) decay, \(^{(2\cdot1)}\)

with

\[ J^I_\mu (x) \cdot L_\mu (x) = J^{(v)}_\mu (x) \cdot L - J^{(0)}_\mu (x) \cdot L_0. \]

Here \( J^{(v)}_\mu \) stands for the weak nuclear vector current, and \( L_\mu \) for the lepton current

\[ L_\mu (x) = i (\bar{\psi} \gamma_\mu (1 + i \gamma_5) / \sqrt{2} \psi). \]

\(^{(2\cdot2)}\)

For the electromagnetic interaction we replace

\[ J^{(v)}_\mu (x) \rightarrow J^{(e)}_\mu (x), \quad L_\mu (x) \rightarrow A_\mu (x). \]

Since the charge density is supposed to be concentrated mostly at the position of each proton, we have for the time component of the current

\[ J_0^{(e)} (x) = e \sum_{k=1} A \frac{1 + \tau_3^{(e)} \cdot \rho (x - x_k)}{2}, \]

where \( \rho (x) \) may be regarded as the proton form factors obtained by Hofstadter experiments. However, for the practical purpose we can put \( \rho (x) = \delta (x) \). The c.v.c. hypothesis demands the relation

\[ J_0^{(v)} (x) = g^{(v)} \sum_{k=1} A \tau_3^{(e)} \rho (x - x_k). \]

\(^{(2\cdot3)}\)
A. Fujii, J. Fujita and M. Morita

On the other hand we are less certain about the explicit form of the space component \( J(V)^\pm \) which depends on the detail of nuclear interactions. It is the Siegert theorem which dodges the ignorance of it.

We divide the nuclear Hamiltonian \( H_N \) into two parts, the charge independent part \( H_0 \) and the rest \( H_1 \),

\[
H_N = H_0 + H_1.
\]

\( H_1 \) includes the static Coulomb potentials and the neutron-proton mass difference,\(^4\)

\[
H_1 = \sum_{k,d} \left( 1 + \tau_{3}^{(k)} \right) \left( 1 + \tau_{5}^{(d)} \right) \frac{e^2}{r_{kl}} + \sum_{k} \frac{M_p - M_n}{2} \tau_{3}^{(k)}. \tag{2.4}
\]

If the charge dependent interactions are switched off, the nuclear Hamiltonian of this fictitious system becomes \( H_0 = H_N - H_1 \), and the continuity equation

\[
\text{div } J(V)^\pm(x) + [H_N - H_1, iJ_0(V)^\pm(x)] = 0 \tag{2.5}
\]

holds as a consequence of the c.v.c. hypothesis. Upon switching on the charge dependent interactions, modifications should arise from the higher order electromagnetic effects, such as the Coulomb distortion of pion clouds. We believe, however, that the corrections are small although we have no quantitative argument.\(^3\)

Therefore, even though the Coulomb effects are very important in determining the nuclear binding energy, Eq. (2.5) is expected to be still valid with a sufficient degree of accuracy for practical purpose.

Expanding the lepton current into rotation-free and divergence-free components

\[
L(x) = \text{grad } U(x) + \text{rot } V(x), \tag{2.6}
\]

we have for the rotation-free component

\[
\left< f \right| \int d^3x J(V)^\pm(x) \cdot \text{grad } U(x) \left| i \right> = - \left< f \right| \int d^3x U(x) \cdot \text{div } J(V)^\pm(x) \left| i \right> = i(E_f - E_i) \left< f \right| \int d^3x U(x) J_0(V)^\pm(x) \left| i \right> = - i \left< f \right| \int d^3x U(x) [H_1, J_0(V)^\pm(x)] \left| i \right>, \tag{2.7}
\]

where \( E_i \) and \( E_f \) are the total nuclear energies of the initial and final nuclei including their masses. It should be noticed that the space component \( J(V)^\pm \) is eliminated and replaced by the better known time component \( J_0(V)^\pm \). If the divergence-free component \( V \) is shown to be small, the ambiguity on nuclear

\(^4\) We use the unit system \( \hbar = c = m_e = 1 \), throughout this paper.
Application of the Extended Siegert Theorem

interactions is minimized. In the theory of beta decay, the conventional matrix element

\[ \langle i^{j-1} r^{j-1} \alpha \cdot Y_{J_{J-1}} \rangle \]

is the part with rank \( J \) of the complete matrix element

\[ \langle f \mid \int d^3x J^{(V)}(x) \cdot (\text{grad} \ U(x) + \text{rot} \ V(x)) \mid i \rangle. \]

As is shown in § 3, we can choose \( \text{rot} \ V \) to be of higher powers of the momentum transfer and neglect it in comparison with \( \text{grad} \ U \). Thus \( \langle i^{j-1} r^{j-1} \alpha \cdot Y_{J_{J-1}} \rangle \) can be written in terms of \( \langle i^r Y_J \rangle \), by Eq. (2·7).

