Quatnerionic Variational Formalism
for Poincaré Gauge Theory and Supergravity

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A previously proposed quaternionic formulation of Poincaré gauge theory is further extended in two
respects. First, the Rarita-Schwinger field is introduced in addition to the Dirac one, thereby enabling us to
discuss supergravity in the quaternion form. Secondly, quaternionic derivative is employed in the variational
principle. The method is illustrated by an elementary derivation of extended Bach-Lanczos identities. It is
shown that Hayashi and Bregman's identities from Poincaré-gauge invariance are extended so that the
Rarita-Schwinger field occupies the same position as Poincaré gauge fields. A role of local supersymmetry in
these identities is then clarified.

§1. Introduction

In a previous paper\textsuperscript{11) to be referred to as I hereafter, we have pointed out that there
exists a new style of description of Dirac spinors, which makes use of Hamilton's
quaternions. It dispenses with the mixed use of matrix-vector notation, and finds an
interesting application to Poincaré gauge theory\textsuperscript{20-10). Thus we have presented in I a
quaternionic formulation of Poincaré gauge theory, where Hayashi's most general
gravitational Lagrangian\textsuperscript{63,10} has been constructed on the basis of quaternions.

It is remarkable that quaternions seem to be a basic algebra over which fundamental
spinor fields, internal-symmetry and Poincaré gauge fields are defined. There arises a
natural question how to describe gravitinos in the quaternionic formalism. It amounts to
asking if supergravity\textsuperscript{11) is formulated over quaternions.

The purpose of this paper is two-fold. Firstly, we shall show that the Rarita-
Schwinger field\textsuperscript{15,19} is introduced into the theory like the Dirac one; one needs vector
spinor-quaternion, whereas scalar spinor-quaternion represents Dirac spinor as in I. This
makes a non-trivial change in the notational convention in supergravity. Secondly, we
shall develop the quaternionic variational principle for Poincaré gauge theory and
supergravity. It is based on quaternionic derivative which is defined in analogy with the
complex case.

Since Poincaré-gauge invariance is local in nature, there are derived \textit{identities} \textsuperscript{7,8}) which have been investigated by Hayashi and Bregman in general case.\textsuperscript{7) We shall
rederive them by our quaternionic method, and find that Hayashi and Bregman's identities
are extended by the presence of the Rarita-Schwinger field so that the latter occupies the
same position as Poincaré gauge fields. In particular, it is shown that local supersymmetry plays the same role as local Lorentz-gauge invariance also in these
identities.

This paper is organized as follows. Basic algebraic properties of quaternions are
summarized and quaternionic derivative is introduced in the next section. In §3 the Dirac
as well as the Rarita-Schwinger fields are defined over quaternions, together with their
respective Lagrangians. A succinct review of a quaternionic formulation of Poincaré
gauge theory will be given in §4. To illustrate the quaternionic-derivative method, we shall present an elementary derivation of extended Bach-Lanczos identities\textsuperscript{(4,15)} in §5. The quaternionic variational formalism will be developed for Poincaré gauge theory in §6, and for supergravity in §7. The last section contains discussions.

§ 2. Algebraic properties of quaternions

This section collects some useful formulas on quaternions. Quaternionic derivative will be introduced toward the end of this section.

Throughout this paper we shall exclusively use the non-division algebra $Q$ of complex quaternions. It turns out that the algebra $Q$ is quite useful in discussing Poincaré gauge theory and supergravity at the Lagrangian level.

Following the notations of I, we denote the basis of $Q$ by

$$b_0=1, \quad b_\pm=\sqrt{-1}i, \quad b_\mp=\sqrt{-1}j, \quad b_\pm=\sqrt{-1}k,$$  \hspace{1cm} (1)*

where $\{i, j, k\}$ are Hamilton’s units of quaternions.\textsuperscript{*} Any $q \in Q$ is decomposed as

$$q = b_\ell q^\ell,$$  \hspace{1cm} (2)**

where we assume $q^\ell (\ell = 0, 1, 2, 3)$ to be either commuting or anti-commuting numbers. In the latter case we call $q$ Grassmannian quaternion. A null quaternion $q=0$ means $q^\ell=0$ for all $\ell$.

The important operations defined over $Q$ are the quaternion conjugation: $q \rightarrow \bar{q} = \bar{b}_\ell q^\ell$; $\bar{b}_0 = b_0$, $\bar{b}_\ell = -b_\ell (r=1, 2, 3)$, and the quaternionic Hermitian conjugation: $q \rightarrow q^* = b_\ell q^\ell \ast$, where the asterisk indicates ordinary Hermitian conjugation. It is easy to see that, for $p, q \in Q$,

$$\bar{pq} = \bar{q} \bar{p} \quad \text{if} \quad [p^\ell, q^\ell] = 0,$$ \hspace{1cm} (3a)***

$$\bar{pq} = -\bar{p} \bar{q} \quad \text{if} \quad [p^\ell, q^\ell] = 0,$$ \hspace{1cm} (3b)***

$$(pq)^* = q^* p^* \quad \text{in both cases}.$$ \hspace{1cm} (3c)**

If $q^* = q$, it is called Hermitian quaternion. If $\bar{q} = -q$, it is named pure quaternion. We shall encounter both kinds of quaternions as well as Grassmannian ones in later sections.

From definition (1) we have

$$b_\ell b_\gamma + b_\gamma b_\ell = -2 \eta_{\ell \gamma},$$ \hspace{1cm} (4a)**

$$\bar{b}_\ell \bar{b}_\gamma + \bar{b}_\gamma \bar{b}_\ell = -2 \eta_{\ell \gamma},$$ \hspace{1cm} (4b)**

where $\eta_{\ell \gamma} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. As usual, $\eta_{\ell \gamma} = \eta^{\ell \gamma}$ is used to lower and raise Latin indices which run from 0 to 3. By repeatedly using Eq. (4), we obtain

$$b^\ell \bar{b}^\gamma b_k + b^k \bar{b}^\gamma b^\ell = -2 \eta^\ell \gamma b_k + 2 \eta^k \gamma b^\ell - 2 \eta^{k \ell} b^\gamma,$$ \hspace{1cm} (5)

\textsuperscript{*} We assume the imaginary unit $\sqrt{-1}$ to commute with Hamilton’s units.

\textsuperscript{**} Summation is always implied over repeated Latin or Greek indices.

\textsuperscript{***} $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

\textsuperscript{†} We use the convention of the footnote on p. 1651 of I.

