Bootstrap Particles and Elementary Particles in the $Z_s=0$ Limit

--- Discussion in a Soluble Model* ---

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Some exactly soluble models, such as the Lee- and Zachariasen-models, have been used to show the equivalence between composite particles defined by the $ND^{-1}$ method and the condition of vanishing wave function renormalization constant imposed upon elementary particles. These models, however, do not allow to discuss composite particles produced through the bootstrap mechanism. Here a soluble model obtained by an extension of the Lee model is shown to allow such discussion, and the equivalence between one of the bootstrap conditions and the condition $Z_s=0$ is explicitly proved. The argument is supplemented by a general remark which shows the equivalence for more general cases, at least, formally.

§1. Introduction

1. There have been many works that have discussed the relations between composite particles defined in various ways. Most of them concern with the definitions of composite particles as the zero of the $D$-function in the $ND^{-1}$ method and as the limit of elementary particles with vanishing wave function renormalization constant. Equivalence of the two definitions has been proved both in the exactly soluble models (Lee- and Zachariasen-models) and in general formulations under some assumptions. They were useful to make the physical concept of composite particles clear.

2. Recently special attention has been paid to the bootstrap mechanism as a way of defining composite particles. It has been shown that the bootstrap mechanism is useful to understand the mutual dynamical relations and even the internal symmetries among the strongly interacting particles. In spite of its successful applications it still remains ambiguous in principle. It is desired to investigate the relation of the bootstrap mechanism to the other ways of defining composite particles.

3. In §2 we take up a soluble model appropriate for a discussion of the bootstrap mechanism and want to prove the equivalence between the bootstrap conditions and the $Z_s=0$ condition imposed on an elementary particle to make it a composite one. Such equivalence is expected from the previous works, because the bootstrap particle is nothing but a composite particle formulated in the $ND^{-1}$ method. Nevertheless, we should note that the bootstrap conditions

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are more subtle than the conditions for mere composite particles of the ND- formalism. For example, the mass and the coupling constant appear explicitly in the latter ones. This cannot generally be expected for the bootstrap conditions.

Generally, we expect two conditions from the bootstrap mechanism, one for the mass and another for the coupling constant, while from the $Z_3=0$ approach to a composite particle only one condition for the mass and the coupling constant. For the particular model under discussion, however, the two bootstrap conditions are shown to coincide, giving the same condition as the $Z_3=0$ condition.

4. As we are treating a soluble model, there is an advantage that the argument, besides being explicit, can allow possible existence of the Castillejo-Dalitz-Dyson poles. CDD poles have never been treated appropriately and usually excluded in general discussions. As already emphasized by Zachariasen in other connection, a certain type of CDD poles have interesting significance. This is also emphasized in this work.

5. In §3 we try to look at the argument of §2 from a general viewpoint. As mentioned above, bootstrap conditions generally supply two conditions which may determine the mass and the coupling constant of the composite particle. Likewise we should have another condition, besides the one $Z_3=0$, in the formulation of composite particles as a limit of elementary particles, if the latter be equivalent to the bootstrap method. Some authors have proposed $Z_1=0$ as such a condition, where $Z_1$ is the vertex function renormalization constant. As far as we know, it has been difficult to formulate this condition in general. We shall discuss another condition which, together with $Z_3=0$, may constitute two independent conditions. In this section we also give a reasoning about why the bootstrap conditions coincide in the present model.

Section 4 contains a comment on unstable bootstrap particles.