For practical purpose, it is convenient to introduce the following approximation which is expected to be good within 10%.

a) Ahrens-Feenberg approximation\(^{39}\)

In this approximation, we have

\[ \langle f \mid \int d^3x [H_1, J_0^{(V)}(x)] \mid i \rangle \]

\[ \approx [\langle f \mid H_1 \mid f \rangle - \langle i \mid H_1 \mid i \rangle] \langle f \mid \int d^3x J_0^{(V)}(x) \mid i \rangle \]

\[ = \left( \mp F \frac{\alpha Z}{R} \pm 2.5 \right) \langle f \mid \int d^3x J_0^{(V)}(x) \mid i \rangle, \]

where \( R \) is the nuclear radius. From Eq. (2·7) with the above relation, we have

\[ \langle f \mid \int d^3x J^{(V)}(x) \cdot \text{grad} \ U(x) \mid i \rangle \]

\[ \approx -i \left( E_i - E_f \mp F \frac{\alpha Z}{R} \pm 2.5 \right) \langle f \mid \int d^3x U(x) \cdot J_0^{(V)}(x) \mid i \rangle, \]

where

\( F = 1.2 \) for uniform charge distribution.

b) Iso-multiplet approximation\(^{27}\)

Experimental data of isobaric state gives an empirical formula\(^{31}\)

\[ H_N J_0^{(V)} \mid i \rangle \approx \left( E_i + 1.15 \frac{\alpha Z}{R} - 2.5 \right) J_0^{(V)} \mid i \rangle. \]

\(^{39}\) \( Y_{J_{J-1} M} \) is the spherical vector harmonics defined by Eq. (3·7) below. The magnetic quantum number \( M \) is dropped for the reduced matrix element of rank \( J \).
This relation holds for the state $J_0^{(v)}|i\rangle$ with the width less than 100 keV. The nuclear matrix element of the $E1$ gamma transition from the isobaric state $J_0^{(v)}|i\rangle$ to the state $|f\rangle$ is given by*\)

$$
\langle f | \int d^3 x J^{(v)}(x) \cdot \text{grad} \ U(x) \ J_0^{(v)}|i\rangle
$$

$$
= i \langle f | \int d^3 x U(x) \cdot [H_S, J_0^{(v)}(x)] J_0^{(v)}|i\rangle
$$

$$
= -i \left( E_f - E_i + 1.15 \frac{\alpha Z}{R} - 2.5 \right) \langle f | \int d^3 x U(x) J_0^{(v)}(x) J_0^{(v)}(x)|i\rangle,
$$

(2·9)

according to the original Siegert theorem. If the total isospin is a good quantum number even for heavy nuclei,\(^{30,32}\) the Wigner-Eckart theorem proves the equivalence of Eqs. (2·8) and (2·9) with $F=1.15$.

The conclusion of this section is summarized in Eq. (2·7) which is obtained by direct application of the extended Siegert theorem. However, Eq. (2·8), though less accurate, is more convenient for practical calculations.

§ 3. Application to muon capture reactions

In this section we apply the method developed in § 2 to muon capture reactions, following the notation in MF.\(^*\)

The vector part of the interaction Hamiltonian density for muon capture in complex nuclei is expressed as

$$
H^{(v)} = -C_v (\mathbf{J} \cdot \mathbf{L} - J_0 \cdot L_0).
$$

(3·1)

Here the lepton currents are given by

$$
L_0 = \psi^\dagger \left[ (1 + \gamma_5) / \sqrt{2} \right] \psi^\mu,
$$

$$
\mathbf{L} = -\psi^\dagger \left[ (1 + \gamma_5) / \sqrt{2} \right] \mathbf{a} \psi^\mu,
$$

