Fuzzy $S^4$ and Its Construction

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We present construction of fuzzy $S^4$, utilizing the fact that $\mathbb{C}P^3$ is an $S^2$ bundle over $S^4$. Fuzzy $S^4$ is obtained by imposing an additional algebraic constraint on fuzzy $\mathbb{C}P^3$. Hence, as in the case of fuzzy $\mathbb{C}P^3 = SU(4)/U(3)$, fuzzy $S^4$ is globally defined on $\mathbb{R}^{\dim SU(4)}$ with the additional constraint on top of the construction of fuzzy $\mathbb{C}P^3$. We consider the commutative limit of fuzzy $S^4$ and find that it naturally gives commutative $S^4$ in terms of homogeneous coordinates on $\mathbb{C}P^3$. We also discuss exact matrix-function correspondence of fuzzy $S^4$. In conclusion, it is proposed that fuzzy $S^4$ is described by a certain form of block-diagonal matrix whose embedding square matrix represents fuzzy $\mathbb{C}P^3$.

§1. Motivation for fuzzy $S^4$

The idea of fuzzy spaces is based on the realization of spaces (or spacetimes) by $(N \times N)$-matrices, with the large $N$ limit leading to the corresponding ordinary (or commutative) spaces. Fuzzy spaces are interesting when one considers solutions to matrix models, in particular, the matrix model of $M$-theory or the so-called M(atrix) theory. Since fuzzy spaces, by definition, represent noncommutative geometry, study of such spaces is also important to investigate properties of noncommutative versions of various field theories. Use of fuzzy spaces has at least one advantage in the study of noncommutative properties, that is, one can predict some physically interesting quantities by numerical simulations.

Recently, partly linked with the developments of string/M-theory, fuzzy spaces and their applications to field theories have been intensively studied. If one focus on the construction of fuzzy spaces, however, there are few types of fuzzy spaces which are known to exist. This is because the construction of fuzzy spaces is essentially equivalent to the quantization of spaces, and if one takes the approach of geometric quantization, it is demanded that the spaces to be quantized have symplectic structure. This is the reason why the fuzzification of complex projective spaces $\mathbb{C}P^k$ ($k = 1, 2, \cdots$) has been successful and why other known fuzzy spaces are constructed by use of fuzzy $\mathbb{C}P^k$. (For details on the relation between construction of fuzzy spaces and geometric quantization of space, see the reference, this contribution is based on Ref. 2.)

From physicists’ point of view, it is of great interest to obtain a four-dimensional fuzzy space. The well-defined fuzzy $\mathbb{C}P^2$ is not suitable for this purpose, since $\mathbb{C}P^2$ does not have a spin structure. Construction of fuzzy $S^4$ is then physically well motivated. (Notice that fuzzy spaces are generally obtained for compact spaces and that $S^4$ is the simplest four-dimensional compact space that allows a spin structure.)

In the following, we shall propose construction of fuzzy $S^4$ by use of the fact

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that $\mathbb{CP}^3$ is an $S^2$ bundle over $S^4$. We shall obtain fuzzy $S^4$, imposing a further constraint on fuzzy $\mathbb{CP}^3$. In the next section, deducing from the cases of fuzzy $\mathbb{CP}^k$, we briefly review algebraic construction of fuzzy $\mathbb{CP}^3$. In §3, we present construction of fuzzy $S^4$, discussing its commutative limit. In §4, to reconfirm our construction, we examine and discuss the matrix-function correspondence of fuzzy $S^4$. Finally, in §5, we conclude our discussion, presenting a natural candidate for the matrix configuration of fuzzy $S^4$ such that it will be obvious that the algebra of fuzzy $S^4$ preserves associativity and closure.

§2. Brief review of fuzzy $\mathbb{CP}^3$

In this section, we present construction of $\mathbb{CP}^3$ in the framework of the creation-annihilation operators.\textsuperscript{3,5} Since fuzzy $\mathbb{CP}^k$ ($k = 1, 2, \cdots$) in general can be constructed systematically, we first present the general case of any $k$ and, in the end, we briefly rephrase the specific $k = 3$ case.

2.1. Algebraic construction of fuzzy $\mathbb{CP}^k$

The coordinates $Q_A$ of fuzzy $\mathbb{CP}^k$ can be defined in terms of $L_A$ as

$$Q_A = \frac{L_A}{\sqrt{C_2^{(k)}}}, \quad (2.1)$$

satisfying the following two constraints

$$Q_A Q_A = 1, \quad (2.2)$$
$$d_{ABC} Q_A Q_B = c_{k,n} Q_C, \quad (2.3)$$

where $d_{ABC}$ is the totally symmetric symbol of $SU(k+1)$, $C_2^{(k)}$ is the quadratic Casimir of $SU(k+1)$ in the $(n,0)$-representation

$$C_2^{(k)} = \frac{n k (n + k + 1)}{2 (k + 1)} \quad (2.4)$$

and $N^{(k)}$ is the dimension of $SU(k+1)$ in the symmetric $(n,0)$-representation given by

$$N^{(k)} = \dim(n,0) = \frac{(n + k)!}{k! n!}. \quad (2.5)$$

In order to determine the coefficient $c_{k,n}$ in (2.3), we now notice that the $SU(k+1)$ generators in the $(n,0)$-representation can be written by

$$A_A = a_i^\dagger (t_A)_{ij} a_j, \quad (2.6)$$

where $t_A$ ($A = 1, 2, \cdots, k^2 + 2k$) are the $SU(k+1)$ generators in the fundamental representation with normalization $\text{tr}(t_A t_B) = \frac{1}{2} \delta_{AB}$ and $a_i^\dagger$, $a_i$ ($i = 1, \cdots, k+1$) are the creation and annihilation operators acting on the states of $\mathcal{H}_N$ which are spanned by

$$| n_1, n_2, \cdots, n_{k+1} \rangle = (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_{k+1}^\dagger)^{n_{k+1}} | 0 \rangle \quad (2.7)$$
with the following relations
\[ a_i^\dagger n_1, n_2, \ldots, n_{k+1} = (n_1 + n_2 + \cdots + n_{k+1}) | n_1, n_2, \ldots, n_{k+1} \rangle \]
\[ a_i | 0 \rangle = 0 \].