§2. Soluble model

6. The model which we take up describes, like the Lee model, the scattering of the $\theta$-particle by the fixed $N$-particle through the intermediate $V$-particle. It is slightly modified from the Lee model. The scattering amplitude in this model obeys the dispersion relation

$$t(w) = -\frac{g^2}{4\pi} \left( \frac{1}{w-d} - \frac{1}{w+d} \right)$$

$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} dw' |f(w')|^2 |t(w')|^2 \sqrt{w^2 - m_v^2} \left( \frac{1}{w' - w} + \frac{1}{w' + w} \right),$$

(2.1)

where $d=m_v-m_N$ and $f(w)$ is a cutoff function. We have retained the $V$ and $N+\theta$ states as the intermediate states. The $V+\theta$ states and all the others are neglected. This dispersion relation can be solved exactly if we adopt $w^0$ as the variable instead of $w$. Because it then reduces to the usual Lee model with non-
relativistic expression for the $\theta$-particle momentum.

7. At the first place, however, let us regard the $V$-particle as a bootstrap particle made up of the $N$- and $\theta$-particles through exchange of the $V$-particle itself. Correspondingly we dismiss the $V$-particle pole at $w=A$ in (2.1). It is easy to find a solution of the dispersion relation thus modified. We get

$$t(w) = -\frac{\theta^2/4\pi}{(w+A)G_0(w)},$$

(2.2)

$$G_0(w) = 1 - \frac{\theta^2}{4\pi^2} (w+A) \int_{w}^{\infty} dw' \rho(w') \left( \frac{1}{(w'+A)^2(w-w')} + \frac{1}{(w'-A)^2(w'+w)} \right),$$

where $\rho(w) = |f(w)|^2 \sqrt{w^2 - m^2}$. One can easily see that $G_0(w)$ has no complex zeros. It has no zero anywhere, if $\theta^2/4\pi < J_0^{-1}$, while it has a single zero on the part of the real axis $A < w < \mu$, if $\theta^2/4\pi > J_0^{-1}$, where

$$J_0 = \frac{\mu + A}{\pi} \int_{w}^{\infty} dw \rho(w) \left( \frac{1}{(w+A)^2(w-w)} + \frac{1}{(w-A)^2(w+w)} \right).$$

(2.3)

The bootstrap conditions $G_0(A) = 0$ and $(\theta^2/8\pi A)/(dG_0(w)/dw)_{w=A} = -\theta^2/4\pi$ coincide with each other and give

$$(\theta^2/4\pi) J = 1,$$

(2.4)

where

$$J = \frac{4A}{\pi} \int_{w}^{\infty} dw \frac{\rho(w)}{(w-A)^2}.$$  

8. It is possible to cast the solution (2.2) into the usual $ND^{-1}$ form, putting $N(w) = (\theta^2/4\pi) H(w)/(w+A)$ and $D(w) = G_0(w) H(w)$, $H(w)$ being the phase function

$$H(w) = \exp \left\{ -\frac{w+A}{\pi} \int_{-\infty}^{w} \frac{\alpha(w')}{(w'+A)(w'-w)} dw' \right\},$$

$$\alpha(w) = \frac{1}{2i} \log \left[ \frac{G_0(w+i\epsilon)}{G_0(w-i\epsilon)} \right]; \quad w \leq -\mu.$$  

(2.5)

Note that $\alpha(-\mu)$ is defined to be zero. If $(\theta^2/4\pi) J < 1$, $\alpha(w)$ tends to zero for $w \to -\infty$, and, with a suitable cutoff function, $H(w)$ goes to a constant for large $w$. In this case both $N$ and $D$ satisfy unsubtracted dispersion relations

$$N(w) = \int_{-\infty}^{w} dw' \frac{\nu(w') D(w')}{w'-w},$$

$$D(w) = D(\infty) - \frac{1}{\pi} \int_{w}^{\infty} dw' \rho(w') N(w') \frac{w'}{w-w},$$

(2.6)

where $\nu(w)$ denotes the left-hand cut discontinuity of the amplitude; it involves the delta function of the pole contribution at $w = -A$. If $(\theta^2/4\pi) J > 1$, on the
other hand, one needs once subtracted dispersion relations for $N$ and $D$. The bootstrap conditions

$$D(\delta) = 0,$$  \hspace{1cm} (2.7a)

$$\left( N(\omega) \int \frac{dD(\omega)}{d\omega} \right)_{\omega = \delta} = -\frac{g^2}{4\pi}$$  \hspace{1cm} (2.7b)

turn out to be satisfied with the critical value $(g^2/4\pi)J = 1$. For this case we probably needs once subtracted dispersion relations. In any case we conclude from the foregoing argument that the two conditions (2.7a, b) coincide with each other and are not independent. This is the situation just expected, if we need once subtracted dispersion relations for $N$ and $D$.