(3·2)

and the nuclear weak currents by

$$
J_0 = \psi^\dagger \psi_t,
$$

$$
\mathbf{J} = -\psi^\dagger \mathbf{a} \psi_t.
$$

(3·3)

In order to find a relation for nuclear matrix elements in each forbidden transition, we decompose the lepton current $\mathbf{L}$ into its spherical component defined by\(^{12}\)

$$
(\kappa \mu | \alpha | -1 \mu')
$$

$$
= \langle \psi^{(\alpha)}_{t,\mu} | \alpha \phi^{(\alpha)}_{1,\mu'} \rangle,
$$

(3·4)

\(^*\) As is Eq. (2·4), we decompose $\mathbf{A}(x) = \text{grad} \ U(x) + \mathbf{rot} \ V(x)$.

\(^*\) Reference 12) is abbreviated as MF hereafter.
where the quantum numbers $\kappa$ and $\mu$ specify the free neutrino state, while $\kappa' = -1$ and $\mu'$ the bound muon state, the K-orbit. We write the $\kappa\mu$-component of $L$ as follows:

$$L \rightarrow -(\kappa\mu|\alpha| - 1\mu') = \hat{j}_i(\rho) e^{-\rho}[A^{i+1} Y_{l+1M} + A^0 Y_{lM} + A^{-1} Y_{l-1M}],$$

where

$$\rho = qr, \quad \varepsilon = \alpha Z m_{\mu}'/q \approx \alpha Z, \quad \text{and} \quad M = \mu' - \mu.$$  

$j_i(\rho)$ is the spherical Bessel function of order $l$, $l$ is the orbital angular momentum of neutrino,$^8$ and $q$ and $m_{\mu}'$ are the momentum of neutrino and the reduced mass of muon, respectively. The numerical coefficients $A$ are given by$^9$

$$A^i = i(-)^{k+l/3} (\alpha Z m_{\mu}')^{1/3}[6(2j+1)/\pi]^{1/3} S_j \times W(1/2 \ 1j; \ 1/2 \ l+\Delta l \ (j \ 1/2 \ -\mu \mu'|l+\Delta') M)$$

for $\Delta l = 0, \pm 1$,

where $j$ and $l$ are the total and orbital angular momentum for $-\kappa$, $S_j$ the sign of $\kappa$. The vector harmonics is defined by$^{34}$

$$Y_{l+\Delta lM} = \sum_m (l+1 \ M-m \ m | l+\Delta' \ M) Y_{lM-m}(\theta, \varphi) e_m,$$

$Y_{lM-m}$ and $e_m$ being the spherical harmonics and the unit vector. Three terms in Eq. (3·5) produce nuclear matrix elements of rank $l+1$, $l$ and $l-1$ respectively, so that each term can be handled separately. In particular the first term with $Y_{l+1M}$ is important, since it gives the nuclear matrix elements$^{*4} [1 \ l-1 \ l \ p]$ an analogue of $j\alpha$, $jA_{ij}$, etc., in the theory of beta decay.

Let us divide it into two parts,

$$A^{i+1} j_i(\rho) e^{-\rho} Y_{l+1M} = \text{grad} \ U + \text{rot} \ V,$$

and determine $U$ and $V$ making use of the following two identities.$^{34}$

$$\frac{1}{q} \text{grad} [\phi Y_{l+1M}(\theta, \varphi)] = -\left( \frac{l+2}{2l+3} \right)^{1/2} \left( \frac{d}{d\rho} - \frac{l+1}{\rho} \right) \phi Y_{l+1l+2M} + \left( \frac{l+1}{2l+3} \right)^{1/2} \left( \frac{d}{d\rho} + \frac{l+2}{\rho} \right) \phi Y_{l+1lM},$$

$$-\frac{i}{q} \text{rot} [\phi Y_{l+1l+1M}] = \left( \frac{l+1}{2l+3} \right)^{1/2} \left( \frac{d}{d\rho} - \frac{l+1}{\rho} \right) \phi Y_{l+1l+2M} + \left( \frac{l+2}{2l+3} \right)^{1/2} \left( \frac{d}{d\rho} + \frac{l+2}{\rho} \right) \phi Y_{l+1lM},$$

$^8$ It corresponds to $-\kappa$ and is $\bar{\kappa}$ in the original notation in MF. The bar is dropped for simplicity. This does not change the results in this paper.

