Fractal Dimension in High Energy Nucleus-Nucleus Collisions

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The pseudorapidity distributions of the high multiplicity events of the HELIOS-emulsion collaboration and the EMU01 collaboration are analysed. The fractal dimensions are obtained analysing each event by the length method of Higuchi, and by the power spectrum analysis together with the maximum entropy method. It is found that the fractal dimension of the EMU01 event is higher than those of the four HELIOS events. The results are compared with those of the JACEE events.

§ 1. Introduction

One expects that quark gluon plasma (QGP) will be formed in high energy nucleus-nucleus (AA) collisions. A possible signal would be large multiplicity fluctuation in small rapidity intervals.1,2)

Takagi applied the power spectrum method3) to the analysis of the pseudorapidity distributions of the JACEE events.4) He concluded that anomalous non-statistical fluctuations are observed in Si-AgBr and Ca-C events.

Bialas and Peschanski proposed the multiplicity moment analysis in small pseudorapidity intervals $\Delta \eta$, and found5) weak intermittency in the Si-AgBr event observed by JACEE.

Analyses of AA collisions6) in accelerator experiments show that the strength of intermittency decreases with increasing multiplicity for constant incident energy. For $^{16}$O-Emulsion collisions, it decreases with increasing incident energy. In S-Ag collisions at 200 GeV/A,7) weak intermittency is observed with almost vanishing slope for $\Delta \eta < 0.1$. In the recent HELIOS-emulsion experiment8) intermittency is not observed. Thus from the simple analysis of intermittency by the moment method, it would be very difficult to extract the characteristic features of high multiplicity events.

In this paper we first analyse the pseudorapidity distributions $d\eta/d\eta$ of the four high multiplicity events in the tail region of the transverse energy ($E_T$) distributions reported by the HELIOS-emulsion collaboration9) at 200 GeV/A. We also analyse the S-Ag event of the EMU01 experiment7) at 200 GeV/A. We show $d\eta/d\eta$ of these events in Fig. 1, for $\Delta \eta = 0.1$.

In order to evaluate the degree of complexity of the events, we apply two methods: (A) the length method proposed by Higuchi10) on the basis of fractal theory, and (B) the power spectrum analysis together with the maximum entropy method (MEM).11)–14)

With these two methods, we estimate the fractal dimensions15) related to $d\eta/d\eta$. In Ref. 16) we already applied these two methods to the three JACEE events. We compare the results obtained here with those of Ref. 16).
Fig. 1. Pseudorapidity distributions of (a) S-W event ($E_p=267 \text{ GeV}$) and (b) S-Ag event.

§ 2. Higuchi's method

We define for the $i$-th bin,

$$X(i) = \frac{d n}{d \eta} \bigg|_{\Delta \eta} \Delta \eta, \quad (i=1, 2, \ldots, N)$$

(1)

where bin $i$ has pseudorapidity $\eta_i = \eta_1 + (i-1) \cdot \Delta \eta$, $\eta_1$ is the minimum, and $\eta_N = \eta_1 + (N-1) \cdot \Delta \eta$ is the maximum of the central values of the bins. Equation (1) gives an ordered set $\{X(i), i=1, 2, \ldots, N\}$.

Following Ref. 9) we define for fixed $k (k < N)$ and $j=1, 2, \ldots, k$,

$$L_j(k) = \frac{1}{k} \sum_{i=1}^{i_m} |X(j+k \cdot i) - X(j+k \cdot (i-1))| \frac{N-1}{i_m \cdot k},$$

(2)

where $i_m = [(N-j)/k]$ is the integer part of $(N-j)/k$. The length $\langle L(k) \rangle$ of the curve associated with $dn/d\eta$ for the interval $k$, is defined as the average $\langle L(k) \rangle = (1/k) \times \sum_{j=1}^{k} L_j(k)$. If $\langle L(k) \rangle$ behaves like a power law of $k$,

$$\langle L(k) \rangle \propto k^{-D}, \quad (1 \leq D \leq 2)$$

(3)

then $D$ is called the fractal dimension of $dn/d\eta$.

§ 3. Data analysis

The results of our analysis of the five events of Refs. 7) and 9) by Higuchi's method are shown in Table I. We show in Figs. 2(a) and (b) the behavior of $\ln \langle L(k) \rangle$ as a function of $\ln k$ for the first event of Ref. 9) (see Table I) and of the event of Ref. 7), respectively. As can be seen, $\ln \langle L(k) \rangle$ depends linearly on $\ln k$ for $\ln k < 3$. The slope parameter $D$ is determined by the method of linear regression in the interval $1 \leq k \leq 15$. We find that the fractal dimensions $D$ of the four HELIOS events are almost the same and are about 1.6 (see Table I). However, the dimension of the EMU01 event is somewhat larger. We also determine $D$ by using the points from $k=1$ to $k=20$. The results are also shown in Table I. As can be seen, the resulting values for $D$ are almost the same.
Fig. 2. Length \( <L(k)\) vs interval \( k \) with Higuchi's method. C.C. denotes the correlation coefficient. (a) S-W \( (E_T=267 \text{ GeV}) \) event and (b) S-Ag event.