9. Alternately we return to the original dispersion relation (2.1) and regard the $V$-particle as an elementary particle. We will pass to the composite $V$-particle by imposing the condition $Z_3 = 0$. The solution of (2.1) is given, admitting a pair of CDD poles, by

$$\Gamma(\omega) = \frac{\Delta g^2/2\pi}{(\omega^2 - \Delta^2)G_1(\omega)},$$  \hspace{1cm} (2.8)

$$G_1(\omega) = 1 + \frac{\Delta g^2}{\pi^2}(\omega^2 - \Delta^2) \int \frac{d\omega'}{(\omega^2 - \Delta^2)^2} \left( \frac{\omega' \rho(\omega')}{(\omega^2 - \omega'^2)} \right) + \frac{(\omega^2 - \Delta^2)R}{(\omega^2 - \Delta^2)(\omega^2 - \omega^3)}. \hspace{1cm} (2.9)$$

It is required that $R$ is non-negative, $\omega_0 > \mu$, and

$$J \cdot \frac{g^2}{4\pi} \cdot \frac{R}{\omega_0^2 - \Delta^2} \leq 1. \hspace{1cm} (2.10)$$

We define the Omnès function $\mathfrak{D}(\omega)$:

$$\mathfrak{D}(\omega) = \exp \left( \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\delta(\omega')}{(\omega - \Delta)(\omega' - \omega)} \right),$$

$$\delta(\omega) = -\frac{1}{2i} \log \left[ \frac{G_1(\omega + \pi i)}{G_1(\omega - \pi i)} \right]. \hspace{1cm} (2.11)$$

The form factor of the $V$-particle and the wave function renormalization constant are given by

$$F(\omega) = g P(\omega) \mathfrak{D}^{-1}(\omega),$$  \hspace{1cm} (2.12)

$$Z_3^{-1} = 1 + \frac{1}{4\pi^2} \int d\omega \frac{\rho(\omega)|F(\omega)|^2}{(\omega - \Delta)^2}, \hspace{1cm} (2.13)$$

where $P(\omega)$ is a polynomial of $\omega$ and normalized as $P(\Delta) = 1$.

10. In the following we do not consider the possible occurrence of CDD poles except for one particular type. This is because we are to compare the amplitude in the limit $Z_3 = 0$ with the $ND^{-1}$ amplitude and the latter usually
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does not admit any CDD poles. The particular CDD poles which we want to allow have the parameters limited by the condition \( G_1(\infty) = 0 \), i.e.

\[
\frac{R}{w^3 - \Delta^3} = 1 - J \frac{g^2}{4\pi}.
\]  

(2.14)

From (2.14) one can easily see that, if the CDD poles exist, the amplitude is rewritten as

\[
t(\omega) = -\left( \lambda + \frac{\Delta g^2/2\pi}{\omega^2 - \Delta^2} \right) \overline{G}_1(\omega),
\]  

(2.15)

\[
\lambda = -\frac{\Delta g^2/2\pi}{\omega^2 - \Delta^2},
\]  

(2.16)

\[
\overline{G}_1(\omega) = 1 + \frac{2}{\pi} \int_0^\infty dw' \left( \lambda + \frac{\Delta g^2/2\pi}{\omega^2 - \Delta^2} \right) \left( \frac{w'}{w^2 - \Delta^2} \right) \left( \frac{w^2}{w'^2 - \Delta^2} \right) \rho(\omega').
\]  

(2.17)

Why we allow the existence of the particular CDD poles will become clear in the following. The significance of such a CDD pole has been emphasized by Zachariasen. From Zachariasen’s argument one may consider it to represent a “direct” \( NN\theta \) interaction.