\textsuperscript{**} We have omitted $b_0$ from the RHS. We shall use this convention throughout this paper.
and its quaternion conjugated form. The minus combination becomes
\[ b^i \bar{b}^j b^k - b^k \bar{b}^j b^i = 2i \epsilon^{ijkl} b_l, \quad i = \sqrt{-1}, \]  
(6a)
where \( \epsilon^{ijkl} \) is the totally antisymmetric tensor with \( \epsilon^{0123} = +1 \). From now on we employ the conventional notation \( i = \sqrt{-1} \) for the imaginary unit. We quote two more useful relations derived from Eqs. (4) and (6a)
\[ b^i \bar{b}^j - b^j \bar{b}^i = i \epsilon^{ijkl} b_k \bar{b}_l, \]
(6b)
\[ \epsilon^{ijkl} b_i \bar{b}_j b_k = -6ib^l. \]  
(6c)
The real part of \( q \in \mathbb{Q} \) is denoted by
\[ \text{Re}[q] = \frac{1}{2}(q + \bar{q}) = \text{Re}[\bar{q}]. \]  
(7)
Note that \( q^0 = \text{Re}[q] \) may still be complex. Although \( \mathbb{Q} \) is the non-commutative algebra, the order within \( \text{Re}[\cdot] \) is irrelevant in the sense,
\[ \text{Re}[pq] = \text{Re}[qp] \quad \text{for} \quad [q^i, q^j] = 0, \]  
(8a)
\[ \text{Re}[pq] = -\text{Re}[qp] \quad \text{for} \quad \{p^i, q^j\} = 0. \]
Equation (4) is equivalent to
\[ \text{Re}[b_i \bar{b}_j] = \text{Re}[\bar{b}_i b_j] = -\eta_{ij}. \]  
(9)
We then derive from Eqs. (5) and (6a) that
\[ \text{Re}[b^i \bar{b}^j b^k \bar{b}^l + \text{h.c.}] = 2(\eta^{ij} \eta^{kl} - \eta^{ik} \eta^{jl} + \eta^{il} \eta^{jk}), \]  
(10a)
\[ \text{Re}[b^i \bar{b}^j b^k \bar{b}^l - \text{h.c.}] = -2i \epsilon^{ijkl}. \]  
(10b)
Here and hereafter \( \text{h.c.} \) designates the quaternionic Hermitian conjugation of the previous terms within the real part \( \text{Re}[\cdot] \); if the latter contain \( i \) explicitly, the sign of \( i \) should also be reversed.
We now proceed to define the quaternionic derivative with respect to \( q \) of Eq. (2) by
\[ \frac{\partial}{\partial q} = \bar{b}_i \frac{\partial}{\partial q^i} = -b^i \frac{\partial}{\partial q^i}. \]  
(11)
This is analogous to \( \partial/\partial z = (1/2)(\partial/\partial x - i\partial/\partial y) \) for complex \( z \). We have chosen the normalization so that
\[ \frac{\partial q}{\partial q} = -2, \quad \frac{\partial \bar{q}}{\partial q} = 4. \]  
(12)
If \( p \in \mathbb{Q} \) is independent of \( q \), then
\[ \frac{\partial (pq)}{\partial q} = \begin{cases} -2p & \text{for} \quad [p^i, q^j] = 0, \\ +2p & \text{for} \quad \{p^i, q^j\} = 0, \end{cases} \]  
(13a)
\[ \frac{\partial (\bar{q}p)}{\partial q} = 4p \quad \text{for both cases}. \]  
(13b)
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quaternions is differentiated. Thus, it follows from Eq. (13) that

\[
\frac{\partial}{\partial \tilde{q}} \Re[\tilde{p} q] = \frac{\partial}{\partial \tilde{q}} \Re[p \tilde{q}] = (\dagger_p \tilde{p})
\]

(14a)

(14b)

with the minus sign occurring only if both \( p \) and \( q \) are (independent) Grassmannian quaternions. Equation (14) immediately follows also from the relation \( \Re[p \tilde{b} \tilde{t}] = -p \tilde{t} \).

Similarly, if \( q_A \), \( A = 1, 2, \cdots \), are ordinary (Grassmannian) quaternions, of which an ordinary \( p \in Q \) is independent,

\[
\frac{\partial}{\partial q_A} \Re[p q c q_B] = \delta^A c p \tilde{q} B(\dagger) \delta^A B \tilde{c} q A .
\]

(15a)

(15b)

Or, in the variational form,

\[
\delta \Re[p q c q_B] = \Re\left[(\dagger) \frac{\partial \Re[p q c q_B]}{\partial q_A} \right] \delta q_A .
\]

(16a)

(16b)

Equations (14) \sim (16) are the basic formulas in the quaternionic variational formalism. In view of Eq. (14), Eq. (16) is valid for more variables like \( \Re[p q c q_B q_C \cdots] \).

There exists a matrix representation (rep) of \( Q: b_i \rightarrow \sigma_i, \tilde{b}_i \rightarrow \tilde{\sigma}_i \), where \( \sigma_i = (\sigma_0, \sigma) \), \( \tilde{\sigma}_i = (\sigma_0, -\sigma) \), with \( \sigma_0 \) being the \( 2 \times 2 \) unit matrix and \( \sigma \) Pauli ones. It is well-known that the spinor formalism\(^{16}\) is based on Pauli matrices while preserving 2-component notation for spinors. We have previously shown\(^{13}\) that this mixed use of matrix-vector notation is not always necessary in Poincaré gauge theory for spin-1/2 particles at the Lagrangian level. The reason is simple: Dirac spinor is also regarded as \( 2 \times 2 \) matrix-valued complex fields, and, therefore, is defined over \( Q \) as a single-component entity. Similarly, Rarita-Schwinger field is defined as a vector over \( Q \). This we shall examine in the next section.

§ 3. The Dirac and Rarita-Schwinger fields in the quaternionic formalism

In this paper we shall consider spin-1/2 and spin-3/2 particles within the framework of Poincaré gauge theory. They are described, respectively, by the Dirac and the Rarita-Schwinger\(^{12}\) fields.

We begin by assuming that spinor and vector-spinor fields are represented by

\[
\psi = b_i \psi^i
\]

(17)

and

\[
\psi_j = b_i \psi_j^i , \quad (j = 0, 1, 2, 3)
\]

(18)

respectively. As in I, we call \( \psi \) spinor-quaternion which is a Grassmannian quaternion; \( \psi_j \) is also a spinor-quaternion with one vector index. The superscript \( i \) on the component fields in Eqs. (17) and (18) stands for spinor index. It is related to Dirac index as follows. In terms of the split basis \{\( v_a, v^*_a \}_{a=0,1}^{**} \) introduced in I, any spinor-quaternion (17) is expressed as

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\(^{*)}\) This is sufficient for our purpose, for only the real part is to be varied when the action principle is applied.

