Quaternionic Gauge Fields on $S^7$ and Yang's $SU(2)$ Monopole

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A regular connection 1-form is introduced on a 7-sphere, and gauge transformations are applied to it to produce two specific singular connections. The gauge potentials which these singular connections embody are shown to be identical with Yang's $SU(2)$ monopole solutions as shown by pulling back the latter by a Hopf map $\gamma_2: S^7 \to S^5$. The study is a genuine generalization of a previous work where Dirac's $U(1)$ monopole was discussed within the realm of the Hopf fibre-map $S^5 \to S^3$: The present discussion hence parallels the previous one; a major difference being only that the quaternions are used in place of the complex numbers.

§ 1. Introduction

In a previous paper\textsuperscript{10} (hereafter referred to as [I]), we showed that the singular point on $S^3$ drilled by the Dirac string can be hidden away if we release the string from inside $S^3$ and embed it in $S^7$: A regular gauge connection 1-form exists on $S^7$ which still works as a tool of describing the Dirac magnetic monopole.\textsuperscript{31}

It is known that the next most important Hopf fibre map after $\gamma_1: S^3 \to S^1$ is $\gamma_2: S^7 \to S^5$,\textsuperscript{11} and so we see it sensible to consider that the argument in [I] naturally admits generalization.

The intention in this paper is hence to construct the non-Abelian regular gauge potentials on $S^7$ from which we shall deduce some specific singular gauge potentials via gauge transformations. These may be mapped on $S^5$ by the Hopf map $\gamma_2$ and expected to prove to be the gauge fields representing Yang's $SU(2)$ monopole.\textsuperscript{31}

The approach we adopt here is based on an analogy consideration and is to transcribe the previous results on $S^7$ term by term into the new formulae on $S^7$ through a certain procedure: It should firstly be recalled that almost all the expressions in [I] are expressible in terms of complex coordinates $(z_1, z_2)$ defined by

\begin{equation}
\begin{aligned}
z_1 &= \frac{1}{R} (x_1 + ix_2), \\
z_2 &= \frac{1}{R} (x_3 + ix_4),
\end{aligned}
\end{equation}

\textsuperscript{10} Refer also to Ryder's article entitled "Dirac Monopoles and the Hopf Map $S^3 \to S^1", \textsuperscript{12} which I learned after the submission of the present paper. See also Ref. 1c).
where \((x_1, x_2, x_3, x_4)\), subjected to \(x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2\), are the coordinate system of a 3-sphere employed in [I]. For example, \(S^3\) itself is represented as
\[
|x_1|^2 + |x_2|^2 = 1,
\]
and the Hopf map \(\gamma_i\) used in [I] is compactly written as
\[
\phi = a(x_1 x_2 - x_3 x_4),
\]
where
\[
\phi = \phi^* + i\phi^2
\]
(cf. (1-11) or (4·3) in [I]).

To represent \(S^7\), it is advisable to introduce two quaternions \((q_a, q_\bar{a})\): Let \(q_a \in H = \mathbb{C} \oplus \mathbb{C}\) be of the form
\[
q_a = q_a^0 + iq_a^1 + jq_a^2 + kq_a^3
\]
with \((i, j, k)\) subjected to
\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]
The quaternionic conjugate \(q_a^*\) is then
\[
q_a^* = q_a^0 - iq_a^1 - jq_a^2 - kq_a^3,
\]
and the length squared of \(q_a\) is defined by
\[
|q_a|^2 = q_a q_a^*.
\]
By means of these, the fibre-space \(S^7\) is represented as
\[
|q_1|^2 + |q_2|^2 = 1.
\]
On the analogy of the complex number case, we sometimes write \(\text{Re} q a\) for \((1/2) (q + q_a^*)\).

The rule (1·6) satisfied by \((i, j, k)\) is reminiscent of the rule governing the Pauli matrices \(\sigma^h (h = 1, 2, 3)\). It is therefore sometimes recommended to send the quaternions into \((2 \times 2)\) matrices by the mapping
\[
\Sigma: (1, i, j, k) \rightarrow (1, -i\sigma^1, -i\sigma^2, -i\sigma^3).
\]
For example,
\[
\Sigma(q) = -i \begin{pmatrix} q^2 + iq^4 & q^1 - iq^3 \\ q^1 + iq^4 & -q^2 + iq^3 \end{pmatrix}.
\]
The Hopf map \(\gamma_i\) we employed in [I] was the simplest one associated with
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a unit Hopf invariant. The map $\gamma_2$ with which we shall be concerned in this work is also of the simplest type, and the case where any inverse images $\gamma_2^{-1}(q_1, q_2)$ for $(q_1, q_2) \in S'$ are the great "spheres" on $S'$: Any two great spheres on $S'$ have linking number $\pm 1$. To be explicit, the Hopf map which we are mainly concerned with is defined by

$$\phi = 2aq_1q_2^*, \quad \phi_i = \alpha (|q_1|^2 - |q_1|^2),$$

where $\phi$ belongs to $H$ such that

$$\phi = \phi_0 + i\phi_1 + j\phi_2 + k\phi_3.$$  \hfill (1.13)

The base-space $S^3$ is then represented by

$$\phi^*\phi + \phi^2 = a^2.$$ \hfill (1.14)

The resemblance between (1.3) and (1.12) should be noticed, but this time it should also be recalled that the algebra of $H$ does not satisfy the commutativity, and hence $\phi = 2aq_1^*q_2$ in (1.12) is not identical with $\phi = 2a^*q_1$: The latter gives rise to the case where orientations of some axes are inversed, although leaving the Hopf invariant unchanged.

The case where the sign of the Hopf invariant is reversed is, however, not trivial, but within the scope of this paper we shall omit mention of the case because the discussion is only parallel (refer, however, to the remarks around (4.15)).

The present work is organized as follows. In § 2, we start by transcribing the previous results in [I] in terms of complex variables $z_1$ and $z_2$ to allow an easy generalization to the case described by the quaternions $q_1$ and $q_2$. In § 3.1, we make this generalization and introduce first a regular gauge connection 1-form on $S'$ and then discuss its associated $Sp(2)$ properties. In § 3.2, we derive two kinds of singular connection 1-forms via gauge transformations and prove that they turn out to be those pulled back by $\gamma_2$ from certain singular connections on $S'$. In § 4 we shall show that the singular connections formally yield Yang’s $SU(2)$ monopole solutions. We end with § 5, where some additional remarks are given.