Generally speaking, if there exist \( n \) pairs of CDD poles and if the inequality

\[
J \frac{g^2}{4\pi} + \sum_{i=1}^n \frac{R_i}{w_i^3 - \Delta^3} < 1
\]  

(2.18)

holds, it can be seen that \( \delta(\omega) \) approaches to \( n\pi \) and \( \overline{\Delta}(\omega) \) diverges like \( \omega^n \) for large \( \omega \). In this case it is necessary to take the degree of \( P(\omega) \) in (2.12) as large as \( n \), because, otherwise, the condition \( Z_3 = 0 \) will never be realized. As seen in the following, we do not need to raise the degree of \( P(\omega) \) for the particular CDD poles just admitted. Thus we may put

\[
F(\omega) = g/\overline{\Delta}(\omega),
\]  

(2.19)

whether the particular CDD poles exist or not.

11. In order to see what the condition \( Z_3 = 0 \) imposes on \( g^2/4\pi \) and \( \Delta \), it is necessary to examine the asymptotic behaviour of \( \delta(\omega) \) and \( G_1(\omega) \). This is done under the assumption that \( \rho(\omega) \) never vanishes for \( \mu < \omega < \infty \) and behaves as \( c \omega^{\alpha} \) for large \( \omega \), where \( \alpha < 1/2 \). Further it is assumed that

\[
\lim_{\omega \to \infty} P \int_0^{\omega} dw' \left( \frac{\rho(\omega') - c \omega'^{\alpha_0}}{\omega^2 - \Delta^2} \right) \left( \frac{\omega'}{\omega^2 - \Delta^2} \right) = O\left( \frac{1}{\omega^3} \right).
\]  

(2.20)

The results are shown in Table I. The third column of Table I refers to the case where there is a pair of CDD poles not of the particular type discussed above. It is not directly related with the following argument, but included for completeness.
Table I. $\alpha$ is defined by $\rho(w) \sim c w^\alpha$ and $\alpha \leq 1/2$. $G_1(\infty) = 1 - J \frac{\theta^2}{4 \pi} - \frac{R}{w^2 - D^2}$.

<table>
<thead>
<tr>
<th></th>
<th>$\delta(\infty)$</th>
<th>Asymptotic behavior of $G_1(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No CDD poles</td>
<td>$G_1(\infty) &gt; 0$</td>
<td>0</td>
</tr>
<tr>
<td>$(R=0)$</td>
<td>$G_1(\infty) = 0$</td>
<td>$\max{-\pi, -\pi(1-\alpha)}$</td>
</tr>
<tr>
<td>CDD poles</td>
<td>$G_1(\infty) &gt; 0$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>$(R \neq 0)$</td>
<td>$G_1(\infty) = 0$</td>
<td>$\max{0, \pi\alpha}$</td>
</tr>
</tbody>
</table>

12. We first assume that

$$\int_0^\infty dw \frac{\rho(w)}{(w - \bar{a})^2}$$

is finite. (2.21)

From (2.11) one can easily derive for the case without the CDD poles

$$F(w) = \frac{1}{G_1(w)} \exp\{K_1(w)\},$$

(2.22)

$$K_1(w) = \frac{w - \bar{a}}{\pi} \int_0^\infty dw' \frac{\delta(w')}{(w' + \bar{a})(\bar{a}' + \bar{a})}.$$  

(2.23)

For a later use we also derive

$$F(w) = \frac{1}{\sqrt{G_1(w)}} \exp\{K_2(w)\},$$

(2.24)

$$K_2(w) = \frac{w}{\pi} \int_0^\infty dw' \frac{\delta(w')}{w'^2 - w^2} - \frac{\bar{a}}{\pi} \int_0^\infty dw' \frac{\delta(w')}{w'^2 - \bar{a}^2}.$$  