\(^{**})\) For completeness we repeat here the definition: \( v_0 = (b_0 + b_3)/2, v_1 = (b_1 - ib_2)/2, v_0^* = (b_0 - b_3)/2 \) and \( v_1^* = -(b_1 + ib_2)/2 \). The last minus sign was absent in Eq. (1.24), i.e., Eq. (24) in I.
\[ \phi = v_a^* \bar{\xi}^a + \nu_a \bar{\eta}^a, \]  \hspace{1cm} (19)

where \( \bar{\xi}^a \) and \( \bar{\eta}^a \) \((\alpha = 0, 1)\) are linear combinations of \( \phi^i \) \((i = 0, 1, 2, 3)\). The notation is such that, in the bispinor rep of gamma matrices,

\[ \gamma_i = \begin{bmatrix} 0 & i \sigma_i \\ i \sigma_i & 0 \end{bmatrix}, \]  \hspace{1cm} (20)

the corresponding 4-component Dirac spinor is given by

\[ \phi = \begin{pmatrix} \xi^a \\ \bar{\eta}^a \end{pmatrix}, \quad \xi^a = (\bar{\xi}^a)^*. \]  \hspace{1cm} (21)

Although we shall use the same symbol \( \phi \) in both formalisms, no confusion would occur since the 4-component notation is always associated with gamma matrices.

The following properties of the split basis are to be noted:

\[ \bar{b}_i b_a^* = v_a^* (\sigma_i^T)_{ba}, \]  \hspace{1cm} (22a)

\[ \bar{b}_i b_a = v_a (\bar{\sigma_i})_{ba}, \]  \hspace{1cm} (22b)

\[ \text{Re}[v_a^* v_b] = \text{Re}[v_a^* v_b^*] = \text{Re}[v_a v_b^*] = \text{Re}[v_a v_b] = \frac{1}{2} \delta_{ab}, \]  \hspace{1cm} (22c)

\[ \text{Re}[v_a^* v_b] = \text{Re}[v_a^* v_b^*] = \text{Re}[v_a v_b^*] = \text{Re}[v_a v_b^*] = 0. \]  \hspace{1cm} (22d)

These equations are helpful to convert the real part of bilinear forms of \( \phi \) (or \( \psi_j \)) to the 4-component version in the rep (20), the result being Pauli-transformed to any rep.

The Lorentz transformation is effected by left translation as in I except that we assume an additional vector transformation property of \( \psi_j \).

The decomposition (19) has a remarkable property:

\[ \phi b_1 = v_a \bar{\xi}_a + v_a^* \bar{\eta}^a, \]  \hspace{1cm} (23)

where our index-shifting rule is \( \xi_a = \omega_{ab} \xi^b, \omega = -i \sigma_2 \). The Majorana constraint \( (\xi = \eta \text{ in the rep (21)}) \) is immediately written:

\[ \phi b_1 = \phi. \]  \hspace{1cm} (24)

This condition is really independent of decomposition (19), and, moreover, linear in contrast to the conventional non-linear condition \( \phi^c = C \phi = \phi, \bar{\phi} = \phi^* \gamma^0 \).

The Dirac and the Rarita-Schwinger Lagrangians turn out to be

\[ L_0(\bar{\phi}) = i \text{Re}[\phi^*(\bar{\phi}) - \text{h.c.}] - m \text{Re}[\bar{\phi} \phi b_5 + \text{h.c.}] \]  \hspace{1cm} (25)*

and

\[ L_{RS}(\bar{\psi}) = -(1/2) \epsilon^{ijkl} \text{Re}[\bar{\psi}_i \partial_j \psi_k^* \bar{b}_l + \text{h.c.}] \]

\[ - (iM/4) \epsilon^{ijkl} \text{Re}[\bar{\psi}_i b_k \bar{b}_l \psi_j b_5 - \text{h.c.}], \]  \hspace{1cm} (26)

respectively, where

* This is identical with Eq. (1-27).
\[ \partial \equiv \vec{\partial}_i \frac{\partial}{\partial x^i} = -b^i \partial_i. \quad (\partial_i \equiv \frac{\partial}{\partial x^i}) \]

Using Eq. (22) we readily recover \[ L_D \equiv -(i/2)(\vec{\phi} \gamma^i \partial_i \phi - \partial_i \vec{\phi} \gamma^i \phi) - im\vec{\phi} \phi \] from Eq. (25), and
\[ L_{RS} \equiv (i/2)\epsilon^{ijkl} \vec{\phi} i\gamma_5 \gamma_j \gamma_k \phi_i - (M/2) \vec{\phi} i\sigma^{ij} \psi_j , \]

plus a total divergence term from Eq. (26), where \( \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) and \( \sigma^{ij} = (2i)^{-1}[\gamma^i, \gamma^j] \). The total divergence term, \( -(i/4)\epsilon^{ijkl}\partial_k(\vec{\phi} i\gamma_5 \gamma_j \gamma_k \phi_i) \), identically vanishes if \( \phi_i \) is Majorana. Henceforth we shall assume this to be the case.

Defining the Euler-Lagrange derivative by
\[ \frac{\delta L}{\delta \phi} \equiv \frac{\partial L}{\partial \phi} - \partial_i \left( \frac{\partial L}{\partial (\partial_i \phi)} \right) , \]
we use the differentiation rule (15b) to obtain
\[ \frac{\delta L_D(\vec{\phi})}{\delta \phi} = K = 2i[\partial_t \vec{\phi} + im\phi b] \]

and
\[ \frac{\delta L_{RS}(\phi)}{\delta \phi} = K^i = -\epsilon^{ijkl} [b_l \partial_j \vec{\phi} k + (iM/2)b_k \vec{\phi} l b]. \]

Free field equations are \( K = 0 \) and \( K^i = 0 \). The latter is to be supplemented by a constraint.\(^{13}\) We impose
\[ \partial_t K^i = \frac{3}{2} iMb^i \vec{\phi} = 0 , \quad \text{(on-shell)} \]
where the divergence has been calculated from Eq. (31) with \( K^i = 0 \) and Eq. (6c). For massive case it leads to the usual constraint, \( \vec{b} \vec{\phi} = 0 \) or \( \gamma^i \phi_i = 0 \). On the other hand, the spinor-current \( K^i \) is exactly conserved for \( M = 0 \): This is due to the gauge invariance, \( \phi_i \rightarrow \partial_i a, \) \( a \) being a spinor-quaternion with \( abl = a \). This feature is inherited in supergravity, for local supersymmetry yields Poincaré-gauge covariant generalization\(^{11}\) of the conservation equation. Without local supersymmetry the generalized constraint puts additional condition in the gravitational sector.