§ 2. Preliminaries: Gauge fields on $S^3$

To make our later generalization plausible, we shall begin by preparing a survey of some ingredients in [I] concerning the gauge fields on $S^3$ in the language of complex coordinates $z_1$ and $z_2$ defined by (1.1).

1° We have introduced in [I] the following regular gauge connection 1-form on $S^3$ as a fundamental quantity:

$$\Omega = A_\ast (x) dx = \frac{2}{eR^3} (x_2dx_1 - x_1dx_2 + x_3dx_3 - x_3dx_4)$$ \hfill (2.1)
We first note that this is simply rewritten as

$$\Omega_i = \frac{i}{e} (\bar{z}_a dz_a - d\bar{z}_a z_a) \quad (2.2)$$

2° The curvature form $\Omega_2$ of $\Omega_1$ is, therefore, given as

$$\Omega_2 = d\Omega_1 = \frac{2i}{e} d\bar{z}_a \wedge dz_a \quad (2.3)$$

for which we usually write $(1/2) F_{\mu \nu} dx^\mu \wedge dx^\nu$.

As shown in [I], $\Omega_2$ is also defined by $\Omega_2 = \gamma_i \gamma_j$, where $\gamma_i$ is the Hopf map of (1.3) and

$$\gamma_i = \frac{1}{2\pi e^2} \epsilon^{ijk} \phi^j d\phi^k / d\phi^i. \quad (2.4)$$

It is easy anyway to see that

$$\int \gamma_2 = -4\pi / e, \quad (2.5)$$

which is just $g$, the magnetic charge, if we recall the Dirac quantization condition, and thus $\Omega_2$ defined at (2.1) well represents the Dirac magnetic monopole.

We will revert to the integral (2.5) once more in the last of this section.

3° Up until now, $(\phi^i, \phi^2, \phi^3)$ is merely a local coordinate system of a three-space. In [I], we, however, proceeded to regard $\phi^i$ as the Higgs fields, and introduced a triple $(\Omega_1, \Omega_2, \Omega_3)$, where

$$\Omega_1 = W_0^i dx_a \quad (2.6)$$

to produce

$$\Omega_i = (\phi^i / a) \Omega_1^i \quad (2.7)$$

considering $W_0^i(x)$ as making $SU(2)$ gauge fields.

Let us denote

$$w_\phi = \frac{1}{2} W^i_\phi d\phi^i \quad \Omega_1 = w_\phi d\phi . \quad (2.8)$$

Let us further define

$$\phi = \phi^i \sigma^i = \begin{pmatrix} \phi^2 \\ \phi^3 \end{pmatrix} \quad (2.9)$$

Then, for instance, Eq. (2.7) implies

$$A_\phi = \text{tr} w_\phi \cdot \phi. \quad (2.10)$$

Let us next define
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\[
\begin{align*}
\nu_1 &= \frac{1}{2} (w_1 + iw_2), \\
\nu_2 &= \frac{1}{2} (w_3 + iw_4).
\end{align*}
\]

Then
\[
\begin{align*}
\text{tr} [\nu_1 \phi] &= \frac{1}{2} (A_1 - iA_2), \\
\text{tr} [\nu_2 \phi] &= \frac{1}{2} (A_3 - iA_4)
\end{align*}
\]
and hence we may write (2.2) as
\[
\Omega_{\alpha} = \frac{1}{\alpha} \text{tr} [\nu_{\alpha} \phi] dz_{\alpha} - \text{c.c.}. \tag{2.13}
\]

On reference to Eq. (4.9) in [I], we explicitly have
\[
\begin{align*}
\nu_1 &= \frac{i}{2e} \begin{pmatrix} z_1 & -2z_2 \\ 0 & -z_1 \end{pmatrix}, \\
\nu_2 &= \frac{i}{2e} \begin{pmatrix} -z_1 & 0 \\ -2z_1 & z_2 \end{pmatrix}
\end{align*}
\]
and so
\[
\text{tr} [\nu_{\alpha} \phi] = \frac{i}{e} a z_{\alpha}, \quad (\alpha = 1, 2) \tag{2.15}
\]
as they should.

Equation (4.11) in [I] also tells us that $\Omega_1$ is written as
\[
\Omega_1 = \frac{i}{e} d g g^{-1}, \tag{2.16}
\]
where
\[
g(x) = -i \begin{pmatrix} z_1 & \bar{z}_1 \\ z_1 - \bar{z}_1 & 1 \end{pmatrix} \in SU(2) \tag{2.17}
\]
under the proviso $|z_1|^2 + |z_2|^2 = 1$. Equations (2.16) and (2.14) imply
\[
\nu_\alpha = \frac{i}{e} \frac{\partial g g^{-1}}{\partial z_\alpha} \tag{2.18}
\]
which informs that $w_\alpha$ are in a pure gauge on $S^7$.

4° In [I] invoked also were the Eulerian angles $(\theta, \phi, \varphi)$ defined through
\[
\begin{align*}
\theta_1 &= i \sin \frac{\theta}{2} \exp \left(-i \frac{\phi - \varphi}{2} \right), \\
\theta_2 &= i \cos \frac{\theta}{2} \exp \left(-i \frac{\phi + \varphi}{2} \right).
\end{align*}
\]

For example, we can use the angles to simplify $\Omega$, as

$$\Omega_1 = \frac{1}{e} (d\phi + \cos \theta \, d\phi). \tag{2.20}$$

Use of them also decomposes the $SU(2)$ gauge group element $g$ defined by (2.17) simply as

$$g = g_3(\phi) g_2(\theta) g_1(\phi) \tag{2.21}$$

with

$$g_3(\phi) = \exp\left( -\frac{i}{2} \sigma^3 \right),$$
$$g_2(\theta) = \exp\left( -\frac{i}{2} \sigma^2 \right). \tag{2.22}$$

The connection $\Omega_0$, regular throughout $S^7$, can yield by a gauge transformation another connection that has singularity on the great circle $P_{12} = \{(z_1, z_2) | z_2 = 0\}$ or on $P_{13} = \{(z_1, z_3) | z_1 = 0\}$.