(2.25)

For the case with the CDD poles one gets analogous expressions with $G_1(w)$ replaced by $\bar{G}_1(w)$ of (2.17). Replacing $\delta(w')$ by $\delta(\infty) + \pi \varepsilon_1$ in (2.23) one gets an upper bound for $\exp\{K_1(w)\}$

$$|\exp\{K_1(w)\}| < a w^{\delta(\infty) + \pi \varepsilon_1},$$

(2.26)

where $\varepsilon_1$ is an arbitrarily small positive number and $a$ is a positive constant. Substituting (2.23) into (2.13), we conclude from (2.21) that, if $G_1(\infty) > 0$, $Z_3^{-1}$ is finite in the case without the CDD poles. In the case with the CDD poles we note that

$$\bar{G}_1(w) = \frac{w_0^2 - w^2}{w_0^2 - D^2} G_1(w).$$

(2.27)

From Table I it is easy to see that the integral of (2.13) is bounded by constant

$$\times \int_0^\infty dw \frac{1}{w^{3-\varepsilon_2}},$$

irrespective of $\delta(\infty)$. Thus $Z_3^{-1}$ is always finite (unless $w_0 \to \infty$).
We now derive a lower bound for $Z_3^{-1}$. One easily sees that $\delta(\omega) \geq -\pi$, whether the CDD poles exist or not. Replacing $\delta(\omega)$ by $-\pi$ in (2·23) we get a lower bound for $\exp\{K_1(\omega)\}$

$$|\exp\{K_1(\omega)\}| > \frac{b}{\omega + \delta},$$

where $b$ is a constant. Therefore, if the CDD poles do not exist,

$$Z_3^{-1} > 1 + \frac{b^2}{8\pi^2} \int_{\omega}^{\infty} \frac{\rho(s)}{s^{1/2}} \frac{ds}{|G_1(s)|^2 (s-\delta^2)^2},$$

where $s = \omega^2$. By making use of the unitarity condition

$$\text{Im} \left( \frac{1}{G_1(s)} \right) = -\frac{\Delta^2}{2\pi} \frac{\rho(s)}{(s-\delta^2)|G_1(s)|^2},$$

and further replacing the integral by the one along a contour which encircles the cut $\mu^2 \leq s < \infty$, we get

$$Z_3^{-1} > 1 + \frac{ib^2}{8\pi \delta} \int_{\epsilon}^{\infty} \frac{ds}{(s-\delta^2)^{1/2} \sqrt{s} G_1(s)}$$

$$= 1 - \frac{b^2}{4\Delta^2} + \frac{b}{4\pi \Delta} \int_{0}^{\infty} \frac{ds}{(s+\delta^2)^{1/2} \sqrt{s} G_1(-s)}. \quad (2·30)$$

We have deformed the contour into the another which encircles the cut $-\infty < s < 0$, and changed the variable $s$ into $-s$. From (2·30) and the second column of Table I it is seen that $Z_3^{-1}$ diverges as $G_1(\infty) \rightarrow 0$.

For the case where the CDD pole exists, we get

$$Z_3^{-1} > 1 - \frac{b^2}{4\Delta^2} + \frac{b^2}{4\pi \Delta} \int_{0}^{\infty} \frac{ds}{(s+\delta^2)^{1/2} \sqrt{s} G_1(-s)} + \frac{b^2}{4\pi \Delta \sqrt{\delta \omega}} (s_0-\delta^2). \quad (2·31)$$

The right-hand side diverges as $R/(s_0-\delta^2) \rightarrow 0$. To summarize, we have proved that $Z_3 \rightarrow 0$ is equivalent to $G_1(\infty) \rightarrow 0$ or $R/(s_0-\delta^2) \rightarrow 0$, according as the CDD poles exist or not. The latter conditions in turn are equivalent to the bootstrap condition (2·4).