$^{*4}$ For notation, see Appendix.
where \( \phi \) is an arbitrary radial function. We put

\[
U = (c\phi_0 + e\phi_1) Y_{l+1M}(\theta, \varphi),
\]

\[
V = (d\phi_0 + f\phi_1) Y_{l+1l+1M},
\]

where \( c, d, e \) and \( f \) are constants. \( \phi_0 \) is the particular solution of

\[
\left( \frac{d}{d\rho} + \frac{l+2}{\rho} \right) \phi_0 = A^{+1}(2l+3)^{-1/2} e^{-i\rho} j_1(\rho),
\]

and \( \phi_1 = \rho^{l+1} \) satisfies the homogeneous equation

\[
\left( \frac{d}{d\rho} - \frac{l+1}{\rho} \right) \phi_1 = 0.
\]

The solution \( \phi_1 \) of the homogeneous equation

\[
\left( \frac{d}{d\rho} + \frac{l+2}{\rho} \right) \phi_1 = 0
\]

is discarded because of the singularity at \( \rho = 0 \). We insert \( U \) and \( V \) into Eq. (3.8), and decompose \( \text{grad} U \) and \( \text{rot} V \) into two terms proportional to \( Y_{l+1l+1M} \) and \( Y_{l+1L+1M} \), by using Eqs. (3.9) and (3.10). We choose the constants \( c \) and \( d \) so that the term with \( Y_{l+1l+1M} \) drops out. A relation between \( e \) and \( f \) is also obtained to cancel the term proportional to

\[
\left( \frac{d}{d\rho} + \frac{l+2}{\rho} \right) \phi_1.
\]

For the moment we impose

\[ \varepsilon = 0 \]

and obtain

\[
\phi_0 = A^{+1}(2l+3)^{-1/2} j_{l+1}(\rho).
\]

Thus, the expressions for \( U \) and \( V \) are given as follows:

\[
U \approx (A^{+1}/q) [(l+1)/(2l+3)]^{1/2} [j_{l+1}(\rho) - (l+2)(l+1)^{-1/2} \rho j_{l+1}^{(1)}] Y_{l+1M}(\theta, \varphi),
\]

\[
V \approx -iA^{+1}/q [(l+2)/(2l+3)]^{1/2} [j_{l+1}(\rho) + b^{l+1}] Y_{l+1l+1M},
\]

where the constant \( b \) is still undetermined. The polynomial \( \rho^{l+1} \) is expanded into the series of the spherical Bessel function

\[
\rho^{l+1} = 2^{l+1} \pi^{-1/2} \sum_{i=0}^{\infty} (l+2i+3/2)\Gamma(l+i+3/2) j_{l+1+2i}(\rho)/i!,
\]

of which we shall retain only the first term

\[
\rho^{l+1} \approx 1 \cdot 3 \cdot 5 \cdots (2l+3) j_{l+1}(\rho),
\]

noting that

\[ \ast \]

\( \ast \) The symbols in the reduced nuclear matrix element \( [\cdot] \) are explained in Appendix.
Application of the Extended Siegert Theorem

\[ [SL + 2Jp] \leq 3 \times 10^{-5} [SL Jp]. \quad (3.18) \]

From Eqs. (3.15)–(3.18), we find

\[ U \approx (A^{+1}/q) [(l+1)/(2l+3)]^{1/2} [1 - 1 \cdot 3 \cdot 5 \cdots (2l+3) (l+2) (l+1)^{-1} b] \times j_{t+1}(\rho) Y_{t+1M}(\theta, \varphi), \]

\[ V \approx - (iA^{+1}/q) [(l+2)/(2l+3)]^{1/2} [1 + 1 \cdot 3 \cdot 5 \cdots (2l+3) b] j_{t+1}(\rho) Y_{t+1+t+1M}. \quad (3.19) \]

The ambiguity due to the constant \( b \) is the gauge in the lepton current. We choose \( b \),

\[ 1 + 1 \cdot 3 \cdot 5 \cdots (2l+3) b = 0, \quad (3.20) \]

such that

\[ A^{+1} j_{t+1}(\rho) Y_{t+1+t+1M} = \text{grad} U \quad (3.21) \]

holds as good as possible. The effects of

\[ \varepsilon \neq 0 \]

and the second term of Eq. (3.16) are easily taken into account as a small correction. The final results are

\[ U = (A^{+1}/q) [(2l+3)/(l+1)]^{1/2} e^{-\left[ j_{t+1}(\rho) + O(j_{t+3}(\rho)) + O(\varepsilon j_{t+1}(\rho)) \right] Y_{t+1+t+1M}, \]

\[ V = (O(j_{t+3}(\rho)) + O(\varepsilon j_{t+1}(\rho))) Y_{t+1+t+1M}. \quad (3.23) \]