Before going into these topics, we shall briefly sketch the quaternionic formulation of Poincaré gauge theory in the next section.

§ 4. Quaternionic formulation of Poincaré gauge theory. I

In I, starting with the Dirac Lagrangian (25), we have reformulated Poincaré gauge theory\(^{23-10}\) based on \( \tilde{G} \)-gauge fields \( b^\mu \) and \( A_\mu \), which are expressed in terms of vierbein \( b^a_i \) and spin connection \( A_{\mu ij}, \) antisymmetric in \( i \leftrightarrow j, \) as
\[
\begin{align*}
b^\mu &\equiv b^i b^\mu_i, \\
A_\mu &\equiv \frac{1}{4} A_{\mu ij} b^i b^j, \\
\vec{A}_\mu &\equiv -A_\mu.
\end{align*}
\]

The second equation of Eq. (33a) is obtained from the first by lowering the Greek index.
with the metric $g_{\mu\nu}$, which is then defined to be

$$b_\mu \bar{b}_\nu + b_\nu \bar{b}_\mu = -2g_{\mu\nu}.$$  

(34)

This is a general-relativistic extension from Eq. (4a). The space-time geometry is determined by the requirement (equivalent to $\mathcal{D}_\nu b_\mu = 0$)

$$b_{\mu;\nu} + A_\nu b_\mu + b_\mu A_\nu^* = 0,$$

(35)

where the semicolon stands for coordinate-covariant derivative with affinity $\Gamma^\lambda_{\mu\nu}$. This of course implies the metric condition $g_{\mu\nu;\lambda} = 0$.

Transformation properties of $G$-gauge fields under local Lorentz group are determined from that of spinor fields,

$$b_\mu' = Ub_\mu U^*,$$

(36a)

$$A_\mu' = UA_\mu \bar{U} + U\partial_\mu \bar{U},$$

(36b)

where $U = U(x) \in Q$ represents a local Lorentz transformation:

$$\phi' = U\phi,$$

(36c)

$$\phi_\mu' = U\phi_\mu, \quad N(U) = 1.$$  

(36d)

In the last equation $N(U) = U\bar{U} = \bar{U}U$ signifies the norm of $U \in Q$ and we have defined

$$\phi_\mu \equiv b^i_\mu \phi_j = b_i^j \phi_{ij}, \quad \phi_{ij} \equiv b_i^j \phi_j^i.$$  

(33c)

Note a similarity in definition for $b_\mu$ and $\phi_\mu$.

Under general coordinate transformations $x'' \rightarrow x'$, $\phi$ is assumed to be scalar, whereas

$$\phi_\mu'(x') = \frac{\partial x'^\nu}{\partial x_\mu} \phi_\nu(x), \quad \phi_\mu = b_\mu, A_\mu \quad \text{and} \quad \phi_\mu.$$

(36e)

In Poincaré gauge theory the total Lagrangian density is assumed to be

$$L = L_G + L_M,$$

(37)

where $L_G$ represents Hayashi's gravitational Lagrangian density, and $L_M$ describes a material system. We denote scalar density by corresponding Gothic letter so that $L = bL$, $b = \text{det}(b_{ij}^\mu)$, with $L$ being a scalar.

A quaternionic expression for $L_G$ was explicitly given in I; $L_G = L_1(K_\mu) + L_2(F_{\mu\nu})$ where $K_\mu$ and $F_{\mu\nu}$ are $G$-gauge field strengths. In particular,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

(38a)

The linear gravity Lagrangian is contained in $L_2$:

$$L_2 = a\Re[F_{\mu\nu}b^\mu b^\nu + \text{h.c.}],$$

(39a)

where $a = (2\chi)^{-1}$, $\chi = 8\pi G$ being the Einstein constant. The rest is quadratic in field strengths. Putting

$$F_{\mu\nu} \equiv \frac{1}{4} F_{\mu
u ij} b^i b^j, \quad F_{\mu\nu} = -F_{\nu\mu},$$

(38b)
and using Eq. (10a), we find \( L_F = -aF \), where \( F \) is a scalar curvature, \( F = \eta^{ij} F_{ij} \), \( F_{ij} = \eta^{kl} F_{ikjl} \) and \( F_{ikjl} = \eta^{rs} b^r_i b^s_j F_{rjsl} \). For later convenience we rewrite Eq. (39a) as

\[
L_F = (ia/2) E^{\mu\nu\rho\sigma} \text{Re}[F_{\mu\nu} b_\sigma \bar{b}_\rho - \text{h.c.}],
\]

where

\[
E^{\mu\nu\rho\sigma} = b_i^\mu b_j^\nu b_k^\rho b_l^\sigma \epsilon^{ijkl}
\]

is the Levi-Civita tensor density with \( E^{0123} = +1 \). In obtaining Eq. (39b) use has been made of the formula

\[
b(b^\mu \bar{b}^\nu - b^\nu \bar{b}^\mu) = iE^{\mu\nu\rho\sigma} b_\rho \bar{b}_\sigma
\]

which follows from Eqs. (6b), (33a) and (40).

The matter part \( L_M \) consists of several terms. The Dirac Lagrangian density is the same as in I; \( L_D(\mathcal{D}) = bL_D(\mathcal{D}) \), where \( \mathcal{D} = -\bar{b}^\mu D_\mu \), \( D_\mu = \partial_\mu + A_\mu \) (we are using different notation for \( \partial_\mu + A_\mu \)). The Majorana-Rarita-Schwinger Lagrangian (26) is generalized to the density,

\[
L_{RS}(D_\mu) = -(1/2) E^{\mu\nu\rho\sigma} \text{Re}[\phi_\mu (D_\nu \phi_\rho)^* \bar{b}_\sigma + \text{h.c.}]
\]

\[
-(iM/4) E^{\mu\nu\rho\sigma} \text{Re}[\bar{\phi}_\mu b_\rho \phi_\nu b_\sigma - \text{h.c.}].
\]

Here we have omitted the affinity term as in Ref. 11). Additional terms in \( L_M \) would include, e.g., internal-symmetry gauge fields, Higgs fields. In what follows we simply assume \( L_M = L_D(\mathcal{D}) + L_{RS}(D_\mu) \).