13. The case, for which the cutoff function $f(\omega) = 1$, is pathological. The $ND^{-1}$ method still gives a meaningful condition. The $Z_3$ condition, however, does not work, unless the CDD poles exist, because it vanishes identically. If the CDD poles exist, it can be seen that $\delta(\omega)$ approaches to $\pi/2$ for large $\omega$, so that $\xi(\omega) \rightarrow \omega^{1/2}$. Thus, $Z_3$ is finite in this case. If we make $Z_3$ vanish, we must put $R/(s_0-\delta^2) \rightarrow 0$. Thus it is possible to get (2·4) from the condition $Z_3 \rightarrow 0$. 

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§3. General remarks

14. The preceding proof of equivalence between the bootstrap condition and the $Z_{\alpha} = 0$ condition cannot be generalized beyond the particular model. However, let us try to find the general features of the argument. The following consideration is based on the fact that the $ND^{-1}$ amplitude obeys the same dispersion relation (2.1) as the amplitude for the elementary $V$-particle. These amplitudes may be distinguished from each other essentially by the asymptotic behaviours at infinity and/or the zero points, because they have the common type of singularities in any finite region of the complex plane. Indeed, the foregoing discussion suggests:

1) The $ND^{-1}$ amplitude may have an extra pole at infinity compared with the elementary particle amplitude.

2) The elementary particle amplitude may have one (or several) extra zero of such a kind as it can be pushed to infinity by the suitable choice of the mass and the coupling constant of the particle. (Of course, the amplitude may have zeros corresponding to the usual CDD poles whose positions are independent on these parameters. We do not consider this possibility.)

In our model the case 1) occurs if the elementary particle amplitude does not involve the particular CDD poles, while the case 2) occurs, if it does. Let us write the elementary particle amplitude as

$$t(w) = R(w)/D(w), \quad (3.1)$$

where $D(w)$ is the Omnès function as defined by (2.11). In the model discussed above, if the CDD poles do not exist, one gets from (2.24)

$$D(\infty) = \sqrt{G_1(\infty)} \exp{-K_0(\infty)} \quad K_0(\infty) = -\frac{A}{\pi} \int_{\mu}^\infty dw' \frac{\delta(w')}{w'^2 - A}.$$  

Therefore, the condition $D(\infty) = 0$ corresponds to $Z_{\alpha} = 0$. We expect from (2.13) and (2.19) that this correspondence will be generally true.

15. It is interesting to imagine the situation, where to push the zero of $N(w)$, say $w_0$, to infinity and to make $D(\infty)$ vanish work as independent conditions on the particle’s mass and coupling constant. We shall show formally that (3.1) becomes identical with the $ND^{-1}$ amplitude, if the two conditions are imposed.

We assume that $D(w)$ obeys the dispersion relation

$$D(w) = 1 - \frac{w - A}{\pi} \int_{\mu}^\infty dw' \frac{\rho(w')R(w')}{(w' - A)(w' - w)}. \quad (3.2)$$

We use the same notation as before, but the argument is evidently general. Putting $R(w) = ((w_0 - w)/(w_0 - A)) \cdot N^*(w)$, we have
If we make \( \omega_0 \to \infty \), the last term of \( D^*(\omega) \) drops. Next, using the condition \( D^*(\infty) = 0 \), we get

\[
D^*(\omega) = -\frac{1}{\pi} \int_\omega^\infty d\omega' \frac{\phi(\omega') N^*(\omega')}{\omega' - \omega}.
\]

Put \( N^*(\omega) = -\left( (g^2/4\pi)/(\omega - \delta) \right) \cdot N(\omega) \), then

\[
t(\omega) = N(\omega)/D(\omega),
\]
\[
D(\omega) = -\frac{\omega - \delta}{\pi} \int_\omega^\infty d\omega' \frac{\phi(\omega') N(\omega')}{(\omega' - \delta)(\omega' - \omega)}
= D(\infty) - \frac{1}{\pi} \int_\omega^\infty d\omega' \frac{\phi(\omega') N(\omega')}{\omega' - \omega}.
\]

It is evident that \( N(\omega) \) obeys the dispersion relation of (2.6). Note that the left-hand cut discontinuity \( v(\omega) \) does not involve the pole contribution at \( \omega = \delta \).