Let us first specifically choose

$$u_z = z_1/|z_2| \tag{2.23}$$
as a $U(1)$ gauge operator, and construct

$$\Omega_{i}^{(s)} = \Omega_1 - \frac{2i}{e} d u_z u_z^{-1}. \tag{2.24}$$

Then the rhs of (2.24) takes the form

$$\Omega_{i}^{(s)} = i e \left[ \bar{z}_1 d z_1 - d \bar{z}_1 z_1 - \frac{|z_1|^2}{|z_2|^2} (\bar{z}_2 d z_1 - d \bar{z}_1 z_2) \right], \tag{2.25}$$

and this just recovers $\Omega_{i}^{(a)}$ presented in (2.7a) of [1]. Note that the form of (2.25) clearly shows the singularity on $P_{13}$.

It is easily seen that the new connection $\Omega_{i}^{(s)}$ is given by $\Omega_{i}^{(s)} = \gamma_i^* (\Gamma_{i}^{(a)})$ where

$$\Gamma_{1}^{(a)} = \frac{i}{2ae} \frac{1}{a + \phi} (\bar{\phi} d\phi - d\bar{\phi} \phi). \tag{2.26}$$

The curvature of $\Gamma_{1}^{(a)}$ proves to be identical with the rhs of (2.4), and so reproduces the result (2.5).

It is also easy to see that $\Omega_{i}^{(s)}$ of (2.25) turns out to be

$$\Omega_{i}^{(s)} = \frac{1}{e} (1 - \cos \theta) d\phi \tag{2.27}$$

by the use of the Eulerian angles. This implies
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\[
A_{r}(\alpha) = -\frac{1}{e} \frac{1 - \cos \theta}{r \sin \theta} \tag{2.28}
\]

in \( E^3 \) and hence the description of the Dirac string.\(^{1, b} \)

6° By the same argument with the \( U(1) \) gauge transition function

\[
u_1 = z_1/|z_1| \tag{2.29}
\]

replacing \( u_3 \) in (2.23), we can construct a third connection

\[
\Omega_{1}^{(3)} = \Omega_{1} - \frac{2i}{e} du_{1}u_{1}^{-1} \tag{2.30}
\]

on \( S^6 \backslash P_{3} \). The rhs of (2.30) is explicitly of the form

\[
\Omega_{1}^{(3)} = \frac{i}{e} \left[ \frac{|z_1|^2}{|z_1|^2} (\bar{\varepsilon}_1 dz_1 - d\bar{\varepsilon}_1 z_1) + \varepsilon_1 dz_2 - d\varepsilon_1 z_2 \right] \tag{2.31}
\]

agreeing again with \( \Omega_{1}^{(0)} \) of (2.7b) in \([1]\). Similar to (2.26) and (2.27), the following formulae hold respectively

\[
\Gamma_{1}^{(3)}(\eta_1^{* \cdot -1} \Omega_{1}^{(3)}) = - \frac{i}{2ae} \cdot \frac{1}{a - \phi^2} (d\phi^2 - \phi d\phi) \tag{2.32}
\]

and

\[
\Omega_{1}^{(3)} = \frac{1}{e} (1 + \cos \theta) d\varphi. \tag{2.33}
\]

7° It follows from (2.27) and (2.33) that

\[
\Omega_{1}^{(a)} - \Omega_{1}^{(3)} = - (2/e) d\varphi \tag{2.34}
\]

implying that \( \Omega_{1}^{(a)} \) and \( \Omega_{1}^{(3)} \) are related by a gauge transformation. In fact, if we write

\[
\Omega_{1}^{(a)} = \Omega_{1}^{(3)} + \frac{2i}{e} \tau^{*} d\tau \tag{2.35}
\]

in conformity with (2.24) and (2.30), then \( \tau \) is given by

\[
\tau = \frac{z_1 \bar{z}_1}{|z_1||z_2|} \tag{2.36}
\]

or simply by

\[
\tau = e^{\phi^2}, \tag{2.37}
\]

if we use (2.19). Definition (1.3) informs us that \( \tau \) and \( \phi \) are related as

\[
\phi = 2a|z_1||z_2|\tau, \tag{2.38}
\]

which is also written...
\[
\phi = a \sin \theta \cdot \tau ,
\]  
(2.39)

because (2.19) implies
\[
|z_1| = \cos \frac{\theta}{2} \quad \text{and} \quad |z_2| = \sin \frac{\theta}{2}.
\]  
(2.40)

In conformity with (3.37) \(u_1\) and \(u_2\) simply read
\[
u_1 = \exp \left( -i \frac{\phi - \varphi}{2} \right) \quad \text{and} \quad u_2 = \exp \left( -i \frac{\phi + \varphi}{2} \right)
\]  
(2.41)

and (2.24) and (2.30) are simply of the forms
\[
\varrho_1^{(a)} - \varrho_1 = \frac{2}{e} \left( \frac{\phi - \varphi}{2} \right)
\]  
\[
\varrho_1^{(b)} - \varrho_1 = \frac{2}{e} \left( \frac{\phi + \varphi}{2} \right).
\]  
(2.42)

consistent with (2.34).