The following is a quaternionic version of the cyclic identity.

\[
\sum'_{\mu,\nu,\rho} (F_{\mu\nu} b_\rho + b_\rho F_{\mu\nu}) = \sum'_{\mu,\nu,\rho} S_{\mu\nu\rho},
\]

where \( \sum' \) denotes a cyclic sum over \( \mu, \nu, \rho \), torsion is defined by

\[
S_{\mu\nu} = b_\mu S_{\mu\nu}^\lambda = b_\lambda S_{\mu\nu}^\lambda,
\]

\[
S_{\mu
u}^\lambda = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda,
\]

and the double bar denotes the Lorentz-gauge covariant derivative,

\[
S_{\mu\nu;\rho} = \partial_\rho S_{\mu\nu} + A_\rho S_{\mu\nu} + S_{\mu\nu} A_\rho.
\]

This definition is valid for any Hermitian quaternion like \( b_\mu \). Due to the metric condition (35), torsion takes an alternative form

\[
S_{\mu\nu} = b_{\mu;\nu} - b_{\nu;\mu}.
\]

Finally, we note the Bianchi identity

\[
\sum'_{\mu,\nu} (\partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]) = 0.
\]
§ 5. Extended Bach-Lanczos identities

An immediate consequence from Eq. (45) is known as the second Bianchi identity (the first one is Eq. (43)). Less immediate is the extended Bach-Lanczos identities, first discovered by Hayashi and Shirafuji\textsuperscript{149} and independently by Nieh.\textsuperscript{150} We shall present below an elementary derivation of them to illustrate the quaternionic method.

Consider the integral

$$J = E^{\mu\nu\rho\sigma} \int \text{Re}[F_{\mu\nu}F_{\rho\sigma}] d^4x.$$  \hspace{1cm} (46)

This depends only on $A_\mu$ but not on $b_\mu$. Hence, we take an arbitrary variation with respect to $A_\mu$; upon the variation both factors $F_{\mu\nu}$ and $F_{\rho\sigma}$ produce the same result, thereby we need vary only the first factor and multiply the result by 2. Furthermore, due to the totally antisymmetric factor $E^{\mu\nu\rho\sigma}$, $F_{\mu\nu}$ is replaced by $2(\partial_\mu A_\nu + A_\mu A_\nu)$:

$$\delta J = 4E^{\mu\nu\rho\sigma} \int \text{Re}[(\partial_\mu A_\nu + A_\mu A_\nu) \cdot F_{\rho\sigma}] d^4x,$$

where the variation acts only on the first factor as indicated by dot, and is performed as follows. (The variation $\delta / \delta A_\lambda$ is defined in the sense of Eq. (29) so that surface terms are neglected.)

$$\delta \text{Re}[(\partial_\mu A_\nu + A_\mu A_\nu) \cdot F_{\rho\sigma}] = \text{Re} \left[ \frac{\delta \{ (\partial_\mu A_\nu + A_\mu A_\nu) \cdot F_{\rho\sigma} \}}{\delta A_\lambda} \right]$$

$$= \text{Re} \left[ \delta A_\lambda \cdot F_{\rho\sigma} + \delta A_\mu A_\nu F_{\rho\sigma} + \partial_\tau (\delta^{\lambda \mu} \delta^{\lambda \nu} F_{\rho\sigma}) \right]$$

which becomes, when multiplied by $E^{\mu\nu\rho\sigma}$,

$$\text{Re}[E^{\mu\nu\rho\sigma} (\partial_\rho F_{\mu\sigma} + [A_\mu, F_{\rho\sigma}]) \delta A_\lambda].$$

This identically vanishes by virtue of the Bianchi identity (45). We thus proved that

$$\delta J = 0,$$ \hspace{1cm} (47)

as a consequence from Eq. (45).

Now compute the real part in Eq. (46) from definition (38) using formula (10). We obtain

$$J = -(1/8) J_1 - (i/16) J_2,$$ \hspace{1cm} (48a)

where

$$J_1 = E^{\mu\nu\rho\sigma} \int F_{\mu\nu\rho\sigma} d^4x,$$ \hspace{1cm} (48b)

$$J_2 = E^{\mu\nu\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \int F_{\mu\nu\rho\sigma} d^4x.$$ \hspace{1cm} (48c)

The identities

$$\delta J_i = 0, \hspace{1cm} (i = 1, 2)$$ \hspace{1cm} (49)

are called the extended Bach-Lanczos ones. The case $i = 2$ was first proved by Hayashi.
and Shirafuji.\textsuperscript{14}

Equation (47) implies that the integrand in Eq. (46) is nothing but a total divergence. It is easy to verify this:

$$E^{\mu\nu\rho\sigma}\text{Re}[F_{\mu\nu}F_{\rho\sigma}] = 4\partial_{\mu}\left[E^{\mu\nu\rho\sigma}\text{Re}\left[A_{\nu}\partial_{\rho}A_{\sigma} + \frac{2}{3}A_{\nu}A_{\rho}A_{\sigma}\right]\right].$$

(50)

This is equivalent to Nieh’s proof.\textsuperscript{15}

For physical implication of the extended Bach-Lanczos identities see Ref. 14). In short, the quadratic part of $L_2(F_{\mu\nu})$ contains only 5 independent parameters instead of 6 as claimed in I.

§ 6. Quaternionic formulation of Poincaré gauge theory. II

In the previous section we have digressed to apply the quaternionic-derivative method to a particular action-like integral (46). We shall again return to the quaternionic formulation of Poincaré gauge theory.

Given the total Lagrangian density $L$ (37) in accord with Poincaré-gauge invariance, there are derived identities\textsuperscript{7,8} which have been classified into (canonical) conservation equations, spin and energy Bianchi identities and one more type.\textsuperscript{7} Since the presence of $\phi_{\mu}$ affects the result in a non-trivial way, we shall rederive them following Hayashi and Bregman.\textsuperscript{7}

Let us consider transformations (36) in the infinitesimal form: Under the infinitesimal transformations,

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu,$$

$$U = 1 + \delta U, \quad \overline{\delta U} = - \delta U,$$

(51a, 51b)

Eq. (36) reads

$$\delta b_{\mu} = - \partial x^\nu b_{\mu\nu} + \delta U b_{\mu} + b_{\mu} \delta U^\dagger,$$

$$\delta A_{\mu} = - \partial x^\nu A_{\mu\nu} + \delta U A_{\mu} - A_{\mu} \delta U - \delta U_{\mu},$$

$$\delta \phi = \delta U \phi,$$

$$\delta \phi_{\mu} = - \delta x^\nu \phi_{\nu} + \delta U \phi_{\mu}.$$  

(52a, 52b, 52c, 52d)

Here the comma ,\textsuperscript{\mu} indicates the partial derivative with respect to $x^\mu$.