Table I. Fractal dimension \( D \) determined by Higuchi's method and by the MEM.

<table>
<thead>
<tr>
<th></th>
<th>Higuchi ( (k=1-15) )</th>
<th>Higuchi ( (k=1-20) )</th>
<th>MEM ( (f=0.2-1.1) )</th>
<th>MEM ( (f=0.3-1.2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-W ( (E_T=267 \text{ GeV}) )</td>
<td>1.63</td>
<td>1.61</td>
<td>1.66 ( (M=2) )</td>
<td>1.63 ( (M=2) )</td>
</tr>
<tr>
<td>S-W ( (E_T=296 \text{ GeV}) )</td>
<td>1.67</td>
<td>1.61</td>
<td>1.64 ( (M=3) )</td>
<td>1.65 ( (M=3) )</td>
</tr>
<tr>
<td>S-Ag ( (E_T=179 \text{ GeV}) )</td>
<td>1.64</td>
<td>1.57</td>
<td>1.67 ( (M=3) )</td>
<td>1.66 ( (M=3) )</td>
</tr>
<tr>
<td>O-Ag ( (E_T=114 \text{ GeV}) )</td>
<td>1.66</td>
<td>1.63</td>
<td>1.64 ( (M=3) )</td>
<td>1.64 ( (M=3) )</td>
</tr>
<tr>
<td>S-Ag ( (E_T=200 \text{ GeV}) )</td>
<td>1.81</td>
<td>1.78</td>
<td>1.76 ( (M=3) )</td>
<td>1.72 ( (M=3) )</td>
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<th>MEM ( (f=0.2-1.0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fe-Pb ( (JACEE) )</td>
<td>1.59</td>
<td>1.53</td>
<td>1.60 ( (M=4) )</td>
<td>1.64 ( (M=4) )</td>
</tr>
<tr>
<td>Si-AgBr ( (JACEE) )</td>
<td>1.72</td>
<td>1.67</td>
<td>1.70 ( (M=3) )</td>
<td>1.64 ( (M=3) )</td>
</tr>
<tr>
<td>Ca-C  ( (JACEE) )</td>
<td>1.93</td>
<td>1.91</td>
<td>1.84 ( (M=5) )</td>
<td>1.82 ( (M=5) )</td>
</tr>
</tbody>
</table>

§ 4. Power spectrum analysis

Next we examine the power spectrum of the data, by first introducing the quantity\(^{13}\)
\[
S(f) = \sum_{k=-\infty}^{\infty} R[k] \exp(-2\pi i k \Delta \eta).
\] (4)

\(R[k]\) is the autocorrelation function, defined as
\[
R[k] = E\{X(i+k)X(i)\}
\] (5)
for the set of \(X(i)\) given in Eq. (1). In Eq. (5) \(E\{\cdots\}\) indicates the ensemble average. As we have only a finite set \(\{X(i), i=1, 2, \ldots, N\}\), we determine the power spectrum by using the maximum entropy method (MEM).\(^{11)}\)

First the \(m\)-th order forward and backward predictor errors are defined as\(^{12})\)
\[
\tilde{e}_m[n] = X(n) - \sum_{k=1}^{m} a_k^m X(n-k),
\] (6)
\[
\tilde{e}_m[n-m] = X(n-m) - \sum_{k=1}^{m} a_k^m X(n-m+k).
\] (7)

\(m\) is an integer to be determined below (see Eq. (15)). The coefficients \(a_k^m\) in Eqs. (6) and (7) are determined by minimizing the mean square estimation error of the forward and backward predictor errors, defined as \(\hat{P}_m = E\{\tilde{e}_m[n]^2\}\) and \(\tilde{P}_m = E\{\tilde{e}_m[n]^2\}\), respectively.

The coefficients \(a_k^m\) are calculated by the following recursion equations: As shown in Ref. 12), the predictor errors satisfy
\[
\tilde{e}_m[n] = \tilde{e}_{m-1}[n] - \tilde{P}_m \cdot \tilde{e}_{m-1}[n-m],
\] (8)
\[
\tilde{e}_m[n-m] = \tilde{e}_{m-1}[n-m] - \tilde{P}_m \cdot \tilde{e}_{m-1}[n],
\] (9)
where \(a_k^m\) and \(\tilde{P}_m\) are related by
\[
a_k^m = a_k^{m-1} - \tilde{P}_m a_m^{m-1} \quad (k=1, 2, \ldots, m-1), \quad a_m^m = \tilde{P}_m,
\] (10)
and \(\tilde{P}_m\) also enters into the recursion relation
\[
\tilde{P}_m = (1-\tilde{P}_m^2) \tilde{P}_{m-1}.
\] (11)

\(\tilde{P}_m\) is expressed by the \((m-1)\)-th order forward and backward predictor errors as
\[
\tilde{P}_m = \frac{2}{\tilde{P}_{m+1}} \sum_{n=m+1}^{N} \tilde{e}_{m-1}[n] \cdot \tilde{e}_{m-1}[n-m] \sum_{n=m+1}^{N} (\tilde{e}_{m-1}[n]^2 + \tilde{e}_{m-1}[n-m]^2).
\] (12)