Notice that the condition (2.9) corresponds to the polarization condition in the context of geometric quantization.

Using the completeness relation for \( t_A \)'s
\[ (t_A)_{ij} (t_A)_{kl} = \frac{1}{2} \left( \delta_{il} \delta_{jk} - \frac{1}{k+1} \delta_{ij} \delta_{kl} \right) \]
and the commutation relation \([a_i, a_j^\dagger] = \delta_{ij}\), we can check \( \Lambda_A \Lambda_A = C_2^{(k)} \), where the creation and annihilation operators act on the states of the form (2.7) from the left. We also find
\[ d_{ABC} \Lambda_B \Lambda_C = (k-1) \left( \frac{n}{k+1} + \frac{1}{2} \right) a_i^\dagger (t_A)_{ij} a_j \]
\[ = (k-1) \left( \frac{n}{k+1} + \frac{1}{2} \right) \Lambda_A . \]

Representing \( \Lambda_A \) by \( L_A \), we can determine the coefficient \( c_{k,n} \) in (2.3) by
\[ c_{k,n} = \frac{(k-1)}{\sqrt{C_2^{(k)}}} \left( \frac{n}{k+1} + \frac{1}{2} \right) . \]

For \( k \ll n \), we have
\[ c_{k,n} \rightarrow c_k = \sqrt{\frac{2}{k(k+1)}} (k-1) \]
and this leads to the constraints for the coordinates \( q_A \) of \( \mathbb{CP}^k \)
\[ q_A q_A = 1 , \]
\[ d_{ABC} q_A q_B = c_k q_C . \]

The second constraint (2.15) restricts the number of coordinates to be \( 2k \) out of \( k^2 + 2k \). For example, in the case of \( \mathbb{CP}^2 = SU(3)/U(2) \) this constraint around the pole of \( A = 8 \) becomes \( d_{8BC} q_8 q_B = \frac{1}{\sqrt{3}} q_C \). Normalizing the 8-coordinate to be \( q_8 = -2 \), we find the indices of the coordinates are restricted to 4, 5, 6, and 7 with the conventional choice of the generators of \( SU(3) \) as well as with the definition \( d_{ABC} = 2 \text{tr}(t_A t_B t_C + t_A t_C t_B) \).

2.1.1. Matrix-function correspondence

The matrix-function correspondence for fuzzy \( \mathbb{CP}^k \) can be expressed by
\[ N^{(k)} \times N^{(k)} = \sum_{l=0}^{n} \dim(l,l) , \]
where \( \text{dim}(l, l) \) is the dimension of \( SU(k + 1) \) in the \((l, l)\)-representation. This expression indicates that the number of matrix elements coincides with the number of coefficients in an expansion series of truncated functions on \( \mathbb{C}P^k = SU(k + 1)/U(k) \).

We need the real \((l, l)\)-representation in order to have an expansion of scalar functions on \( \mathbb{C}P^k \). Symbolically the correspondence is written as

\[
(n, 0) \bigotimes (0, n) = \bigoplus_{l=0}^{n} (l, l)
\]

in terms of the dimensionality of \( SU(k + 1) \). The left-hand side of (2.17) can be interpreted from the fact that \( A_A = a_i^+(t_A)_{ij}a_j \sim a_i^+a_j \) transforms like \((n, 0) \otimes (0, n)\).

The right-hand side of (2.17), on the other hand, can be interpreted by a usual tensor analysis, i.e., \( \text{dim}(l, l) \) is the number of ways to construct tensors of the form \( T_{i_1, i_2, \ldots, i_l}^{j_1, j_2, \ldots, j_l} \) such that the tensor is traceless and totally symmetric in terms of \( i, j = 1, 2, \ldots, k+1 \).

2.2. Algebraic construction of fuzzy \( \mathbb{C}P^3 \)

Here we briefly rephrase the above construction in the case of \( k = 3 \). The coordinates \( Q_A \) of fuzzy \( \mathbb{C}P^3 \) can be defined by

\[
Q_A = \frac{L_A}{\sqrt{C_2^{(3)}}},
\]

where \( L_A \) are \( N^{(3)} \times N^{(3)} \)-matrix representations of \( SU(4) \) generators in the \((n, 0)\)-representation. The coordinates satisfy the following constraints:

\[
Q_A Q_A = 1, \quad d_{ABC} Q_A Q_B = c_{3,n} Q_C.
\]

As discussed before, in the large \( n \) limit these constraints become constraints for the coordinates on \( \mathbb{C}P^3 \) as embedded in \( \mathbb{R}^{15} \). In (2.18)–(2.20), \( C_2^{(3)} \), \( 1 \), \( d_{ABC} \) and \( c_{3,n} \) are all defined above, including the relation

\[
N^{(3)} = \frac{1}{6} (n + 1)(n + 2)(n + 3).
\]

§3. Construction of fuzzy \( S^4 \)

In this section, we present construction of fuzzy \( S^4 \). As discussed earlier, the construction is carried out by imposing an additional constraint to the fuzzy \( \mathbb{C}P^3 = SU(4)/U(3) \). We now consider the decomposition, \( SU(4) \rightarrow SU(2) \times SU(2) \times U(1) \), where the two \( SU(2) \)'s and one \( U(1) \) are defined by

\[
\left( \begin{array}{ccc}
SU(2) & 0 \\
0 & 0
\end{array} \right), \quad \left( \begin{array}{cc}
0 & 0 \\
0 & SU(2)
\end{array} \right), \quad \left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right)
\]

in terms of the \((4 \times 4)\)-matrix generators of \( SU(4) \) in the fundamental representation. (Each \( SU(2) \) denotes the algebra of \( SU(2) \) group in the \((2 \times 2)\)-matrix representation.)
As we shall see in the following, functions on $S^4$ are functions on $\mathbb{C}P^3 = SU(4)/U(3)$ which are invariant under transformations of $H \equiv SU(2) \times U(1)$, $H$ being relevant to the above decomposition of $SU(4)$. In order to obtain functions on fuzzy $S^4$, we thus need to require