Poincaré-gauge invariance of $L$ leads to the identity

$$\delta^* L + (\delta x^i L)_{,i} \equiv 0,$$

(53)

where $\delta^* L$ is the variation of $L$ with respect to the substantial variation

$$\delta^* \phi \equiv \delta \phi - \delta x^i \phi_{,i}.$$  

(54)

That is, noting Eq. (16),

$$\delta^* L = \text{Re}\left[\frac{\partial L}{\partial \phi} \delta^* \phi + \frac{\partial L}{\partial \phi_{,\mu}} \delta^* (\phi_{,\mu})\right].$$

(55a)

For typographical reason we assume only a single Hermitian field $\phi$ over $Q$ till full
expression is necessitated to elucidate physical content. Since $\delta^*$ commutes with the derivative $\partial_\nu$, we employ definition (29) to rewrite Eq. (55a) as

$$\delta^* L = \text{Re} \left[ \frac{\delta L}{\delta \bar{\phi}} \delta^* \phi + \left( \frac{\delta L}{\delta \phi_\nu} \delta^* \phi \right)_{\nu} \right].$$  

(55b)

Substituting this into Eq. (53) yields

$$\text{Re} \left[ \frac{\delta L}{\delta \phi} \delta^* \phi \right] + J^\mu,\nu = 0,$$

(56a)

where

$$J^\mu = \text{Re} \left[ \frac{\partial L}{\partial \phi_\mu} \delta^* \phi \right] + \delta x^\mu L$$

$$\equiv \text{Re} \left[ \frac{\partial L}{\partial \phi_\mu} \delta \bar{\phi} \right] + \bar{T}^\nu_* \delta x^\nu$$

(56b)

with the canonical energy-momentum complex

$$\bar{T}^\nu_* = - \text{Re} \left[ \frac{\partial L}{\partial \phi_\mu} \bar{\phi} \right] + \delta^\nu,\nu L.$$

(57)

Now, in actuality, $\phi = \phi_\mu$ and $\bar{\phi}$ in our theory. Among them only $b_\mu$ is Hermitian. For other fields the Hermitian-conjugated terms should be added to within the real part $\text{Re} [\cdot]$. Moreover, $\psi_\nu$ and $\bar{\psi}$ are Grassmannian quaternions, for which the minus sign in the variational formula (16b) has to be taken care of. Remember also that $A_\mu$ is pure quaternion, Eq. (33b). Keeping these points in mind, we substitute Eqs. (52) and (54) into the LHS of Eq. (56a) and express the result as

$$- \delta x^\nu I_\nu + \text{Re} [i \delta U + \text{h.c.}] - D^\mu,\nu = 0,$$

(58a)

where, in full expression,

$$I_\nu = \text{Re} \left[ \frac{\delta L}{\delta b_\mu} b_\mu,\nu - \left( \frac{\delta L}{\delta A_\mu} A_\mu,\nu + \frac{\delta L}{\delta \bar{\psi}} \bar{\psi}_\nu + \frac{\delta L}{\delta \phi_\mu} \phi_\mu,\nu + \text{h.c.} \right) \right].$$

(58b)

$$I = - \delta L \delta b_\mu - \delta L A_\mu - \delta L \bar{\psi}_\nu + \delta L \phi_\mu - \left( \delta L A_\mu \right)_{\text{h.c.}}$$

(58c)

and

$$D^\mu = \delta x^\nu \text{Re} \left[ \frac{\delta L}{\delta b_\mu} \bar{b}_\nu - \left( \frac{\delta L}{\delta A_\mu} A_\nu + \frac{\delta L}{\delta \phi_\nu} \phi_\nu + \text{h.c.} \right) \right] - \text{Re} \left[ \delta L \delta U + \text{h.c.} \right] - J^\mu.$$

(58d)

According to the usual reasoning, Eq. (58) leads to the identities:

$$D^\mu,\nu = 0,$$

(59a)
The LHS of Eq. (59a) still contains arbitrary functions \( \delta x^\mu \), \( \delta U \) and their derivatives up to the second. Equating respective coefficients to nought, we find the (canonical) conservation equations, superpotentials\(^*\) which determine the conserved quantities, and, finally,

\[
\text{Re} \left[ \frac{\partial L}{\partial b_{(\mu,\lambda)}} \vec{b}_\nu - \left( \frac{\partial L}{\partial A_{(\mu,\lambda)}} A_\nu + \frac{\partial L}{\partial \bar{\phi}_{(\mu,\lambda)}} \bar{\phi}_\nu + \text{h.c.} \right) \right] = 0,
\]

Equations (60a, b) stem from the arbitrariness of \( \delta x^{\nu,\mu,\lambda} \) and \( \delta U_{\nu,\mu,\lambda} \), respectively. Inserting Eq. (60b) into Eq. (60a) yields

\[
\text{Re} \left[ \frac{\partial L}{\partial b_{(\mu,\lambda)}} \vec{b}_\nu - \left( \frac{\partial L}{\partial \bar{\phi}_{(\mu,\lambda)}} \bar{\phi}_\nu + \text{h.c.} \right) \right] = 0.
\]

Were it not for the Rarita-Schwinger field \( \phi_{\mu} \), we could conclude from Eqs. (60b, c) that the first derivatives of \( b_\mu \) and \( A_\mu \) should appear in \( L \) only through the antisymmetric combination.\(^1\) Due to the presence of \( \phi_{\mu} \), we cannot make the usual statement\(^7\)

\[
\frac{\partial L}{\partial b_{(\mu,\lambda)}} = 0,
\]

unless the following is derived from a further gauge symmetry:

\[
\frac{\partial L}{\partial \bar{\phi}_{(\mu,\lambda)}} = 0.
\]

We shall see in the next section that it is local supersymmetry that implies the validity of Eq. (60e).

The identity (59b) becomes

\[
\Delta_\mu \left( \frac{\partial L}{\partial A_\mu} \right) = \frac{1}{2} \left[ - \frac{\partial L}{\partial b_\mu} \bar{b}_\mu + \frac{\partial L}{\partial \bar{\phi}} \bar{\phi} + \frac{\partial L}{\partial \phi} \phi - \text{q.c.} \right],
\]

where q.c. designates the quaternion conjugation of the previous terms and

---

\(^*\) Since \( \delta U \) is pure quaternion, the second term on the LHS of Eq. (58a) reads

\[
\text{Re} \left[ \delta U + \text{h.c.} \right] = (1/2) \text{Re} \left[ (I - \bar{I}) \delta U + \text{h.c.} \right],
\]

which justifies the validity of identity (59b).