8° We shall end this section by showing a (rather trivial) method of calculating the integral (2.5) for later comparison: The method is one not using the coordinates of \(S^3\) but making use of a stereographic projection. Let us define \(\varpi_i : (\chi, \xi) \to S^3\) by
\[
\phi^i = a \frac{\chi_t}{1 + \chi^2}, \quad (i = 1, 2)
\]  
(2.43)

where \(\chi^2 = \chi_1^2 + \chi_2^2\). In this space the rhs of (2.26) takes the form
\[
\varpi_i^*(T_i^{(a)}) = \frac{1}{e} \frac{1}{1 + \chi^2} (\chi_1 d\chi_1 - d\chi_2 \wedge \chi_2),
\]  
(2.44)

so that
\[
\tilde{\varpi}_1 = d(\varpi_1^* T_i^{(a)}) = - \frac{4}{e} \frac{1}{(1 + \chi^2)^2} d\chi_1 \wedge d\chi_2.
\]  
(2.45)

By passing into the polar coordinate system, we are readily led to
\[
\int \tilde{\varpi}_1 = - \frac{4}{e} \frac{2\pi}{1 + \chi^2} \frac{\chi d\chi}{(1 + \chi^2)^2} = - \frac{4\pi}{e}.
\]  
(2.46)

\section*{3. Gauge fields on \(S^3\)}

3.1. A regular connection

Now that we have listed the ingredients of [I] in terms of the complex \(z_\alpha\)
(α=1,2) variables, we are ready, adhering as closely as possible to the list, to construct gauge quantities on $S'$: The clue is only to replace $z_u \in \mathbb{C}$ by $q_u \in \mathbb{H}$.

1° Our starting expression is

$$\mathcal{Q}_1^\beta = \text{Re} \frac{h}{e} [q_1^* dq_1 - dq_1^* q_1 + q_2^* dq_2 - dq_2^* q_2]$$  \hspace{1cm} (3·1·1)$$

which just corresponds to (2·2). The “imaginary” number $h$ corresponds to $i = \sqrt{-1}$ in (2·2), but this time this $h$ can be any one of $i, j, k$, and hence we additionally need “Re” in (3·1·1). When we use the letter $h$ as the superfix, it shall run over 1, 2 and 3 corresponding respectively to $h = i, j, k$. And thus we have three components $(\mathcal{Q}_1^1, \mathcal{Q}_1^2, \mathcal{Q}_1^3)$, each of which is a regular 1-form on $S'$. To be more explicit, they are paraphrased as follows:

$$\mathcal{Q}_1^1 = \frac{2}{e} (q_a^2 dq_a^1 - q_a^1 dq_a^2 + q_a^3 dq_a^4),$$  \hspace{1cm} (3·1·2)$$

$$\mathcal{Q}_1^2 = \frac{2}{e} (q_a^3 dq_a^1 + q_a^4 dq_a^1),$$  \hspace{1cm} (3·1·3)$$

$$\mathcal{Q}_1^3 = \frac{2}{e} (q_a^1 dq_a^1 + q_a^2 dq_a^2 - q_a^3 dq_a^3).$$

It should further be remarked that, since the algebra of $\mathbb{H}$ loses the commutativity, the definition of $\mathcal{Q}_1^\beta$ given by (3·1·1) is nowhere unique, but we can assure that there may arise no essential difference if we follow another definition.

Let us next compose the $SU(2)$ gauge connection 1-form $\mathcal{Q}_1$ in a $(2 \times 2)$ matrix form by

$$\mathcal{Q}_1 = \mathcal{Q}_1^a \sigma^a / 2. \hspace{1cm} (3·1·4)$$

Then we have

$$\mathcal{Q}_1 = - \frac{i}{2e} \Sigma (q_1^* dq_1 - dq_1^* q_1 + q_2^* dq_2 - dq_2^* q_2),$$

where $\Sigma$ stands for the mapping defined at (1·10).

2° Even if we never recall the theory of characteristic classes, we can anticipate that, in view of (2·4) and (2·5), the integral which may express the topological number associated with $\mathcal{Q}_1$, be given by

$$\int \tilde{\phi} = 1 \hspace{1cm} (3·1·5)$$

with

$$\tilde{\phi} = \frac{3}{8\pi^2} \frac{1}{a^2} \sum_{\mu=0}^{1} \phi_\mu d\phi_\mu$$

$$= \frac{3}{8\pi^2} \frac{1}{a^2} \sum (-)^\phi_\mu d\phi_\mu \wedge \cdots \wedge d\phi_\mu \wedge \cdots \wedge d\phi_1. \hspace{1cm} (3·1·6)$$

* One should not confuse these with those appearing in (2·6).
The problem is then to know how $\eta_i^* \delta$ is related with the curvature matrix $\Omega_2$ of $\mathcal{O}_1$ defined by

$$\Omega_2 = d\Omega_1 + ie\Omega_1 \wedge \Omega_1$$  \hspace{1cm} (3.1.7)

or with

$$\text{tr } \Omega_1 \wedge \Omega_1 = ed(\Omega_1^1 \wedge \Omega_1^2 \wedge \Omega_1^3) + (1/2) d\Omega_1^3 \wedge \Omega_1^3.$$  \hspace{1cm} (3.1.8)

It is, however, quite burdensome to directly verify the conjecture, and we shall here bypass and defer the calculation until we compile some other formulae (up until § 3.2, §).  

3° We here generalize the matrix $g(x)$ of the gauge group given by (2.17) to

$$g(q) = -k \begin{pmatrix} q_2 & q_1 \* \* q_1 & -q_2 \end{pmatrix}$$ \hspace{1cm} (3.1.9)

under the proviso $|q_1|^2 + |q_2|^2 = 1$. It should be kept in mind that the entries in (3.1.9) are such quaternions that $g(q)$ belongs to $Sp(2)$. Similar to (2.18) quaternionic gauge fields $V^a$ can then be defined by

$$V^a_a = \frac{h}{e} \frac{\partial g}{\partial q_{a\*}^a} g^{-1}, \hspace{1cm} (\alpha = 1, 2)$$ \hspace{1cm} (3.1.10)

that is,

$$V^1_a = \frac{h}{2e} \begin{pmatrix} q_1 & -2q_2 \\ 0 & -q_1 \end{pmatrix}$$ \hspace{1cm} (3.1.11)

and

$$V^2_a = \frac{h}{2e} \begin{pmatrix} -q_2 & 0 \\ -2q_1 & q_2 \end{pmatrix}.$$ \hspace{1cm} (3.1.12)