$$[F(Q), L_\alpha] = 0,$$

where $F$ denote matrix-functions of $Q_A$'s and $L_\alpha$ are generators of $H$ represented by $N^{(3)} \times N^{(3)}$-matrices. Construction of fuzzy $S^4$ can be carried out by imposing the additional constraint (3.2) onto the functions on fuzzy $\mathbb{C}P^3$. What we claim is that the further condition (3.2) makes the functions $F(Q_A)$ become functions on fuzzy $S^4$. This does not mean that fuzzy $S^4$ is a subset of fuzzy $\mathbb{C}P^3$. Notice that $Q_A$'s are defined in $\mathbb{R}^{15}$ ($A = 1, \cdots, 15$) with the algebraic constraints (2.19) and (2.20). While locally, say around the pole of $A = 15$ in (2.20), one can specify the six coordinates of fuzzy $\mathbb{C}P^3$, globally they are embedded in $\mathbb{R}^{15}$. Equation (3.2) is a global constraint in this sense. So the algebra of fuzzy $S^4$ is given by a subset of $SU(4)$. The emerging algebraic structure of fuzzy $S^4$ will be clearer when we consider the commutative limit of our construction.

3.1. Commutative limit

As shown in §2, in the large $n$ limit we can approximate $Q_A$ to the commutative coordinates on $\mathbb{C}P^3$:

$$Q_A \approx \phi_A = -2 \text{tr}(g^\dagger t_A g t_{15})$$

which indeed obey the following constraints for $\mathbb{C}P^3$

$$\phi_A \phi_A = 1, \quad d_{ABC} \phi_A \phi_B = \sqrt{\frac{2}{3}} \phi_C.$$  

(3.4)

Algebraic constraints for $\mathbb{C}P^k$ are in general given in (2.13)–(2.15). In (3.3), $t_A$'s are the generators of $SU(4)$ in the fundamental representation and $g$ is a group element of $SU(4)$ given as a $(4 \times 4)$-matrix. Truncated functions on $\mathbb{C}P^3$ are then written as

$$f_{\mathbb{C}P^3}(u, \bar{u}) \sim f_{j_1 j_2 \cdots j_l}^{i_1 i_2 \cdots i_l} \bar{u}_{i_1} \bar{u}_{i_2} \cdots \bar{u}_{i_l} u_{j_1} u_{j_2} \cdots u_{j_l},$$

(3.5)

where $l = 0, 1, 2, \cdots, n$, $u_j = g_{j4}$, $\bar{u}_i = (g^\dagger)_i^4$ and $\bar{u}_i u_i = 1$ ($i, j = 1, 2, 3, 4$). One can describe $\mathbb{C}P^3$ in terms of four complex coordinates $Z_i$ with the identification $Z_i \sim \lambda Z_i$ where $\lambda$ is a nonzero complex number ($\lambda \in \mathbb{C} - \{0\}$). Following Penrose, \(^6\) we now write $Z_i$ in terms of two spinors $\omega$, $\pi$ as

$$Z_i = (\omega_{\dot{a}}, \pi_{\dot{a}}) = (x_{a\dot{a}} \pi_{\dot{a}}, \pi_{\dot{a}}),$$

(3.6)

where $a = 1, 2, \dot{a} = 1, 2$ and $x_{a\dot{a}}$ can be defined with the coordinates $x_\mu$ on $S^4$ via $x_{a\dot{a}} = (1 x_4 - i \vec{\sigma} \cdot \vec{z})$, $\vec{\sigma}$ being $(2 \times 2)$ Pauli matrices. The scale invariance $Z_i \sim \lambda Z_i$ leads to the invariance $\pi_{\dot{a}} \sim \lambda \pi_{\dot{a}}$. The $\pi_{\dot{a}}$'s then describe a $\mathbb{C}P^1 = S^2$. This shows the fact that $\mathbb{C}P^3$ is an $S^2$ bundle over $S^4$, or Penrose’s projective twistor space. Note that we can parametrize $u_i$ of (3.5) by the homogeneous coordinates $Z_i$, i.e.,

$$u_i = \frac{Z_i}{\sqrt{Z \cdot Z}}.$$
Functions on $S^4$ can be considered as functions on $\mathbb{CP}^3$ which satisfy
\[ \frac{\partial}{\partial \pi_\alpha} f_{\mathbb{CP}^3}(Z, \bar{Z}) = \frac{\partial}{\partial \bar{\pi}_\alpha} f_{\mathbb{CP}^3}(Z, \bar{Z}) = 0. \] (3-7)
This implies that $f_{\mathbb{CP}^3}$ are further invariant under transformations of $\pi_\alpha$, $\bar{\pi}_\alpha$. In terms of the four-spinor $Z$, such transformations are expressed by
\[ Z \to e^{it_\alpha \theta_\alpha} Z, \] (3-8)
where $t_\alpha$ represent the algebra of $H = SU(2) \times U(1)$ defined previously in regard to the decomposition of $SU(4)$ in (3-1). The coordinates $\phi_A$ in (3-3) can be written by $\phi_A(Z, \bar{Z}) \sim \bar{Z}_i(t_\alpha)_ijZ_j$. Under an infinitesimal ($\theta_\alpha \ll 1$) transformation of (3-8), the coordinates $\phi_A(Z, \bar{Z})$ transform as
\[ \phi_A \to \phi_A + \theta_\alpha f_{\alpha AB} \phi_B, \] (3-9)
where $f_{ABC}$ is the structure constant of $SU(4)$. The constraint (3-7) is then rewritten as
\[ f_{\alpha AB} \phi_B \frac{\partial}{\partial \phi_A} f_{\mathbb{CP}^3} = 0, \] (3-10)
where $f_{\mathbb{CP}^3}$ are seen as functions of $\phi_A$'s rather than that of $(Z, \bar{Z})$. Note that $\phi_A$'s in (3-10) are defined by (3-3), i.e., they are globally defined on $\mathbb{R}^{15}$.