An application of the Siegert theorem to muon capture reactions is now straightforward. We put \( U \) and \( J_{\alpha} \) (given in Eqs. (3.23) and (3.3)) into Eq. (2.8). The left-hand side of Eq. (2.8) is found to be

\[ - \langle f | \alpha \cdot A^{+1} j_{t}(\rho) e^{-\left[ j_{t+1}(\rho) + O(j_{t+3}(\rho)) + O(\varepsilon j_{t+1}(\rho)) \right] Y_{t+1+t+1M} | i \rangle = (A^{+1}/iM_N) (4\pi/3)^{1/3} [1 l + 1 p] (J_{l+1} M_{l+1} M_{l} M_{l}), \quad (3.24) \]

\[ M_N: \text{nucleon mass}, \]

while the right-hand side is approximately equal to

\[ - i \left( E_{l+1} - E_{l} - F \frac{\alpha Z}{R} + 2.5 \right) \langle f | A^{+1} \left( \frac{2l+3}{l+1} \right)^{1/3} j_{t+1}(\rho) e^{-\left[ j_{t+1}(\rho) + O(j_{t+3}(\rho)) + O(\varepsilon j_{t+1}(\rho)) \right] Y_{t+1+t+1M} | i \rangle = - i \left( E_{l+1} - E_{l} - F \frac{\alpha Z}{R} + 2.5 \right) A^{+1} \left( \frac{2l+3}{l+1} \right)^{1/3} (4\pi)^{1/3} [0 l + 1 l + 1] (J_{l+1+l+1} M_{l+1} M_{l+1} M_{l+1}). \quad (3.25) \]

The reduced nuclear matrix elements \([1 l + 1 p]\) and \([0 l + 1 l + 1]\) appear in the \( n \)-th forbidden transition, where the forbiddenness \( n \) equals \( l+1 \). Thus the velocity-dependent (relativistic) matrix element \([1 n - 1 n p]\) in the \( n \)-th
forbidden transition is found in terms of the velocity-independent matrix element $[0 \ n \ n]$, 

$$[1 \ n-1 \ n \ p]$$

$$\approx -\left[ W_0 + F \frac{\alpha Z}{R} - 2.5 \right] M_n \left[ \frac{3(2n+1)}{n} \right]^{\frac{1}{2}} [0nn] \quad (\text{for } n \geq 1, \text{in units of } \hbar = c = m_e = 1),$$

where $W_0 = E_f - E_i$ is the maximum energy of $\beta^-$ in the beta decay process $|f\rangle \rightarrow |i\rangle$. If the Ahrens-Feenberg approximation is not introduced, Eq. (3.25) may read from Eq. (2.7)

$$-i A^+ \left( \frac{2l+3}{l+1} \right)^{\frac{1}{2}} \{(E_i - E_f) (4\pi)^{\alpha/3} (J_i l + 1 M_i M | J_f M_f) [0 l + l M] \}$$

$$- \langle f | e^{-\varphi} j_{l+1} (\rho) Y_{l+1 M} | H_i, J_0 (\rho) | i \rangle \}.$$

In the beta decay $|f\rangle \rightarrow |i\rangle$, the relationship for nuclear matrix elements is identical with Eq. (3.26) except for the sign which should read plus. The relations for $\alpha$ and $\beta$ and their higher rank matrices can be obtained from Eq. (3.26) in the low $q$ limit.

### § 4. Concluding remarks

Assuming the conserved vector current theory, we found the relation (3.26) for the nuclear matrix elements in muon capture reactions. In derivation of this relation, we omitted the terms $j_{l+1} (\rho)$ and $\varepsilon j_{l+2} (\rho)$, since the correction due to these terms is less than, say, three percent. As is mentioned in § 2, Eq. (3.26) is subject to about 10% theoretical error due to the Ahrens-Feenberg approximation.