These really generalize $(1/2) (w_1 + iw_2)$ and $(1/2) (w_3 + iw_4)$ of (2.11): For example, $h V^1_a$ decomposes as

$$h V^1_a = W^1_a + i W^1_1 + j W^1_2 + k W^1_3$$

with

$$W^1_a = -\frac{1}{2e} \begin{pmatrix} q_1 & -q_1^* + i q_2^* \\ q_2 & -q_2^* + i q_1^* \end{pmatrix},$$

$$W^1_1 = -\frac{1}{2e} \begin{pmatrix} q_1^* & -q_1 + i q_2 \\ q_2^* & -q_2 + i q_1 \end{pmatrix},$$

$$W^1_2 = -\frac{1}{2e} \begin{pmatrix} q_1^* & -q_1 + i q_2 \\ -q_2^* & -q_1 + i q_2 \end{pmatrix},$$

$$W^1_3 = -\frac{1}{2e} \begin{pmatrix} q_1^* & -q_2^* + i q_1 \\ q_2 & -q_1^* + i q_2 \end{pmatrix},$$  \hspace{1cm} (3.1.13)
The Higgs variables $\Phi$ of (2.9) also generalize as

$$\Phi = \begin{pmatrix} \phi_a & \phi^* \\ \bar{\phi} & -\bar{\phi}_i \end{pmatrix}$$

with quaternion entries. It is easy to see that

$$\text{tr}[(V^*_a)^*\Phi] = \frac{\hbar}{e} a q_a^*, \quad (a=1, 2)$$

and hence we can also write

$$\Omega_i^a = \text{Re} \left\{ \frac{1}{a} \left[ \text{tr}[(V^*_a)^*\Phi] dq_a - \text{h.c.} \right] \right\}$$

in conformity with (2.13).

4. The following is a way to associate $q_a$ on $S^7$ with the seven Eulerian angles $\theta, \theta_1, \theta_2, \psi_1, \phi_1, \phi_2$ and $\phi_3$:

$$q_a^a = |q_a| \cos \frac{\theta_a}{2} \cos \frac{\phi_1 + \phi_2}{2},$$
$$q_a^1 = |q_a| \sin \frac{\theta_a}{2} \sin \frac{\phi_1 - \phi_2}{2},$$
$$q_a^2 = |q_a| \sin \frac{\theta_a}{2} \cos \frac{\phi_1 - \phi_2}{2},$$
$$q_a^3 = |q_a| \cos \frac{\theta_a}{2} \sin \frac{\phi_1 + \phi_2}{2},$$

for $a=1, 2$ with

$$|q_2| = \cos \frac{\theta}{2} \quad \text{and} \quad |q_1| = \sin \frac{\theta}{2}$$

(cf. (2.40) for the latter). By this change of variables, $\Omega_i^a$'s given by (3.1.2) become

$$\Omega_i^1 = -\frac{1}{e} \sin^2(\theta/2) \left[ \sin \phi_1 d\theta_1 - \sin \theta_1 \cos \phi_1 d\phi_1 \right]$$
$$\Omega_i^2 = -\frac{1}{e} \cos^2(\theta/2) \left[ \sin \phi_1 d\theta_1 - \sin \theta_1 \cos \phi_1 d\phi_1 \right],$$
$$\Omega_i^3 = -\frac{1}{e} \sin^2(\theta/2) \left[ \cos \phi_1 d\theta_1 + \sin \theta_1 \cos \phi_1 d\phi_1 \right]$$
$$\Omega_i^4 = -\frac{1}{e} \cos^2(\theta/2) \left[ d\phi_1 + \cos \theta_1 d\phi_2 \right].$$

Note that $\Omega_i^a$ leaves some traces of $\Omega_i$ given by (2.20).

Corresponding to the decomposition of the gauge operator $g$ of (2.21), $g(q)$ given by (3.1.9) must decompose in such a way that
where \( g_i(\theta_i) \) represents a rotation by \( \theta_i \) around the \( i \)-th axis: \( g \) is thus generated by ten bases belonging to \( \mathfrak{sp}(2) \cong \mathfrak{so}(5) \) (in place of \( \mathfrak{su}(2) \)), and hence, contrary to the previous case, the relations between angles \( \theta_j (j = 1, 2, \ldots, 10) \) and seven Euler angles are not simple. This is partially due to the fact the \( S' \) makes no group while \( S \) is rather accidentally homeomorphic to \( SU(2) \).

3.2. Singular connections

5° As a generalization of (2.23), we introduce the unitary transition operator

\[
U_t = q_t /|q_t|, \quad (3.2.1)
\]

which is, however, no longer \( c \)-number.

By this, let us produce the gauge transformation and construct

\[
\Omega^{(a)b}_t = U_t Q^a u^{-1}_z + \frac{2i}{e} dU_t u^{-1}_z, \quad (3.2.2)
\]

corresponding to (2.24). Then explicitly we have

\[
\Omega^{(a)b}_t = \Re \frac{\hbar}{e} \frac{1}{|q_t|^2} \left[ q_t (q_t^* d q_t - d q_t^* q_t) q_t^* + |q_t|^2 (q_t dq_t^* - d q_t q_t^*) \right], \quad (3.2.3)
\]

Notice the similarity with the rhs of (2.25), but this time this is no more simplified because of the non-commutability of \( q_a \). The singularity which \( \Omega^{(a)b}_t \) embodies is now on the great "sphere" \( Q = \{ (q_t, q_z) | q_z = 0 \} \).

As in (3.1.3), let us compose the following as an \( \mathfrak{sp}(2) \)-symmetric \( SU(2) \)-gauge connection 1-form

\[
\Omega^{(a)b}_t = \Omega^{(a)b}_t \sigma^a /2. \quad (3.2.4)
\]

Then

\[
\Omega^{(e)}_t = - \frac{i}{2e} \frac{1}{|q_t|^2} \Sigma [q_t (q_t^* d q_t - d q_t^* q_t) q_t^* + |q_t|^2 (q_t dq_t^* - d q_t q_t^*)]. \quad (3.2.5)
\]

The gauge transformation (3.2.2) also takes the form

\[
\Omega^{(e)}_t = U_t Q_t U_t^{-1} + \frac{i}{e} dU_t U_t^{-1}, \quad (3.2.6)
\]