From the relation $\phi_A \sim \bar{Z}_i(t_\alpha)_ijZ_j$, we find $f_{\alpha AB} \phi_B \sim \bar{Z}_i([t_A, t_\alpha])_{ij}Z_j$ where $t_\alpha$ are the generators of $H = SU(2) \times U(1) \subset SU(4)$ as before. The constraint (3-7) or (3-10) is then realized by $[t_A, t_\alpha] = 0$, which can be considered as a commutative implementation of the fuzzy constraint (3-2). Specifically, we may choose $t_\alpha = \{t_1, t_2, t_3, \sqrt{\frac{2}{3}}t_{18} + \sqrt{\frac{1}{3}}t_{15}\}$ in the conventional choices of the generators of $SU(4)$ in the fundamental representation. The constraint $[t_A, t_\alpha] = 0$ then restricts $A$ to be $A = 8, 13, 14, 15$. This is, of course, a local analysis. The constraint $[t_A, t_\alpha] = 0$ does globally define $S^4$ as embedded in $\mathbb{R}^{15}$ similarly to how we have defined $\mathbb{CP}^3$. The number of $\mathbb{CP}^3$ coordinates $\phi_A$ is locally restricted to be six because of the algebraic constraints in (3-4). On top of these, the constraint $[t_A, t_\alpha] = 0$ further restricts the number of coordinates to be four, which is correct for the coordinates on $S^4$.

Functions on $S^4$ are polynomials of $\phi_A = -2\text{tr}(g^t g t_A g t_{15})$ which obey $[t_A, t_\alpha] = 0$. Products of functions are determined by the products of such $t_A$'s. Extension to the fuzzy case is essentially done by replacing $t_A$ with $L_A$, where $L_A$ is the matrix representation of the algebra of $SU(4)$ in the totally symmetric $(n, 0)$-representation. The algebra of fuzzy $S^4$ naturally becomes associative in the commutative limit, while associativity of fuzzy $S^4$, itself, will be discussed in the next section, where we shall present a concrete matrix configuration of fuzzy $S^4$ so that the associativity is obviously seen. Even without any such matrix realizations, we can extract another property of the algebra from the condition (3-2), that is, closure of the algebra; since functions on fuzzy $S^4$ are represented by matrices, it is easily seen that products of such functions also obey the condition (3-2). In what follows, we shall clarify these points in some detail.
The additional condition (3.2) to fuzzy $\mathbb{CP}^3$ plays an essential part in our construction of fuzzy $S^4$. Imposing such an additional condition to obtain a fuzzy space from another is first considered by Nair and Randjbar-Daemi in the construction of fuzzy $S^4/\mathbb{Z}^2$ out of fuzzy $S^2 \times S^2$.\(^7\) Our construction provides another example of such construction.

§4. Matrix-function correspondence

In this section, we examine our construction of fuzzy $S^4$ by confirming its matrix-function correspondence. To show a one-to-one correspondence, one needs to say two things: (a) a matching between the number of matrix elements for fuzzy $S^4$ and the number of truncated functions on $S^4$; and (b) a correspondence between the product of functions on fuzzy $S^4$ and that on $S^4$. It is now suggestive to take a moment to review how (a) and (b) are fulfilled in the case of fuzzy $S^2 = SU(2)/U(1)$.\(^8\) Let $D_{mn}^{(j)}(g)$ be Wigner $D$-functions for $SU(2)$. These are the spin-$j$ matrix representations of an $SU(2)$ group element $g$: $D_{mn}^{(j)}(g) = \langle jm|g|jn \rangle$ $(m, n = -j, \ldots, j)$. Functions on $S^2$ can be expanded in terms of particular Wigner $D$-functions, $D_{m0}^{(j)}(g)$, which are invariant under a $U(1)$ right-translation operator acting on $g$. Since the state $|j0\rangle$ has no $U(1)$ charge, right action of the $U(1)$ operator, $R_3$, on $g$ makes $D_{m0}^{(j)}(g)$ vanish, $R_3D_{m0}^{(j)}(g) = 0$; in fact one can choose any fixed value $(m = -j, \ldots, j)$ for this $U(1)$ charge. The $D$-functions are essentially the spherical harmonics, $D_{m0}^{(l)} = \sqrt{4\pi \over 2l+1} (-1)^m Y_{lm}$, and so a truncated expansion can be written as $f_{S^2} = \sum_{l=0}^{n} \sum_{m=-l}^{l} f_{ml}^{l} D_{m0}^{(l)}$. The number of coefficients $f_{ml}^{l}$ are counted by $\sum_{l=0}^{n} (2l+1) = (n+1)^2$. This relation implements the condition (a) by defining functions on fuzzy $S^2$ as $(n+1) \times (n+1)$ matrices. The product of truncated functions at the same level of $n$ is also expressed by the same number of coefficients. Therefore, the product corresponds to $(n+1) \times (n+1)$ matrix multiplication. This implies the condition (b). One can show an exact correspondence of products, following the reference.\(^9\) Let $f_{mn}$ $(m, n = 1, \ldots, n+1)$ be an element of matrix function-operator $\hat{f}$ on fuzzy $S^2$. We define the symbol of the function as

$$\langle \hat{f} \rangle = \sum_{m,n} f_{mn} D_{mj}^{*}(g) D_{nj}^{(j)}(g), \quad (4.1)$$

where $D_{mj}^{*}(g) = D_{jm}^{(j)}(g^{-1})$. We here consider $|jj\rangle$ as the highest weight state. The star product of fuzzy $S^2$ is defined by $\langle \hat{f} \hat{g} \rangle = \langle \hat{f} \rangle \ast \langle \hat{g} \rangle$. From (4.1), we can write

$$\langle \hat{f} \hat{g} \rangle = \sum_{mnl} f_{mn} g_{nl} D_{mj}^{*}(g) D_{lj}^{(j)}(g)$$

$$= \sum_{mnl} f_{mn} g_{kl} D_{mj}^{*}(g) D_{nr}^{(j)}(g) D_{kr}^{*}(g) D_{lj}^{(j)}(g), \quad (4.2)$$
where we use the orthogonality of $\mathcal{D}$-functions $\sum_{r} \mathcal{D}^{(j)}_{mn}(g)\mathcal{D}^{*(j)}_{kr}(g) = \delta_{nk}$. Let $R_-$ be the lowering operator in right action, we then find

$$R_-\mathcal{D}^{(j)}_{mn}(g) = \sqrt{(j+n)(j-n+1)}\mathcal{D}^{(j)}_{mn-1}(g). \quad (4.3)$$

