On Rephasing Invariants $\Delta_{ia}$ in Three Generations of Quarks

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Assuming that there are three generations of quarks, we express $[\text{Im}(\Delta_{ia})]^2$ and the ratio of $|U_{i6}|$ in terms of $\text{Re} \Delta_{ia}$ to find relations among $\Delta_{ia}$ and further we examine a restriction on $\Delta_{ia}$ according to the present data, where $\Delta_{ia}=U_{ia}U_{ia}^*$ are rephasing invariants for the quark mixing matrix $U$.

The six quark mixing scheme serves as a useful parametrization of the connection between generations of quarks. In this scheme, the weak currents are given by

$$\left(\bar{u} \, c \, \bar{t} \right) \gamma_\mu (1-\gamma_5) U \begin{pmatrix} d \\ s \\ b \end{pmatrix}.$$  \hfill (1)

The $3 \times 3$ matrix $U$ must be unitary, due to the constraint of universality of the charged weak interaction. As usual $U$ is defined by

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix},$$  \hfill (2)

where $U_{11}=U_{ud}$, $U_{12}=U_{us}$ and others. Then the unitarity relations read

$$\sum_a U_{ia}U_{ib}^* = \delta_{ij}, \quad \sum_i U_{ia}U_{ib}^* = \delta_{ab}.$$  \hfill (3)

The original Kobayashi-Maskawa matrix$^1$ is well known as one of parametrizations of $U$, where $s_i=\sin\theta_i$ and $c_i=\cos\theta_i$.

On the other hand, $|U_{i6}|$ in (2) is known from measurements on weak decays of light and heavy quarks and the unitarity relation (3), although a $t$ quark has not been discovered. For example, Particle Data Group$^{31}$ gives the quark mixing matrix in absolute values as follows,

$$|U| = \begin{pmatrix} 0.9748 \text{ to } 0.9761 & 0.217 \text{ to } 0.223 & 0.003 \text{ to } 0.010 \\ 0.217 \text{ to } 0.223 & 0.9733 \text{ to } 0.9754 & 0.030 \text{ to } 0.062 \\ 0.001 \text{ to } 0.023 & 0.029 \text{ to } 0.062 & 0.9980 \text{ to } 0.9995 \end{pmatrix},$$  \hfill (5)

which is almost the same as in others.$^{3,40}$ Since the quark mixing matrix $U$ suffers from a rephasing ambiguity, it is necessary to examine rephasing invariants$^3$ defined
by
\[
\Delta_{ia} = U_{ij} U_{ik} U_{ij}^* U_{ik}^*
\]
with \( i, j, k \) and \( \alpha, \beta, \gamma \) cyclic. Directly from (3) and (6), we have
\[
0 = U_{11} U_{12}^* (U_{11}^* U_{12} + U_{22}^* U_{32})
\]
\[
= |U_{11} U_{12}|^2 + \Delta_{13} + \Delta_{23}^2
\]
(7)
and others, which leads to that \( \text{Im}(\Delta_{ia}) \) are the same up to a sign.