\( \star \) \( S \) consists of unit quaternions and \( S' \) does of unit Cayley numbers. Multiplications of unit quaternions make unit quaternions, while product of unit Cayley numbers does not necessarily. I am indebted to J. Sekiguchi for this observation.
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where $U_1$ implies a $(2 \times 2)$ matrix $\in \text{SU}(2)$ defined by

$$U_1 = \Sigma(u_2) = \frac{1}{|q_1|} \begin{pmatrix} q_1^* - iq_1^* & -i(q_1^* - q_1) \\ -iq_1^* + q_1 & q_1^* + iq_1^* \end{pmatrix}. \quad (3.2.7)$$

We are now in a position to invoke the Hopf map $\gamma_1$ introduced at (1.12). By the use of $\phi \in \mathcal{H}$ and $\phi_4$, $\Omega^{(a)}_1$ simply reads as follows:

$$\Omega^{(a)}_1 = \gamma_2^*(\gamma_1^{(a)}) \quad (3.2.8)$$

with

$$\gamma^{(a)}_1 = -\frac{i}{4ae} \frac{1}{a + \phi_4} \Sigma(\phi^*d\phi - d\phi^*\phi) \quad (3.2.9)$$

and thus $\gamma^{(a)}_1$ becomes a gauge field on the base-space $S'$ having singularity on the point $\phi_4 = a$ (to which the great "sphere" $Q_1$ has been Hopf mapped).

6° Quite the same argument with the unitary operator $u_2$ of (3.2.1) replacing $u_1$ of (3.2.1) yields a third connection

$$\Omega^{(b)}_1 = u_1 \Omega^{(a)}_1 u_1^{-1} + \frac{2i}{e} du_1 u_1^{-1}, \quad (3.2.10)$$

which explicitly paraphrases as

$$\Omega^{(b)}_1 = -Re \frac{\hbar}{e} \frac{1}{|q_1|^2} \left[ |q_1|^2 (d_1 q_1^* q_1^* - q_1^* d_1 q_1) \right. \right.$$  
$$+ q_1 (d_1 q_1^* q_1 - q_1^* d_1 q_1^*) \left. |q_1|^2 \right] \quad (3.2.12)$$

defined on $S^7 \setminus Q_3$ where $Q_3 = \{ (q_1, q_3) | |q_1| = 0 \}$. In the $(2 \times 2)$ form, this connection is written

$$\Omega^{(b)}_1 = \frac{i}{2e} |q_1|^2 \Sigma[|q_1|^2 (d_1 q_1^* q_1^* - q_1^* d_1 q_1)] \right.$$  
$$+ q_1 (d_1 q_1^* q_1 - q_1^* d_1 q_1^*) |q_1|^2 \left. \right] \quad (3.2.13)$$

and the gauge transformation (3.2.11) takes the form

$$\Omega^{(b)}_1 = U_1 \Omega^{(a)}_1 U_1^{-1} + \frac{i}{e} du_1 U_1^{-1}, \quad (3.2.14)$$

where $U_1 = \Sigma(u_2)$.

Finally, corresponding to (3.2.9), we have

$$\Omega^{(b)}_1 = \gamma_2^* \left( \frac{i}{4ae} \frac{1}{a + \phi_4} (\phi d\phi^* - d\phi^*\phi) \right). \quad (3.2.15)$$
It should be remarked that the order of the product of \( \phi^* \) (or \( \phi \)) and \( d\phi \) (or \( d\phi^* \)) in (3.2.15) differs from the order in (3.2.9).

7° Connection forms \( \Omega_i^{(a)} \) and \( \Omega_i^{(b)} \) are also related by gauge transformation:

\[
\Omega_i^{(a)} = T^{-1}\Omega_i^{(b)}T - \frac{i}{e} T^{-1}dT,
\]

(3.2.16)

where

\[
T = \Sigma(\tau) \text{ with } \tau = q_1q_2^*/|q_1||q_2|.
\]

(3.2.17)

Definition (3.2.17) clearly shows that, as in (2.38) and (2.39), \( \phi \in H \) and \( \tau \in H \) are related:

\[
\phi = 2a|q_1||q_2|\tau = a \sin \theta_1 \tau,
\]

(3.2.18)

where \( \tau \) is unitary.

Note that \( U_1, U_2 \) and \( T \) belong to \( SU(2) \) and if we use, as in (2.37) and (2.41), the Eulerian angles to represent them, they take familiar forms: For instance

\[
U_\alpha = \begin{pmatrix}
\cos \frac{\theta_\alpha}{2} \exp\left(-i\frac{\phi_\alpha + \varphi_\alpha}{2}\right) & \sin \frac{\theta_\alpha}{2} \exp\left(i\frac{\phi_\alpha - \varphi_\alpha}{2}\right) \\
\sin \frac{\theta_\alpha}{2} \exp\left(-i\frac{\phi_\alpha - \varphi_\alpha}{2}\right) & \cos \frac{\theta_\alpha}{2} \exp\left(i\frac{\phi_\alpha + \varphi_\alpha}{2}\right)
\end{pmatrix}
\]

(3.2.19)

which is compactly written

\[
U_\alpha = g_1(\varphi_\alpha)g_2(\theta_\alpha)g_3(\phi_\alpha) \quad (\alpha = 1, 2)
\]

(3.2.20)

with \( g_\alpha(\phi) \) given in (2.22). \( T = \Sigma(\tau) \) is paraphrased as \( T = U_1U_2^{-1} \) so that

\[
T = g_1(\varphi_1)g_2(\theta_1)g_3(\phi_1 - \varphi_1)g_4(-\theta_1)g_5(-\varphi_2).
\]

(3.2.21)

8° We shall close the present study by alluding to the topological integral corresponding to (2.5) or (2.46).