By iteration, (4.2) may be rewritten as

$$\langle \hat{f}\hat{g} \rangle = \sum_{s=0}^{2j} (-1)^s \frac{(2j-s)!}{s!(2j)!} R^s_+ \langle \hat{f} \rangle R^s_- \langle \hat{g} \rangle \equiv \langle \hat{f} \rangle \ast \langle \hat{g} \rangle, \quad (4.4)$$

where we use the relation $R^s_- = -R^s_+$. In the large $j$ limit, the term with $s = 0$ in (4.4) dominates and this leads to an ordinary commutative product of $\langle \hat{f} \rangle$ and $\langle \hat{g} \rangle$. By construction, the symbols of functions on fuzzy $S^2$ can be regarded as commutative functions on $S^2$. The expression (4.4) therefore shows a one-to-one correspondence between the product of fuzzy functions and the product of truncated functions on $S^2$.

From (4.2) and (4.4), it is easily seen that the square-matrix configuration, in addition to the orthogonality of the $\mathcal{D}$-functions or of the states $\ket{jm}$, is the key ingredient for the condition (b) in the case of fuzzy $S^2$. Associativity of the star product is direct consequence of this matrix configuration. Suppose the number of truncated functions on some compact space is given by an absolute square. Then, following the above procedure, one may establish the matrix-function correspondence. As shown in (2.16), this is true for fuzzy $\mathbb{CP}^k$ in general. In the case of fuzzy $\mathbb{CP}^3$, the absolute square appears from

$$N^{(3)} \times N^{(3)} = \sum_{l=0}^{n} \dim(l, l), \quad (4.5)$$

$$\dim(l, l) = \frac{1}{12} (2l + 3)(l + 1)^2(l + 2)^2, \quad (4.6)$$

where $\dim(l, l)$ is the dimension of $SU(4)$ in the real $(l, l)$-representation. This arises from the fact that functions on $\mathbb{CP}^3 = SU(4)/U(3)$ can be expanded by $\mathcal{D}^{(l)\delta}_{M\delta}(g)$, Wigner $\mathcal{D}$-functions of $SU(4)$ in the $(l, l)$-representation ($l = 0, 1, 2, \cdots$). Here, $g$ is an element of $SU(4)$. The lower index $M (M = 1, \cdots, \dim(l, l))$ labels the state in the $(l, l)$-representation, while the index $\delta$ represents any suitably fixed state in this representation. Like in (4.1), the symbol of fuzzy $\mathbb{CP}^3$ is defined in terms of $\mathcal{D}^{(n,0)}_{IN^{(3)}}(g)$ and its complex conjugate, where $\mathcal{D}^{(n,0)}_{IN^{(3)}}(g) = \langle (n, 0), I \vert g \vert (n, 0), N^{(3)} \rangle$ are the $\mathcal{D}$-functions belonging to the symmetric $(n, 0)$-representation. The states of fuzzy $\mathbb{CP}^3$ are expressed by $\ket{(n, 0), I}$. The index $I (I = 1, 2, \cdots, \dim(n, 0) = N^{(3)})$ labels these states and the index $N^{(3)}$ indicates the highest or lowest weight state. Notice that one can alternatively express the states by $\phi_{i_1, i_2, \cdots, i_n}$ where the sequence of $i_m = \{1, 2, 3, 4\}$ ($m = 1, \cdots, n$) is in a totally symmetric order.

We now return to the conditions (a) and (b) of fuzzy $S^4$. In the following subsections, we present (i) different ways of counting the number of truncated functions on $S^4$, (ii) a one-to-one matrix-function correspondence for fuzzy $S^4$. In (ii), the condition (a) is shown; we find the number of matrix elements for fuzzy $S^4$ agrees with
the number calculated in (i). The condition (b) is also shown in (ii) by considering the symbols and star products on fuzzy $S^4$ in the commutative limit.

4.1. Ways of counting

A direct counting of the number of truncated functions on $S^4$ can be made in terms of the spherical harmonics $Y_{l_1 l_2 l_3 m}$ on $S^4$ with a truncation at $l_1 = n$:

$$N^{S^4}(n) = \sum_{l_1=0}^{n} \sum_{l_2=0}^{l_1} \sum_{l_3=0}^{l_2} (2l_3 + 1) = \frac{1}{12}(n+1)(n+2)^2(n+3). \quad (4.7)$$

Alternatively, one can count $N^{S^4}(n)$ by use of a tensor analysis. The number of truncated functions on $CP^3$ is given by the totally symmetric and traceless tensors $f_{j_1 \cdots j_l}^{i_1 \cdots i_l} (i, j = 1, \cdots, 4)$ in (3-5). Now we split the indices into $i = a, \hat{a}$ ($a = 1, 2$, $\hat{a} = 3, 4$), and similarly for $j = b, \hat{b}$. The additional constraint (3-7) for the extraction of $S^4$ from $CP^3$ means that the tensors are independent of any combinations of $\hat{a}$’s in the sequence of $i$’s. In other words, in terms of the transformation (3-8), $Z \rightarrow e^{i a_\hat{a} a} Z$, functions on $S^4$ are invariant under the transformations involving $(t_\alpha)_{\hat{a} a}$ where $t_\alpha$ are the $(4 \times 4)$ matrix representations of the algebra of $H = SU(2) \times U(1)$. There are $N^{(2)}(l) = \frac{1}{2}(l+1)(l+2)$ ways of having a symmetric order $i_1, i_2, \cdots, i_l$ for $i = \{1, \hat{a}\}$ ($\hat{a} = 3, 4$). This can be regarded as an $N^{(2)}(l)$-degeneracy due to an $S^2$ internal symmetry for the extraction of $S^4$ out of $CP^3 \sim S^4 \times S^2$. This internal symmetry is relevant to the above $(t_\alpha)_{\hat{a} a}$-transformations. Since the number of truncated functions on $CP^3$ is given by (4-6), the number of those on $S^4$ may be calculated by

$$N^{S^4}(n) = \sum_{l=0}^{n} \frac{\dim(l, l)}{N^{(2)}(l)} = \sum_{l=0}^{n} \frac{1}{6} (l+1)(l+2)(2l+3) = \frac{1}{12}(n+1)(n+2)^2(n+3) \quad (4.8)$$

which reproduces (4.7). This is also in accordance with a corresponding calculation in the context of $S^4 = SO(5)/SO(4)$.