Then, in this paper, we express \([\text{Im}(\Delta_{ia})]^2\) and the ratio of \(|U_{ij}|\) in terms of \(\text{Re} \Delta_{ia}\) to find relations among \(\Delta_{ia}\) and further we examine a restriction on \(\Delta_{ia}\) according to the present analysis (5). If we combine (7) with \(|U_{11}/U_{22}| = (U_{11} U_{12})/(U_{22} U_{21})|\)
\[
= |(U_{11} U_{21})/(U_{22} U_{21})| \quad \text{and others, we have}
\]
\[
|U_{11}/U_{22}|^2 = \text{Re}(\Delta_{23} + \Delta_{33})/|\text{Re}(\Delta_{31} + \Delta_{33})| = \text{Re}(\Delta_{32} + \Delta_{33})/|\text{Re}(\Delta_{31} + \Delta_{33})|,
\]
\[
|U_{12}/U_{21}|^2 = \text{Re}(\Delta_{31} + \Delta_{33})/|\text{Re}(\Delta_{13} + \Delta_{33})| = \text{Re}(\Delta_{23} + \Delta_{33})/|\text{Re}(\Delta_{21} + \Delta_{33})|,
\]
\[
|U_{13}/U_{21}|^2 = \text{Re}(\Delta_{23} + \Delta_{22})/|\text{Re}(\Delta_{33} + \Delta_{22})| = \text{Re}(\Delta_{21} + \Delta_{22})/|\text{Re}(\Delta_{31} + \Delta_{22})|,
\]
\[
|U_{23}/U_{31}|^2 = \text{Re}(\Delta_{31} + \Delta_{11})/|\text{Re}(\Delta_{31} + \Delta_{11})| = \text{Re}(\Delta_{21} + \Delta_{11})/|\text{Re}(\Delta_{21} + \Delta_{11})|,
\]
\[
|U_{33}/U_{11}|^2 = \text{Re}(\Delta_{21} + \Delta_{22})/|\text{Re}(\Delta_{23} + \Delta_{22})| = \text{Re}(\Delta_{31} + \Delta_{22})/|\text{Re}(\Delta_{33} + \Delta_{22})|.
\]

The relations among \(\Delta_{ia}\) are written from (8),
\[
\Delta_{21} \Delta_{31} - \Delta_{12} \Delta_{13} = (\Delta_{12} + \Delta_{13} - \Delta_{21} - \Delta_{31}) \Delta_{11}^2,
\]
\[
\Delta_{12} \Delta_{32} - \Delta_{21} \Delta_{23} = (\Delta_{21} + \Delta_{23} - \Delta_{12} - \Delta_{32}) \Delta_{22}^2,
\]
\[
\Delta_{13} \Delta_{23} - \Delta_{31} \Delta_{32} = (\Delta_{31} + \Delta_{32} - \Delta_{13} - \Delta_{23}) \Delta_{33}^2.
\]

Next we express \([\text{Im}(\Delta_{ia})]^2\) in terms of only \(\text{Re} \Delta_{ij}\). According to Ref. 4, \([\text{Im}(\Delta_{ia})]^2\) is written by
\[
[\text{Im}(\Delta_{ia})]^2 = |\Delta_{11}|^2 - [\text{Re}(\Delta_{11})]^2
\]
\[
= |U_{11} U_{13} U_{23} U_{21}|^2 - (|U_{11} U_{21}|^2 - |U_{12} U_{22}|^2 - |U_{13} U_{23}|^2)^2 / 4.
\]

A further simplification can now be made from (7),
\[
[\text{Im}(\Delta_{ia})]^2 = |U_{11} U_{12} U_{21} U_{22}|^2 + |U_{12} U_{13} U_{22} U_{32}|^2 + |U_{13} U_{11} U_{23} U_{21}|^2
\]
\[
- (|U_{11} U_{21}|^2 + |U_{12} U_{22}|^2 + |U_{13} U_{23}|^2)^2 / 4
\]
\[
= |\Delta_{31}|^2 + |\Delta_{32}|^2 + |\Delta_{33}|^2 - [\text{Re}(\Delta_{31} + \Delta_{32} + \Delta_{33})]^2
\]
\[
= 3[\text{Im}(\Delta_{ia})]^2 - 2(\text{Re} \Delta_{31} \text{Re} \Delta_{32} + \text{Re} \Delta_{32} \text{Re} \Delta_{33} + \text{Re} \Delta_{33} \text{Re} \Delta_{31})
\]

(11)