We first regard \( S^4 \) as \( E^4 \cup \{\infty\} \), and send the connection and all that to those on \( E^4 \) by stereographic projection.\(^*\) Let us define \( \mathfrak{g}_i: (\chi_0, \chi_1, \chi_2, \chi_3) \rightarrow S^i \) by

\[
\phi_i = a \frac{2\chi_i}{1 + \chi^2}, \quad (i = 0, 1, 2, 3)
\]

(3.2.22)

\[
\phi_1 = a \frac{1 - \chi^2}{1 + \chi^2}
\]

\(^*\) Frequently used in studying the O(5) symmetry of the instanton solution (see for example Ref. 7).
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where $\chi^2 = \chi_0^2 + \chi_1^2 + \chi_2^2 + \chi_3^2$. Then, denoting $\omega_1^{(a)}(Y_1^{(a)}) = \omega_1^{(n)}$, we readily have

$$\omega_1^{(n)} = -\frac{1}{e} \cdot \frac{1}{1 + \chi^2} \omega^a,$$

(3.2.23)

where

$$
\begin{align*}
\omega^a &= 2(\chi_0 d\chi_1 - \chi_1 d\chi_0 - \chi_3 d\chi_3 + \chi_2 d\chi_2), \\
\omega^b &= 2(\chi_2 d\chi_3 - \chi_3 d\chi_2 - \chi_0 d\chi_0 + \chi_1 d\chi_1), \\
\omega^c &= 2(\chi_3 d\chi_0 - \chi_0 d\chi_3 - \chi_1 d\chi_1 + \chi_2 d\chi_2).
\end{align*}
$$

(3.2.24)

Let us further write $\omega$ for $\omega^a/2$. Then (3.2.23) defines

$$\omega_1^{(n)} = -\frac{1}{e} \cdot \frac{1}{1 + \chi^2} \omega.$$  

(3.2.25)

The curvature $\omega_1^{(n)}$ of $\omega_1^{(n)}$ is then given by

$$\omega_2^{(n)} = d\omega_1^{(n)} + i e \omega_1^{(n)} / \omega_1^{(n)}.$$  

(3.2.26)

When given a curvature matrix $\omega_2$, the so-called invariant polynomial $P_i(\omega_2)$ is to be defined through

$$\det\left(\lambda I + \frac{1}{2\pi i} \omega_2\right) = \sum_{i=1}^{3} (-1)^i P_i(\omega_2) \lambda^{3-i}.$$  

(3.2.27)

Then the $i$-th Chern class $c_i$ when defined as a differential form is given by

$$c_i = P_i(\omega_2),$$  

(3.2.28)

whence we have

$$
\begin{align*}
c_1 &= (2\pi i)^{-1} \det \omega_2, \\
c_1 &= -(2\pi i)^{-1} \text{tr} \omega_2.
\end{align*}
$$

(3.2.29)

On the other hand, the first Pontrjagin class $p_1$ is composed as

$$p_1 = -2c_1 + c_1^2$$  

(3.2.30)

(cf. Th. 4.6.1 in Ref. 9)), so that

$$p_1 = \frac{1}{(2\pi i)^2} \text{tr} \omega_2 / \omega_2.$$  

(3.2.31)

It is natural for the Pontrjagin index to enter also into our problem as an indicator of topological charge and it is thus expected that $\int p_1$ or $(1/2)\int p_1$ becomes the integral corresponding to (2.5).

It is easily verified that

...
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\[ \text{tr} \, \omega_1^{(0)} \wedge \omega_2^{(0)} = ed (\omega^1 \wedge \omega^2 \wedge \omega^3) + (1/2) d \omega^8 \wedge d \omega^8 \]
\[ = - \frac{1}{e^2} \cdot \frac{48}{(1 + \chi^2)^4} d\chi_8 \wedge d\chi_8 \wedge d\chi_8, \quad (3.2.32) \]
and hence
\[ (e^2/2) \int P_1 = 1. \quad (3.2.33) \]

It is apparent that \( P_1 \) of (3.2.31) is invariant against the gauge transformations, and therefore (3.2.33) remains unchanged if we start from \( Q_1^{(0)} \) or originally from \( Q_1 \). As to the relation between the integral in \( E_1 \) and the surface integral considered in \( \S 3.1, 2^2 \), the reader may be referred to an ingenious discussion in Ref. 7).

§ 4. Yang’s monopole

In 1978, Yang published a paper, in which Dirac’s \( U(1) \) monopole was generalized in the framework of the \( SU(2) \) gauge theory: By making a scrutiny into the so-called non-integrable phase factors, he judiciously constructed the \( SU(2) \) gauge fields which might be singular at the origin of a 5-space. The work is also a successful application of ideas of Wu and Yang.’

To obtain a rough understanding, imagine an instanton in \( E^4 \) to be sticked on the surface of the compactified space \( E^4 \cup \{ \infty \} = S^4 \). Then the instanton necessarily drills a singularity on a point of \( S^4 \) or on its antipodal point. And thus Wu and Yang’s idea of local covering \( \{ R_a, R_b \} \) of \( S^4 \) and transition functions finds its place.

Yang defines a four-sphere by
\[ x_1^2 + x_2^2 + \cdots + x_5^2 = r^2 \quad (4.1) \]
and passes to the coordinate system \( (r, \theta, \xi_1, \xi_2, \xi_3) \) defined by
\[
\begin{align*}
x_i &= r \sin \theta \cdot 2 \zeta (1 + \xi^2)^{-1}, \quad (i = 1, 2, 3) \\
x_4 &= r \sin \theta \cdot (1 - \xi^2) (1 + \xi^2)^{-1}, \\
x_5 &= r \cos \theta,
\end{align*}
\]
where \( \xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \). Then Yang’s monopole gauge potentials are given, for example, by \( b_k^i \) subjected to
\[ \frac{b_k^i}{2} \frac{1}{2} (1 - \cos \theta) \frac{\partial R(T)}{\partial \xi^k} \quad (4.3) \]
on \( R_a \). (See Eq. (28) in Ref. 3.) The representation of the group element is of the form
\[ R(T) = \frac{1 - \xi^2 + 2i \xi \sigma^i}{1 + \xi^2}. \quad (4.4) \]
Our purpose in this paper is to show that our \( \gamma_1^{(a)} \) or \( \gamma_1^{(b)} \) on \( S^7 \) Hopf-mapped from \( \Omega^{(a)}_1 \) or \( \Omega^{(b)}_1 \) on \( S^7 \) does formally produce the Yang monopole solutions.