4.2. One-to-one matrix-function correspondence

As mentioned earlier in this section, the states of fuzzy $CP^3$ can be denoted by $\phi_{i_1 i_2 \cdots i_n}$ where the sequence of $i_m = \{1, 2, 3, 4\}$ ($m = 1, \cdots, n$) is in a totally symmetric order. Let $(\hat{F})_{IJ}$ ($I, J = 1, 2, \cdots, N^{(3)}$) denote a matrix-function on fuzzy $CP^3$. Matrix elements of the function $\hat{F}$ on fuzzy $CP^3$ can be defined by $\langle I | \hat{F} | J \rangle$, where we denote $\phi_{i_1 \cdots i_n} = | i_1 \cdots i_n \rangle \equiv | I \rangle$. We need to find an analogous matrix expression $(\hat{F}^{S^4})_{IJ}$ for a function on fuzzy $S^4$. We now consider the states on fuzzy $S^4$ in terms of $\phi_{i_1 i_2 \cdots i_n}$. Splitting each $i$ into $a$ and $\hat{a}$, we may express $\phi_{i_1 i_2 \cdots i_n}$ as

$$\phi_{i_1 i_2 \cdots i_n} = \{ \phi_{a_\hat{a} a_\hat{a} \cdots a_{\hat{a} \hat{a}}}, \phi_{a_1 a_\hat{a} \cdots a_{\hat{a} \hat{a}}}, \cdots, \phi_{a_1 \cdots a_{\hat{a} \hat{a}} a_\hat{a}} \} \quad (4.9)$$

From the analysis in the previous section, one can obtain the states corresponding to fuzzy $S^4$ by imposing an additional condition on (4.9), i.e., the invariance under the transformations involving any $\hat{a}_m$ ($m = 1, \cdots, n$). Transformations of the states
on fuzzy $S^4$, under this particular condition, can be considered as follows. On the set of states $\phi_{a_1 \cdots a_{n-1}}$, which are $(n+1)$ in number, the transformations must be diagonal because of (3.7), but we can have an independent transformation for each state. (The number of the states is $(n+1)$, since the sequence of $a_m = \{3, 4\}$ is in a totally symmetric order.) Thus we get $(n+1)$ different functions proportional to identity. On the set of states $\phi_{a_1 \cdots a_{n}}$, we can transform the $a_1$ index (to $b_1 = \{1, 2\}$ for instance), corresponding to a matrix function $f_{a_1,b_1}$ which have $2^2$ independent components. But we can also choose the matrix $f_{a_1,b_1}$ to be different for each choice of $(\hat{a}_1 \cdots \hat{a}_{n-1})$ giving $2^2 \times n$ functions in all, at this level. We can represent these as $f_{\hat{a}_1 \cdots \hat{a}_{n-1}}$, the extra composite index $(\hat{a}_1 \cdots \hat{a}_{n-1})$ counting the multiplicity. Continuing in this way, we find that the set of all functions on fuzzy $S^4$ is given by

$$\langle \hat{F}S^4 \rangle_{IJ} = \{ f_{\hat{a}_1 \cdots \hat{a}_n} \delta_{a_1 \cdots a_n,b_1 \cdots b_n}, f_{\hat{a}_1 \cdots \hat{a}_{n-1}} \delta_{a_1 \cdots a_{n-1},b_1 \cdots b_{n-1}},$$

$$f_{\hat{a}_1 \hat{a}_2,b_1 b_2} \delta_{a_1 \cdots a_2,b_1 \cdots b_2 \cdots b_{n-2}}, \cdots, f_{\hat{a}_1 \cdots \hat{a}_n,b_1 \cdots b_n} \},$$

(4.10)

where we split $i_m$ into $a_m$, $\hat{a}_m$ and $j_m$ into $b_m$, $\hat{b}_m$. Each of the operators $\delta_{\hat{a}_1 \cdots \hat{a}_m,b_1 \cdots b_m}$ indicates an identity operator such that the corresponding matrix is invariant under transformations from $\{\hat{a}_1 \cdots \hat{a}_m\}$ to $\{\hat{b}_1 \cdots \hat{b}_m\}$. The structure in (4.10) shows that $\hat{F}S^4$ is composed of $(l+1) \times (l+1)$-matrices $(l = 0, 1, \cdots, n)$, with the number of these matrices for fixed $l$ being $(n+1-l)$. Thus the number of matrix elements for fuzzy $S^4$ is counted by

$$NS^4(n) = \sum_{l=1}^{n} (l+1)(n+1-l) = \frac{1}{12}(n+1)(n+2)^2(n+3).$$

(4.11)

This relation satisfies the condition (a). In order to show the precise matrix-function correspondence, we further need to show the condition (b), the correspondence of products. We carry out this part in analogy with the case of fuzzy $S^2$ in (4.1)–(4.4). The symbol of the function $\hat{F}$ on fuzzy $\textbf{CP}^3$ can be defined as

$$\langle \hat{F} \rangle = \sum_{I,J} \langle N|g|I \rangle \langle \hat{F} \rangle_{IJ} \langle J|g|N \rangle,$$