\(^*\) The superpotentials do not include internal-symmetry gauge fields, since the latter are treated as world scalars over \( Q \), transforming homogeneously under local Lorentz group.

\(^*\) The round bracket denotes symmetrization:

\[
\langle \phi_{(\mu,\lambda)} \rangle = (1/2) (\phi_{\mu,\lambda} + \phi_{\lambda,\mu}).
\]

\(^1\) By our construction of \( L \) in §4, this is automatically satisfied, i.e., Eq. (60) are trivially obeyed. The above discussion is not confined to a specific construction.
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\[ \Delta_\mu \left( \frac{\delta L}{\delta A_\mu} \right) = \partial_\mu \left( \frac{\delta L}{\delta A_\mu} \right) + A_\mu \left( \frac{\delta L}{\delta A_\mu} \right) - \left( \frac{\delta L}{\delta A_\mu} \right) A_\mu, \]  

(61b)

is the Lorentz-gauge covariant divergence for pure quaternion \( \delta L/\delta A_\mu \); it is coordinate-invariant, because \( \delta L/\delta A_\mu \) is a vector density. The on-shell identity

\[ \Delta_\mu \left( \frac{\delta L}{\delta A_\mu} \right) = 0, \quad \text{(on-shell)} \]  

(61c)

is a consistency equation for the equation of motion

\[ \frac{\delta L}{\delta A_\mu} = 0. \]  

(62a)

Putting \( L = L_G \) and \( L_M \) in Eq. (61a), we obtain the spin Bianchi identities\(^*)\) in the gravitational and matter sectors, respectively.

After a little calculation, the last identity (59c) is brought into the form:

\[ \text{Re} \left[ - \left( \frac{\delta L}{\delta b_\mu} \right)_\nu + \{ (D_\mu K^\nu) \bar{\phi}_\nu + \text{h.c.} \} \right] \]

\[ \equiv - \text{Re} \left[ \frac{\delta L}{\delta b_\mu} S_{\mu\nu} + \left\{ \frac{\delta L}{\delta A_\mu} F_{\mu\nu} - K(D_\nu \phi) + K^\nu \bar{\phi}_{\mu\nu} + \text{h.c.} \right\} \right], \]

(63a)

where the double bar indicates the Lorentz-gauge covariant derivative (44c),\(^*)\) \( S_{\mu\nu} \) is the torsion (44), \( F_{\mu\nu} \) the field strength (38a), and we have defined

\[ \frac{\delta L}{\delta \phi} \equiv K, \]  

(64a)

\[ \frac{\delta L}{\delta \phi_\mu} \equiv K^\mu \]  

(64b)

and

\[ \phi_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu. \]  

(65)

In contrast to the spin Bianchi identity (61), the on-shell identity

\[ \text{Re} \left[ - \left( \frac{\delta L}{\delta b_\mu} \right)_\nu + \{ (D_\mu K^\nu) \bar{\phi}_\nu + \text{h.c.} \} \right] = 0, \quad \text{(on-shell)} \]  

(63b)

is not sufficient to provide us with the consistency equations for the equations of motion:

\[ \frac{\delta L}{\delta b_\mu} = 0, \]  

(62b)

\[ K^\mu = 0; \]  

(62c)

namely,

\[ \left( \frac{\delta L}{\delta b_\mu} \right)_\nu = 0, \quad \text{(on-shell)} \]  

(63c)

\[ D_\mu K^\mu = 0. \quad \text{(on-shell)} \]  

(63d)

\(^*)\) The divergence (\( \delta L/\delta b_\mu \)) is coordinate-invariant as Eq. (61b).
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The situation is similar to what we have encountered in relation to identity (60). As Deser and Zumino\textsuperscript{11} first proved, the consistency equation (63d) originates from local supersymmetry.

In Poincaré gauge theory without local supersymmetry, one is forced to impose Eq. (63d) as a constraint, and, then, identity (63c) follows from Eq. (63b).

One may set $\mathbf{L} = \mathbf{L}_G$ and $\mathbf{L}_M$ in Eq. (63a), thereby arriving at the energy Bianchi identities\textsuperscript{7} in the gravitational and matter sectors, respectively.

§ 7. Simple $D=4$ supergravity

It is by now clear that simple $D=4$ supergravity\textsuperscript{11} is also tractable in our quaternionic method; notational simplification achieved deserves special emphasis.

As is well-known, local supersymmetry is defined only up to a divergence:

$$\delta_{\phi} \mathbf{L} = d^{\mu\nu} \phi,$$

(66)

Local supersymmetry transformation on $\phi_{\mu}$ is usually written as

$$\delta_{\phi} \phi_{\mu} = 2D_{\mu}\alpha,$$

(67a)

where, in our notation, $\alpha$ is an arbitrary spinor quaternion with $ab_1 = a$. It is assumed that $\delta_{\phi} b_\mu$ and $\delta_{\phi} A_\mu$ do not involve derivative of $\alpha$, and $\delta_{\phi} \phi = 0$, hence we put aside the Dirac field in this section; $\mathbf{L} = \mathbf{L}_\text{RS}(D_{\mu})$. Inserting Eq. (67) into Eq. (66), converting the action of $D_\mu$ on from $\alpha$ to $K^{\nu}$, and using the same reasoning as in the previous section, we derive the following identities, in full expression except the first:

$$J^{\nu,\mu} = 0,$$

(68a)

$$\text{Re}[2(D_{\nu} K^{\nu}) \bar{a} + \text{h.c.}] = \text{Re}\left[ -\frac{\delta L}{\delta b_{\mu}} \delta_{\phi} \delta_{\mu} + \left( \frac{\delta L}{\delta A_{\mu}} \delta_{\phi} A_{\mu} + \text{h.c.} \right) \right],$$

(68b)

$$d^{\mu} = 2\text{Re}[K^{\nu} \bar{a} + \text{h.c.}].$$

(68c)

Without specifying $\delta_{\phi} b_\mu$ and $\delta_{\phi} A_\mu$, which will be determined so as to validate Eq. (68b), there emerges one identity from Eq. (68a) by virtue of the arbitrariness of $\alpha_{\nu 1}$; it is Eq. (60e). Similarly, Eq. (68b) implies the on-shell identity (63d).