Let us first identify our \( \phi_\alpha \) and Yang's \( x_\alpha \) in such a way that
\[
\phi_0 = x_0, \quad \phi_i = x_i \quad (i = 1, 2, 3) \quad \text{and} \quad \phi_4 = x_4
\]
and also \( a = r \). Then (4·2) implies
\[
\begin{align*}
\phi_0 &= a \sin \theta \cdot 2\xi^1_1 (1 + \xi^2)^{-1}, \\
\phi_0 &= a \sin \theta \cdot (1 - \xi^3) (1 + \xi^3)^{-1}.
\end{align*}
\]

Let us define a quaternion \( \Xi \) by
\[
\Xi = 1 + i\xi^1_j + j\xi^2_j + k\xi^3_j.
\]
Then (4·6) takes the form
\[
\phi = a \sin \theta \cdot \Xi/\Xi^*.
\]
On referring to (3·2·18), we can readily identify
\[
\tau = \Xi/\Xi^* \quad \text{or} \quad \tau^* = \Xi^*/\Xi
\]
so that our \( T = \Sigma(\tau) \) in (3·2·17) becomes
\[
T = 1 - \xi^2 - 2i\xi^1 \theta^j \xi^j
\]

Note next that by the relation \( \phi = a \sin \theta \cdot \tau \), (3·2·18), we have the formula
\[
\phi^* d\phi - d\phi^* \phi = 2a^2 \sin^2 \theta \cdot \tau^* d\tau.
\]
Hence \( \gamma_1^{(a)} \) from (3·2·9) simply takes the form
\[
\gamma_1^{(a)} = \frac{i}{2e} (1 - \cos \theta) d\tau^* \tau.
\]

By applying the \( \Sigma \)-map of (1·10) to (4·12), recalling (4·10) and comparing (4·12) with (4·3), we can easily convince ourselves that \( \gamma_1^{(a)} \) really recovers the monopole fields \( b_1^{(a)} \) of solution \( a \).

Similarly it follows from the relation
\[
d\phi^* - \phi d\phi^* = 2a^2 \sin^2 \theta \cdot d\tau^*
\]
that \( \gamma_1^{(b)} \) from (3·2·13) becomes
\[
\gamma_1^{(b)} = \frac{i}{2e} (1 + \cos \theta) d\tau^*
\]
and it will also soon turn out that this reproduces Yang's fields \( b_1^{(b)} \) of solution \( a \).
To obtain the set of Yang's solutions which are associated with the opposite topological number, we must alternatively start with the other Hopf map whose invariant differs in sign from the Hopf invariant associated with \((1\cdot 12)\); that is, with the map defined by

\[
\phi = 2aq_i q_i^*, \quad \phi_i = a (|q_i|^2 - |q_i|^2).
\]  
\[(4\cdot 15)\]

The argument will quite be parallel and therefore we shall avoid repeating it.

For other details of Yang's \(SU(2)\) monopole solutions, the reader is referred to the original paper,\(^31\) where the properties of the solutions are exhaustively expounded in terms of the \(\xi\) variables. Yang has also completed a study on the \(SU(2)\) monopole harmonics.\(^{10}\)

§ 5. Remarks

a° Although we have used the terminology “quaternionic” for the gauge fields on \(S^7\), the 3-sphere \(S^3\) is already quaternionic since it is represented by \(|x|^2 = 1\) once we define the quaternion \(x\) by

\[
dx = x_1 + ix_2 + jx_3 + kx_4.
\]  
\[(5\cdot 1)\]

The element \(g\) given by \((2\cdot 17)\) of the \(SU(2)\) gauge group is also simply

\[
g = \Sigma (x/|x|).
\]  
\[(5\cdot 2)\]

In this sense, the gauge quantities associated with \(S^7\) are to be describable by the Cayley numbers.

b° The remaining non-trivial Hopf map (that is, the maps containing elements of Hopf invariant 1) is \(\eta_1: S^5 \rightarrow S^8\).\(^{21}\) Given a pair of Cayley numbers \((c_1, c_2)\), then \(S^5\) is represented by

\[
|c_1|^2 + |c_2|^2 = 1.
\]  
\[(5\cdot 3)\]

And therefore there is hope for further generalization; but it is rather discouraging that the algebra of octonions loses the property of associativity in addition to the commutativity.

c° We admit that we have not covered all the subjects which had been contained in \([1]\). One of them is a systematic discussion of the quantities on \(S^7\) or in \(E^8\): The discussion of the Higgs vacuum composed by \(\phi_e\) as well as the pure-gauge fields \(W_\alpha^\mu (\alpha = 1, 2, \mu = 0, 1, \cdots, 3)\) on \(S^7\) (cf. \((3\cdot 1\cdot 13)\)) could have been interesting in itself.

d° Another important lack is that we have failed to refer to the Whitehead-type integral representation on \(S^7\) for the topological charge. This is partly due to our fail to find the corresponding formula on \(S^7\) in J. H. C. Whitehead's mathemat-
ical works, but mainly due to our feeling that a much more bulk of advanced languages must be compiled. It should, however, be remarked that the integral itself does certainly exist. Let $\Theta_1$ be given by

$$\Theta_1 = \frac{\sqrt{3}}{2\pi} \sum_{l=0}^{3} (-1)^l \, [d_q^L d_q^M \wedge \cdots \wedge d_q^N].$$

(5·4)

Then

$$d\Theta_1 = 2\frac{\sqrt{3}}{\pi} \, dq^a \wedge dq^b \wedge dq^c \wedge dq^d$$

(5·5)

and so we have

$$\int_{S^7} \Theta_1 \wedge d\Theta_1 = 1$$

(5·6)

corresponding to

$$\int_{S^7} Q_1 \wedge dQ_1 = 1$$

(5·7)

(cf. (3·4) in [I]). The problem is then to relate $\Theta_1$ with $Q_1$ or $Q_2$. Recall, however, that it is only in the Abelian case that $dQ_1$ becomes a curvature $Q_1$ of $Q_1$, and so (5·6) is not so simple as (5·7). It is rather suggestive to consider that $\Theta_1$ is in connection with the invariant polynomial because it satisfies $P(Q_1', \cdots) = dP(Q_1', \cdots)$, and the integral (5·6) will turn out to be related with the second Pontrjagin class (or 4th Chern class) when converted to the integral in $F$. These subjects will be treated elsewhere.

References