(4.12)

where $|N\rangle \equiv |(n,0),N^{(3)}\rangle$ is the highest or lowest weight state of fuzzy $\textbf{CP}^3$ and $\langle J|g|N \rangle$ denotes the previous D-function, $D_{JN^{(3)}}^{(n,0)}(g)$. The symbol of a function on fuzzy $S^4$ is defined in the same way except that $\langle \hat{F} \rangle_{IJ}$ is replaced with $\langle \hat{F}S^4 \rangle_{IJ}$ in (4.12). We now consider the product of two functions on fuzzy $S^4$. As we discussed above, a function on fuzzy $S^4$ can be described by $(l+1) \times (l+1)$-matrices. From the structure of $\hat{F}S^4$ in (4.10), we are allowed to treat these matrices independently. The product is then considered as a set of matrix multiplications. This leads to a natural definition of the product preserving closure, since the product of functions also becomes a function, retaining the same structure as in (4.10). The star product of fuzzy $S^4$ is written as

$$\langle \hat{F}S^4 \hat{G}S^4 \rangle = \sum_{IJK} \langle \hat{F}S^4 \rangle_{IJ} \langle \hat{G}S^4 \rangle_{JK} \langle N|g|I \rangle \langle K|g|N \rangle \equiv \langle \hat{F}S^4 \rangle \ast \langle \hat{G}S^4 \rangle$$

(4.13)
where the product \((\hat{F}^{S^4})_{IJ}(\hat{G}^{S^4})_{JK}\) is given by the set of matrix multiplications. This fact, along with the orthogonality of the \(D\)-functions, leads to associativity of the star products.

The symbols and star products of fuzzy \(S^4\) can be obtained from those of fuzzy \(\text{CP}^3\) by simply replacing the function operator \(\hat{F}\) with \(\hat{F}^{S^4}\). So the correspondence between fuzzy and commutative products on \(S^4\) can be shown in the large \(n\) limit as we have seen in §2.2. We can in fact directly check this correspondence even at the level of finite \(n\) from the following discussion.

Let us consider functions on \(S^4\) in terms of the homogeneous coordinates on \(\text{CP}^3\), \(Z_i = (\omega_a, \pi_{\hat{a}}) = (x_{a\hat{a}} \pi_{\hat{a}}, \pi_{\hat{a}})\), as in (3-6). Functions on \(S^4\) can be constructed from \(x_{a\hat{a}}\) under the constraint (3-7), which implies that the functions are independent of \(\pi_{\hat{a}}\) and \(\pi_{\hat{a}}\). Expanding in powers of \(x_{a\hat{a}}\), we may express the functions by the following set of terms; \(\{1, x_{a\hat{a}}, x_{a1}, x_{a2\hat{a}}, x_{a1} x_{a2\hat{a}}, x_{a3\hat{a}}, \cdots\}\), where the indices \(a\)'s and \(\hat{a}\)'s are symmetric in their order. Owing to the extra constraint (3-7), one can consider that all the factors involving \(\pi_{\hat{a}}\) and \(\pi_{\hat{a}}\) can be absorbed into the coefficients of these terms. By iterative use of the relations, \(x_{a\hat{a}} \pi_{\hat{a}} = \omega_a\) and its complex conjugation, the above set of terms can be expressed in terms of \(\omega\)'s and \(\bar{\omega}\)'s as

\[
1 , \begin{pmatrix} \bar{\omega}_a \\ \omega_b \end{pmatrix}_{2 \times 2} , \begin{pmatrix} \bar{\omega}_a \bar{\omega}_b \\ \omega_a \omega_b \end{pmatrix}_{3 \times 3} , \begin{pmatrix} \bar{\omega}_a \bar{\omega}_b \bar{\omega}_c \\ \omega_a \omega_b \omega_c \end{pmatrix}_{4 \times 4} , \cdots ,
\]

(4.14)

where the indices \(a\) and \(b\) are used to distinguish \(\bar{\omega}\) and \(\omega\). Because the indices are symmetric, the number of independent terms in each column should be counted as indicated in (4.14).

Notice that even though functions on \(S^4\) can be parametrized by \(\omega\)'s and \(\bar{\omega}\)'s, the overall variables of the functions should be given by the coordinates on \(S^4\), \(x_\mu\), instead of \(\omega_a = \pi_{\hat{a}} x_{a\hat{a}}\). The coefficients of the terms in (4.14) need to be chosen accordingly. For instance, the term \(\omega_a\) with a coefficient \(c_a\) will be expressed as \(c_a \omega_a = c_a \pi_{\hat{a}} x_{a\hat{a}} \equiv h_{a\hat{a}} x_{a\hat{a}}\), where \(h_{a\hat{a}}\) is considered as some arbitrary set of constants. We now define truncated functions on \(S^4\) in the present context. Functions on \(S^4\) are generically expanded in powers of \(\bar{\omega}_a\) and \(\omega_b\) \((a = 1, 2\) and \(b = 1, 2)\)

\[
f_{S^4}(\omega, \bar{\omega}) \sim f_{b_1 b_2 \cdots b_\beta}^{a_1 a_2 \cdots a_\alpha} \bar{\omega}_a \bar{\omega}_b \cdots \bar{\omega}_a \omega_b \cdot \omega_b \cdot \omega_b ,
\]

(4.15)

where \(\alpha, \beta = 0, 1, 2, 3, \cdots\) and the coefficients \(f_{b_1 b_2 \cdots b_\beta}^{a_1 a_2 \cdots a_\alpha}\) should be understood as generalizations of the above-mentioned \(c_a\). The truncated functions on \(S^4\) may be obtained by putting an upper bound for the value \((\alpha + \beta)\). We choose this by setting \(\alpha + \beta \leq n\). In (4.14), this choice corresponds to a truncation at the column which is to be labeled by \((n + 1) \times (n + 1)\). In order to count the number of truncated functions in (4.15), we have to notice the following relation between \(\omega_a\) and \(\bar{\omega}_a\)

\[
\bar{\omega}_a \omega_a \sim x_\mu x_\mu = x^2 .
\]

(4.16)

Using this relation, we can contract \(\bar{\omega}_a\)'s in (4.14). For example, we begin with the contractions involving \(\bar{\omega}_{a_1}\) with all terms in (4.14), which yield the following new set
of terms

\[
1, \begin{pmatrix}
\bar{\omega}_{a_2} \\
\omega_{b_1}
\end{pmatrix}, \begin{pmatrix}
\bar{\omega}_{a_2} \bar{\omega}_{a_3} \\
\omega_{a_2} \omega_{b_1}
\end{pmatrix}, \ldots
\]  

(4.17)

The coefficients for the terms in (4.17) are independent of those for (4.14), due to the scale invariance \( \bar{\pi}_a \pi_a \sim |\lambda|^2 \) (\( \lambda \in \mathbb{C} - \{0\} \)) in the contracting relation (4.16). Conseuqently, we can make similar contractions at most \( n \)-times. The total number of truncated functions on \( S^4 \) is then counted by

\[
N_{S^4}(n) \equiv \sum_{l=0}^{n} [1^2 + 2^2 + \cdots + (l+1)^2] = \frac{1}{12} (n + 1)(n + 2)^2(n + 3) \quad (4.18)
\]

which indeed equals to the previously found results in (4.7) and (4.8).