Through a similar procedure, \([\text{Re}(\Delta_{ia})]^2\) is expressed by
\[
[\text{Re}(\Delta_{ia})]^2 = \text{Re} \Delta_{11} \text{Re} \Delta_{32} + \text{Re} \Delta_{12} \text{Re} \Delta_{33} + \text{Re} \Delta_{13} \text{Re} \Delta_{31}
\]
\[
= \text{Re} \Delta_{11} \text{Re} \Delta_{32} + \text{Re} \Delta_{12} \text{Re} \Delta_{33} + \text{Re} \Delta_{13} \text{Re} \Delta_{31}
\]

(12)
with $k, \beta = 1, 2, 3$. It is easily known that (9) is included in (12). From (12), $|\text{Im}(\Delta_{1a})|^2$ does not exceed $\sum_i (\text{Re}\Delta_{ia})^2$ and $\sum_a (\text{Re}\Delta_{ia})^2$.

Finally we examine a restriction on $\Delta_{ia}$ from the present theoretical and experimental analysis (5). Before we discuss this restriction, we consider how the magnitudes of $|U_{ij}|$ in (5) are classified. For simplicity, we deal with a case of

$$|U_{11}| \approx |U_{22}| \approx |U_{33}|,$$
$$|U_{12}| \approx |U_{21}|, \quad |U_{13}| \approx |U_{31}|, \quad |U_{23}| \approx |U_{32}|.$$  

(13)

Further imposing the condition

$$|U_{11}|^2 \gg |U_{12}|^2 \gg |U_{23}|^2 \gg |U_{13}|^2$$  

(14)
on (13), we investigate the validity of these summarizations. It is assumed from (13) that the deviations of $|U_{12}/U_{21}|^2$ and others from unity are very small. For convenience, we introduce the following quantities $\xi$ and others,

$$\frac{(\Delta_{33} - \Delta_{31})}{(\Delta_{32} - \Delta_{23})} = 1 + \xi,$$
$$\frac{(\Delta_{31} - \Delta_{12})}{(\Delta_{32} - \Delta_{23})} = 1 + \epsilon,$$
$$\frac{(\Delta_{12} - \Delta_{21})}{(\Delta_{32} - \Delta_{23})} = -1 - \Delta,$$
$$\frac{(\Delta_{21} - \Delta_{12})}{(\Delta_{32} - \Delta_{23})} = -1 - \eta.$$  

(15)

Using these notations together with (8), we rewrite the condition (13) in the following forms,

$$|U_{23}/U_{11}|^2 = 1 + \Delta - \epsilon + \eta,$$
$$|U_{33}/U_{11}|^2 = 1 + \Delta,$$
$$|U_{21}/U_{12}|^2 = (1 + \xi - \Delta + \epsilon - \eta)/(1 + \xi),$$
$$|U_{31}/U_{13}|^2 = (1 + \xi + \epsilon)/(2 + \xi + \epsilon - \Delta),$$
$$|U_{32}/U_{23}|^2 = (1 + \eta)/(1 + \epsilon).$$  

(16)

which are almost equal to unity. Combining these relations with $|U_{23}|^2 + |U_{13}|^2 = |U_{33}|^2 + |U_{31}|^2$ from the unitarity relation (3), we have

$$0 = (\eta - \epsilon)(2 + \xi + \epsilon - \Delta)|U_{23}|^2 + \Delta(1 + \xi)|U_{13}|^2.$$  

(17)

Similarly we have

$$0 = (\epsilon - \eta)(2 + \xi + \epsilon - \Delta)|U_{23}|^2 + \Delta(1 + \xi)|U_{13}|^2.$$  

(18)

Substituting

$$\Delta = (\epsilon - \eta)(2 + \xi + \epsilon)|U_{23}|^2/[(1 + \epsilon)|U_{13}|^2 + (\epsilon - \eta)|U_{23}|^2]$$  

(19)

from (17) into (18), we have either

$$\epsilon = \eta.$$  

(20)
or
\[
(1 + \varepsilon)[(2 + \xi + \eta)|U_{23}|^2 - (1 + \varepsilon)|U_{13}|^2]|U_{12}|^2
= (1 + \xi)[(1 + \varepsilon)|U_{13}|^2 + (\varepsilon - \eta)|U_{23}|^2]|U_{23}|^2.
\]