Above we have first made \( \omega_0 \to \infty \) and then \( \mathcal{D}(\infty) \to 0 \). The conditions may be used in a reversed order. From \( \mathcal{D}(\infty) \to 0 \), \( \mathcal{D}(\omega) \) will obey an unsubtracted dispersion relation like (3.5). Then, introducing \( N^*(\omega) \) and making \( \omega_0 \to \infty \), we are led to (3.5) and finally to (3.7).

16. The relation of the general argument to the argument of §2 is as follows. First, we consider the case where the particular CDD poles exist in the elementary \( V \)-particle amplitude. Multiplying the numerator and the denominator of (2.15) by

\[
\exp\left\{ -\frac{\omega - \delta}{\pi} \int_\omega^\infty d\omega' \frac{\phi(\omega')}{(\omega' + \delta)(\omega' + \omega)} \right\},
\]

we may put

\[
\mathcal{R}(\omega) = \left(1 + \frac{4g/2\pi}{\omega^2 - \delta^2}\right) \exp\{-K_1(\omega)\},
\]
\[
\mathcal{D}(\omega) = \mathcal{G}_1(\omega) \exp\{-K_1(\omega)\}.
\]

\( \mathcal{R}(\omega) \) has zeros at \( \omega = \pm \omega_0 \). Define

\[
N^*(\omega) = \frac{\omega_0^2 - \delta^2}{\omega_0^2 - \omega^2} \mathcal{R}(\omega),
\]
\[
D^*(\omega) = \frac{\omega_0^2 - \delta^2}{\omega_0^2 - \omega^2} \mathcal{D}(\omega).
\]
Letting $\omega_0 \to \infty$, we have the relation (2.4) from (2.14). From (2.27) we get, after $\omega_0 \to \infty$,

$$D^*(\omega) = g_1(\omega) \exp\{-k_1(\omega)\}, \quad (3.9)$$

$$g_1(\omega) = \frac{4g^2}{\pi^2} \int_0^\infty d\omega' \frac{\omega' \rho(\omega')}{(\omega'^2 - D)(\omega'^2 - \omega^2)}, \quad (3.10)$$

where $k_1(\omega)$ is $K_1(\omega)$ in (2.23) with the phase shift given by

$$\delta(\omega) = -\frac{1}{2i} \log\left[g_1(\omega + i\varepsilon)/g_1(\omega - i\varepsilon)\right]. \quad (3.11)$$

It can be seen that $\delta(\omega)$ goes to $-\pi$ for large $\omega$, if a reasonable cutoff function is used. Then, $|\exp\{-k_1(\omega)\}| \sim \omega$ and $D^*(\omega) \to 0$ for large $\omega$. Therefore, the condition $D^*(\infty) = 0$ does not work any more.

Conversely, in the case where the amplitude does not have the CDD poles, the condition $S_D(\infty) = 0$ works, but the condition $\omega_0 \to \infty$ does not, because $\Re(\omega)$ has no zero.

17. That $\omega_0$ does not occur in the present model, unless we introduce the particular CDD poles, is due to the peculiarity of the model that the amplitudes be symmetric for the direct (or positive $\omega$) and crossed channel (or negative $\omega$). This may also make an explanation of why the bootstrap conditions (2.7a, b) turn out to coincide.