From (4.14)–(4.18), we find that all the coefficients in \( f_{S^4}(\omega, \bar{\omega}) \) correspond to the number of the matrix elements for \( \hat{F}^{S^4} \) given in (4.11). Further, since any products of fuzzy functions do not alter their structure in (4.10), such products correspond to commutative products of \( f_{S^4}(\omega, \bar{\omega}) \)'s. This leads to the precise correspondence between the functions on fuzzy \( S^4 \) and the truncated functions on \( S^4 \) at any level of truncation.

§5. Block-diagonal matrix realization of fuzzy \( S^4 \)

In this concluding section, we confirm the one-to-one correspondence of fuzzy \( S^4 \) by proposing a block-diagonal matrix realization of fuzzy \( S^4 \). Along these arguments, it will become clear that the algebra of fuzzy \( S^4 \) is closed and associative.

So far, we have analyzed the structure of functions on fuzzy \( S^4 \) and their products in some detail, however, we have not presented an explicit matrix configuration for those fuzzy functions. But, by now, it is obvious that we can use a block-diagonal matrix to represent them, which naturally leads to associativity of the algebra of fuzzy \( S^4 \). Let us write down the equation (4.11) in the following form:

\[
N_{S^4}(n) = 1 + 1 + 2^2 + 1 + 2^2 + 3^2 + 1 + 2^2 + 3^2 + 4^2 + \cdots + (n + 1)^2.
\]

If we locate all the squared elements block-diagonally, then the dimension of an embedding matrix is given by

\[
\sum_{l=0}^{n} [1 + 2 + \cdots + (l+1)] = \frac{1}{6} (n + 1)(n + 2)(n + 3) = N^{(3)}. \quad (5.2)
\]
Coordinates of fuzzy $S^4$ are then represented by these $N^{(3)} \times N^{(3)}$ block-diagonal matrices, $X_A$, which satisfy

$$X_A X_A \sim 1,$$

where 1 is the $N^{(3)} \times N^{(3)}$ identity matrix and $A = 1, 2, 3, 4$ and 5, four of which are relevant to the coordinates of fuzzy $S^4$. The fact that $N^{S^4}$ is a sum of absolute squares does not necessarily warrant associativity of the algebra. (Every integer is a sum of squares, $1 + 1 + \cdots + 1$, but this does not mean any linear space of any dimension is an algebra.) It is the structure of $\hat{F}^{S^4}$ as well as the matching between (4.18) and (4.11) that lead to the block-diagonal configuration $X_A$ to represent fuzzy $S^4$.

Of course, $X_A$ are not the only matrices that describe fuzzy $S^4$. Instead of diagonally locating every block one by one, we can also put the same-size blocks into a single block, using matrix multiplication or matrix addition. Then, the final form has a dimension of $\sum_{l=0}^{n} (l+1) = \frac{1}{2}(n+1)(n+2) = N^{(2)}$. This implies an alternative description of fuzzy $S^4$ in terms of $N^{(2)} \times N^{(2)}$ block-diagonal matrices, $\overline{X}_A$, which are embedded in $N^{(3)}$-dimensional square matrices and satisfy $\overline{X}_A \overline{X}_A \sim \mathbb{1}$, where $\mathbb{1} = \text{diag}(1, 1, \cdots, 1, 0, 0, \cdots, 0)$ is an $N^{(3)} \times N^{(3)}$ diagonal matrix, with the number of 1’s being $N^{(2)}$. Our choice of $X_A$ is, however, convenient in comparison with fuzzy $\text{CP}^3$. The number of 1’s in $X_A$ is $(n+1)$. This corresponds to the dimension of an $SU(2)$ subalgebra of $SU(4)$ in the $N^{(3)}(n)$-dimensional matrix representation. (Notice that fuzzy $S^2 = SU(2)/U(1)$ is conventionally described by $(n+1) \times (n+1)$ matrices in this context.) Using the coordinates $X_A$, we can then confirm the constraint in (3.2), i.e.,

$$[\mathcal{F}(X), L_\alpha] = 0,$$

where $\mathcal{F}(X)$ are matrix-functions of $X_A$’s and $L_\alpha$ are the generators of $H = SU(2) \times U(1) \subset SU(4)$, represented by $N^{(3)} \times N^{(3)}$ matrices. If both $\mathcal{F}(X)$ and $G(X)$ commute with $L_\alpha$, so does $\mathcal{F}(X)G(X)$. Thus, there is closure of such “functions” under multiplication. This indicates that fuzzy $S^4$ follows a closed and associative algebra.

Lastly it needs to be emphasized that the emerging algebra of fuzzy $S^4$ is not a subalgebra of fuzzy $\text{CP}^3$, since the algebra of fuzzy $\text{CP}^3$ is defined globally by $SU(4)$ algebra with the algebraic constraints given in (2.20). The algebra of fuzzy $S^4$ is also globally obtained from $SU(4)$, but with the extra constraint (3.2) on top of this fuzzy $\text{CP}^3$ constraint. The extra constraint is a ‘fuzzy’ analog of the commutative constraint to obtain a compact spacetime ($S^4$) from twistor space ($\text{CP}^3$). The algebra of fuzzy $S^4$ can therefore be given by a subset of $SU(4)$ such that it preserves closure and associativity of the algebra.

**Acknowledgements**

The author would like to thank the organizers of the 21st Nishinomiya-Yukawa Memorial Symposium on theoretical physics (at Nishinomiya and Kyoto, Japan, November 11-15, 2006) for giving him an opportunity of participation.
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