But it is apparent that (21) does not agree with (13) and (14). Then we must have \(\varepsilon = \eta\) and \(\Delta = 0\) or
\[
|U_{ii}| = |U_{ji}| \quad \text{and} \quad |U_{ij}| = |U_{ij}|
\]
from (16). But, from the unitarity relations (3), this means that all of the off-diagonal elements of \(|U|\) must be the same. Judging from (5), the condition (13) with (14) should be modified.

Then, to find another possibility, we decompose condition (13) into the following three cases,
\[
|U_{11}| \approx |U_{22}| \quad \text{and} \quad |U_{12}| \approx |U_{21}| \quad \text{for} \quad \Delta_{13} + \Delta_{23} \approx \Delta_{31} + \Delta_{32},
\]
\[
|U_{22}| \approx |U_{33}| \quad \text{and} \quad |U_{23}| \approx |U_{32}| \quad \text{for} \quad \Delta_{21} + \Delta_{31} \approx \Delta_{12} + \Delta_{13},
\]
\[
|U_{33}| \approx |U_{11}| \quad \text{and} \quad |U_{31}| \approx |U_{13}| \quad \text{for} \quad \Delta_{31} + \Delta_{32} \approx \Delta_{32} + \Delta_{12}
\]
from (8). Comparing these relations with (5), we know that only a case of
\[
\Delta_{13} + \Delta_{23} \approx \Delta_{31} + \Delta_{32}
\]
is compatible with the present analysis of weak decays of quarks. This condition (26) is equivalent to
\[
\Delta \approx \varepsilon - \eta
\]
from (8) and (16). In a restriction (27), \(\Delta = \varepsilon - \eta\) or a combination of \(\Delta = 0\) and \(\varepsilon = \eta\) is not included because of (22). But, as is expected from (21), we can show that an approximation (27) does not always mean \(|U_{13}| \approx |U_{31}|\) and \(|U_{23}| \approx |U_{32}|\). Under (27), we can rewrite (16) as follows,
\[
|U_{22}/U_{11}|^2 \approx |U_{21}/U_{12}|^2 \approx 1,
\]
\[
|U_{33}/U_{11}|^2 = 1 + \Delta,
\]
\[
|U_{31}/U_{13}|^2 \approx (2 + \xi + \varepsilon)/(2 + \xi + \eta),
\]
\[
|U_{32}/U_{23}|^2 = (1 + \eta)/(1 + \varepsilon).
\]
Since \(\xi\) and others in (28) are arbitrary except (21) and (27), \(|U_{31}/U_{13}|^2\) and others can take the values which are largely deviated from unity. Here we give an example within (5),
\[
|U| = \begin{pmatrix}
0.975580 & 0.2195 & 0.007963 \\
0.2188 & 0.974218 & 0.055000 \\
0.019241 & 0.052136 & 0.998455
\end{pmatrix},
\]
which satisfies the unitarity relation (3) and the positivity\(^4\) of \([\text{Im}(\Delta_{ia})]^2\). This
example can be explained by

$$\Delta = +0.0475, \quad \xi = -1.438, \quad \epsilon = -0.505, \quad \eta = -0.555.$$

(30)

Of course, this example satisfies (21).

The quark mixing matrix $U$ suffers from a rephasing ambiguity, which makes the reality or complexity of any-specific element of the matrix uncertain. Since $\Delta_{ia}$ are rephasing invariants, it is important for the future analysis to examine a restriction on $\Delta_{ia}$ from the present analysis and the relations among $\Delta_{ia}$. But we leave a definite determination of $\Delta$ and others in (28) for further theoretical investigation of weak decay of a $b$ quark together with the discovery of a $t$ quark.