Let us destroy the symmetric property of the model using different cutoff functions for the direct and crossed channels. Then, the bootstrap conditions (2.7a, b) become independent, giving

$$\frac{4g^2}{2\pi^2} \int_0^\infty \left(\frac{\rho_1(\omega')}{\omega' + D} + \frac{\rho_1(\omega')}{\omega' - D}\right) \frac{d\omega'}{\omega'^2 - D^2} = 1, \quad (3.12a)$$

$$\frac{4g^2}{2\pi^2} \int_0^\infty \left(\frac{\rho_1(\omega')}{\omega' - D} + \frac{\rho_1(\omega')}{\omega' + D}\right) \frac{d\omega'}{\omega'^2 - D^2} = 1. \quad (3.12b)$$

For the elementary $V$-particle amplitude we have the expression

$$t(\omega) = -\frac{g^2}{4\pi} \left(\frac{1}{\omega - D} - \frac{\gamma}{\omega + D}\right)/G_3(\omega), \quad (3.13)$$

$$G_3(\omega) = 1 + \frac{g^2}{4\pi^2} (\omega - D) \int_0^\infty \left(\frac{1}{\omega' - D} - \frac{\gamma}{\omega' + D}\right)$$

$$\times \left(\frac{\rho_1(\omega')}{\omega' - D}(\omega' - \omega) - \frac{\rho_1(\omega')}{\omega' + D}(\omega' + \omega)\right) d\omega'. \quad (3.14)$$

$\gamma$ is determined from

$$\gamma/G_3(-D) = 1. \quad (3.14)$$
The zero of the numerator of (3·13) is
\[ \varpi_0 = -4 \left( \frac{1 + \gamma}{1 - \gamma} \right). \]  
(3·15)
Thus the condition \( \varpi_0 \to \infty \) is that \( \gamma = 1 \), while the condition \( \mathcal{D}(\infty) = 0 \) may correspond to \( G_3(\infty) = 0 \). These are easily seen to be equivalent to (3·12a, b).

§4. Conclusion

18. The foregoing argument is merely formal. To be rigorous we must have precise knowledge about the asymptotic behaviors of the amplitudes. Probably we need to take into account the Regge asymptotic behaviors of scattering amplitude. It should be noted that the complete equivalence between the bootstrap conditions and \( Z_3 = 0 \) does not hold even within the soluble model. Because there is a pathological amplitude, for which \( Z_3 \) vanishes identically. In order to maintain the complete equivalence we would have to use some assumption about the asymptotic behaviour to exclude such pathological amplitude. An analogous situation may occur in the fully relativistic theory.\(^8\)

19. In §3 the condition \( \mathcal{D}(\infty) = 0 \) has been seen to correspond to the condition \( Z_3 = 0 \). The physical meaning of the condition \( \varpi_0 \to \infty \) has not been clarified. It is a question if this has any relation with \( Z_3 = 0 \).

20. If the equivalence between the \( ND^{-1} \) and the \( Z_3 = 0 \) methods to formulate composite particles be shown to hold widely, it will be useful not only to the basic problem to understand the concept of composite particles but also to the formalism of composite particles. Until now most authors have used the \( ND^{-1} \) or the bootstrap method. As for the formulation of composite particles starting from elementary particles we should note extensive works by S. Weinberg.\(^9\)

21. Finally we comment on unstable bootstrap particles. The present model also give an example of an unstable bootstrap particle. Along with the argument of Gell-Mann and Zachariasen\(^10\) we introduce an elementary unstable \( V \)-particle in terms of a CDD pole. If we require \( Z_3 \) of the particle to be zero, we get a relation between the mass and the coupling constant (or the decay width), which is just expected for the bootstrap unstable particle. A detailed discussion will be published elsewhere.

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References

Earlier references are found in these works.
The latter authors also give a discussion of the bootstrap conditions on the same model, but
their emphasis is different from ours.
6) A. Salam, Nuovo Cim. 25 (1962), 224.