The recursion starts with
\[
\tilde{P}_0 = \frac{1}{N} \sum_{i=1}^{N} X(i)^2 \quad \text{and} \quad \tilde{e}_0[n] = \tilde{e}_0[n] = X(n). \quad (n=1, 2, \ldots, N)
\] (13)

The most probable value \(M\) of the order \(m\) is determined by the Akaike information criterion (AIC):\(^{17})\)
\[
(AIC)_m = N \ln(2\pi \tilde{P}_m) + N + 2(m+1), \quad (m=1, 2, \ldots)
\] (14)
has its minimum at \(m=M\).
Then the $m$-th order estimation $\tilde{S}_m(f)$ of the power spectrum $S(f)$ is written as\(^{12}\)

$$
\tilde{S}_m(f) = \frac{\tilde{P}_m \Delta \eta}{\sqrt{1 - \sum_{k} a_k^n \exp(-2\pi i k M \Delta \eta)}}.
$$

(15)

The best estimate of $S(f)$ is given by $\tilde{S}_m(f)$ for $m=M$.

If the power spectrum of $dn/d\eta$ behaves as

$$
S(f) \propto f^{-\gamma},
$$

(16)

its fractal dimension $D$ is given by\(^{10,15}\)

$$
D = (5 - \gamma) / 2.
$$

(17)

\section{Analysis of data by the MEM}

For the analysis of the data we replace $X(i)$ in Eq. (13) by $X(i)-\bar{X}$ ($i=1, \cdots, N$), where $\bar{X}=(1/N)\sum_{i=1}^{N}X(i)$. The power spectra of the same two events of Refs. 7) and 9) are shown in Fig. 3. Again a power law behavior of $S(f)$ is obtained. The exponent $\gamma$ for each event is determined with the method of linear regression in the frequency intervals $0.2 \leq f \leq 1.1$ and $0.3 \leq f \leq 1.2$. The fractal dimensions of the five events of Refs. 7) and 9) determined by the MEM are also shown in Table I. The results are almost the same as those obtained by Higuchi's method. The differences are within 0.10.

\section{Energy density dependence of fractal dimension}

One expects that QGP is formed in AA collisions if a state with an energy density of a few GeV/fm$^3$ is reached. Here we examine whether for the five events of Refs. 7) and 9) there is any correlation between the fractal dimension $D$ and the energy density $\varepsilon$. The energy density is calculated by the formula,\(^{18}\)

$$
\varepsilon = 1.5\sqrt{\langle p_T^2 \rangle} + m_A^2 (dn/d\eta_c)/(2\pi A_{\text{min}}^{1/3}) \text{ GeV/fm}^3,
$$

\footnote{The value of $\langle p_T^2 \rangle$ is calculated from the transverse momentum distribution of charged hadrons.}
where $<p_T>$ is the average transverse momentum of the produced particles, $m_\pi$ is pi-meson mass, and $dn/d\eta_c$ is the particle number density in the central region of the event. $A_{mn}$ is the minimum of the mass numbers of the two colliding nuclei. We calculate $<p_T>$ of the four HELIOS events from the $E_T$ observed in the range $0.1 \leq \eta \leq 3.0$, and take $dn/d\eta_c$ as the average of $dn/d\eta$ in the same pseudorapidity interval. For the EMU01 event, we assume $<p_T>=0.4$ GeV/$c$, and take $dn/d\eta_c$ also as the average of $dn/d\eta$ in $0.1 \leq \eta \leq 3.0$.

We plot in Fig. 4 the fractal dimensions of the five high multiplicity events of Refs. 7) and 9) together with the three JACEE events[16,19] as a function of $\epsilon$. The three JACEE events have an energy density higher than 2 GeV/fm$^3$, with the Ca-C event having the highest fractal dimension.

§ 7. Concluding remarks

Four HELIOS events and one EMU01 event are analysed by Higuchi's method and by the MEM. It is found that the fractal dimension of each event deduced from both methods is almost the same. Our analysis shows that these methods can be applied to classify the pseudorapidity distributions of high multiplicity events. We also demonstrate that the fractal dimension is a useful concept to measure the complexity of the distributions.

The fractal dimensions of the four HELIOS events in the tail region of $E_T$ distributions are about 1.6. To confirm whether this value is characteristic of the tail region or not, we have to analyse other data in various $E_T$ regions. Very recently, it is informed from one of the HELIOS collaborators[20] that the fractal dimension of the event (No. 50/424) in the shoulder $E_T$ region is estimated as 1.30 by Higuchi's method. The value $D=1.6$ in the tail region is very close to that of Fe-Pb event observed by JACEE.[16] We also found that the fractal dimension of the EMU01 event is higher than those of the four HELIOS events. Furthermore, the Ca-C event has a fractal dimension of about 1.93, almost the highest value possible, different from the situation in the $\epsilon-<p_T>$ plot in Ref. 19).

If large fluctuations in $dn/d\eta$ are a good signature for QGP,[1,2] our methods will be useful tools to select the good candidates for this signal: As the fluctuations become larger, the fractal dimension $D$ approaches 2.

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References