The second term with $Y_{l+1 M}$ in Eq. (3.5) can be expressed as a linear combination of rot $(j_{l+1} (\rho) Y_{l+1 M})$ and rot $(j_{l-1} (\rho) Y_{l-1 M})$. The third term with $Y_{l-1 M}$ in Eq. (3.5) can also be expressed in a form grad $U + \text{rot} V$ with

$$U = (A^{-1}/q) \left[ l/(2l-1) \right]^{\frac{1}{2}} e^{-\varphi} \left[ j_{l-1} (\rho) + c \varphi^{l-1} + \ldots \right] Y_{l-1 M} (\theta, \varphi),$$

$$V = (i A^{-1}/q) \left[ (l-1)/(2l-1) \right]^{\frac{1}{2}} e^{-\varphi} \left[ j_{l-1} (\rho) + c \varphi^{l-1} + \ldots \right] Y_{l-1 M}.$$

Here the same constant $c$ appear in $U$ and $V$ so that rot $V$ is, by no means, negligible compared with grad $U$.

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Appendix

Definition of reduced nuclear matrix elements
in muon capture reactions

We summarize in this Appendix the reduced nuclear matrix elements in muon capture reactions introduced in MF.\textsuperscript{12} The reduced nuclear matrix element $[SLJ]$ is defined by

$$[SLJ](J_i, J M_i; M|J_f, M_f) = \sum_{A} \exp(-\alpha Z m_r^* r_k) \Omega_k \tau^{(k)} U_{J_i J_f} dx_1 dx_2 \cdots dx_A. \quad (A.1)$$

Similar relations hold for $[SLJ^+]$, $[SLJ^-]$ and $[SLJP]$. Here $U_{J_i J_f}$ and $U_{J_i J_f}$ are nuclear wave functions of the initial and final states specified by the spin $J$ and its projection $M$. $\tau^{(k)}$ and $\Omega_k$ are the isospin and operator for the $k$-th nucleon. The symbols $S$, $L$ and $J$ in $[\ ]$ are the resultant spin, the effective orbital angular momentum and the resultant total angular momentum of lepton system, respectively. $J$ specifies the rank of the matrix element. The orbital angular momentum of neutrino is $L \pm 1$ for the symbol $\pm$, and $L$ otherwise. The symbol $p$ means that the matrix element includes the differential operator $p$ acting on the nuclear wave function. The parity change is given by $(-)^L$ for $[SLJ]$ and $[SLJ^\pm]$, and $(-)^{L+1}$ for $[SLJP]$. Nucleon operators $\Omega_k$ in Eq. (1) are summarized in Table I, where the vector harmonics $\gamma^0_{LJ}$ is defined by

$$\gamma^0_{LJ}(r_1', \sigma) = \sum_{m} (S L M M - m |JM) Y_{L M - m} (\theta, \varphi) \gamma^0_{m}(\sigma), \quad (A.2)$$

with

$$\gamma^0_{00}(\sigma) = (1/4\pi)^{1/2},$$

$$\gamma^0_{10}(\sigma) = (3/4\pi)^{1/2}\sigma_z, \text{ etc.}$$

$\gamma^0_{LJ}(r_1', p)$ has a similar expression.

The statistically averaged muon capture rate from $|i\rangle$ to $|f\rangle$ is given by
\[ W = 8 P_o (\alpha Z m^2) \left[ \frac{(2 J_f + 1)}{(2 J_i + 1)} \right] \times \left[ 1 - q (m_\mu + A M_\gamma)^{-1} \right] q^2, \]  
\hspace{1cm} (A·3)

in units of \( h = c = m_e = 1 \), where \( m_\mu \) and \( m'_\mu \) are the muon mass and its reduced mass in the parent muonic atom, respectively. The neutrino momentum \( q \) is given by

\[ q = (m_\mu - W_o) \left[ 1 - (1/2) m_\mu (m_\mu + A M_\gamma)^{-1} \right]. \]  
\hspace{1cm} (A·4)

The expression for \( P_o \) contains the coupling constants, \( C_v, C_A, C_F \), and the nuclear matrix elements of various types. It is given for each forbiddenness in MF. There are 9, 17 and 14 different matrix elements in the allowed, first forbidden and \( n \)-th forbidden (\( n \geq 2 \)) transitions, respectively. Equation (3·26) holds for each forbiddenness except for the allowed transition.

References

11) Private communications from E. Segrè and D. Jenkins, to M. Morita.
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