It goes without saying that local supersymmetry (66) is enjoyed by supergravity:

$$L_{\text{sg}} = L_G + L_{\text{RS}}(D_{\mu})|_{M=0}.$$

(69)

A mass term in the Rarita-Schwinger Lagrangian density (42) may be allowed without losing local supersymmetry if a cosmological term is added to $L_G$ with a special relation imposed.\textsuperscript{11} For simplicity of presentation, we shall consider the original massless version (69).

In what follows, we shall verify identity (68b) for $L = L_{\text{sg}}$. Our purpose is to determine $\delta_{\phi} b_\mu$ and $\delta_{\phi} A_\mu$ in the quaternion form, since only the 4-component version is quoted in the literature.\textsuperscript{11}

First of all, we compute the necessary variations from Eqs. (39b) and (42):

$$G^{\mu} = (-1)^{\frac{1}{2}} \delta L_{\text{RS}} \delta b_{\mu} = (i/2)E^\rho_{\nu\rho\sigma}(F_{\rho\sigma} b_\nu - b_\nu F^\perp_{\rho\sigma}),$$

(70a)
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\[ T^\mu = \frac{\delta L_{RS}(D_\mu)}{\delta A_\mu} |_{M=0} = (1/2)E^{\mu\nu\rho\sigma}\{\phi_\sigma(D_\nu\phi_\rho)' + (D_\nu\phi_\rho)\phi_\sigma'\}, \]  
(70b)

\[ H^\mu = \chi^2\frac{\delta L_E}{\delta A_\mu} = (i/4)E^{\mu\nu\rho\sigma}(\bar{S}_{\sigma\nu}\bar{b}_\rho - b_\rho S_{\sigma\nu}), \]  
(70c)

\[ S^\mu = -\frac{\delta L_{RS}(D_\mu)}{\delta A_\mu} = -(1/4)E^{\mu\nu\rho\sigma}(C_{\sigma\nu}\bar{b}_\rho - b_\rho \bar{C}_{\sigma\nu}), \]  
(70d)

\[ K^\mu = \frac{\delta L_{RS}(D_\mu)}{\delta \phi_\mu} |_{M=0} = -E^{\mu\nu\rho\sigma}\left\{b_\sigma(D_\nu\phi_\rho)'+\frac{1}{4}S_{\sigma\nu}\bar{\phi}_\rho', \right\}, \]  
(70e)

\[ D_{\mu}K^\mu = \frac{1}{2}G^\mu\bar{\phi}_\mu' - \frac{1}{8}E^{\mu\nu\rho\sigma}S_{\sigma\nu}\bar{\phi}_\rho'. \]  
(70f)

In Eq. (70d), we have put

\[ C_{\mu\nu} = \frac{1}{2}(\phi_\mu\phi_\nu' - \phi_\nu\phi_\mu'), \]  
(70g)

and, in obtaining Eq. (70f), use has been made of the cyclic identity (43). The identity (68b) then turns out to be

\[ \text{Re}[iG^\mu(\bar{\phi}_\mu' + \bar{a} - a\bar{\phi}_\mu) - \chi^{-1}G^\mu\delta_q\bar{b}_\mu] = \text{Re}[-T^\mu\delta_q\bar{b}_\mu + \{\chi^{-1}H^\mu\delta_qA_\mu - S^\mu\delta_qA_\mu + (1/4)E^{\mu\nu\rho\sigma}S_{\sigma\nu}\bar{\phi}_\rho'\bar{a} + \text{h.c.}\}]. \]  
(71)

The LHS is deleted by choosing

\[ \delta_qb_\mu = -ix(\alpha\phi_\mu' - \phi_\mu\alpha'). \]  
(67b)

The RHS of Eq. (71) should then vanish for Eqs. (70b, c, d). We next appeal to the Fierz identity in the quaternion form

\[ E^{\mu\nu\rho\sigma}\{\bar{\phi}_\rho(\alpha\phi_\nu' - \phi_\nu\alpha') - \bar{a}\bar{C}_{\mu\nu}\} = 0, \]  
(72)

where \( C_{\mu\nu} \) is defined by Eq. (70g). It helps us eliminate \( \delta_q\bar{b}_\mu \) term from the RHS of Eq. (71), and the resulting equation is satisfied if

\[ \delta_qA_\mu = (ix/2)\phi_\mu\alpha'\bar{b}'' + (ix/12)\phi_\rho\alpha'\bar{b}''\bar{b}_\rho - \text{Re}[\cdot], \]  
(67c)

where \( \text{Re}[\cdot] \) stands for the quaternionic real part of the previous terms.

The process of proof is exactly the same as in the original Ref. 11), and we have verified that Eq. (67) are identical with the familiar transformation laws there.

A related comment concerning the on-shell commutator algebra was made in a previous note.\(^8\) Not mentioned there is a direct proof of the on-shell\(^*) \) identity (63d) from Eq. (70), which hinges upon the Fierz identity (72) with \( \alpha = i\psi_{\sigma\nu}. \)

If a mass term is included by lifting the constraint \( M=0 \) in Eq. (69), the on-shell identity differs from Eq. (63d) by the mass term:

\(*\) Equations of motion are \( G^\mu = \chi T^\mu, \ H^\mu = \chi S^\mu \) and \( K^\mu = 0, \) the second being solved by

\[ S_{\mu\nu} = ixC_{\mu\nu}. \]  
(73)
\[ D_\mu K^\mu = \frac{3}{2} i M^2 b b^\mu \bar{\phi} \mu^\dagger, \quad \text{(on-shell)} \] (74)

because local supersymmetry is broken, unless an appropriate cosmological term is added to \( L_F \).\(^{17}\) We remark that the constraint \( D_\mu K^\mu = 0 \) leads to \( b^\mu \bar{\phi} \mu^\dagger = 0 \) but nothing else as Eq.\,(32) does, only if the gravitational Lagrangian density is given by \( L_F \). For general \( L_G \), the constraint puts additional condition in the gravitational sector. We will return to this problem in the future.

§ 8. Discussion

Quaternionic formulation of Poincaré gauge theory developed in I and the previous sections is predicated upon an observation that Dirac spinor is regarded as an object over quaternions. This is made more precise if one starts with \( SL(2, H) \) spinor of Kugo and Townsend.\(^{18}\) The latter is defined as a two-component object over the division algebra \( H \) of quaternions. (Mathematicians proved that \( H \) is the last number system beyond the real and complex fields.) Kugo and Townsend introduced \( SL(2, H) \) spinor in relation to \( D=6 \) supersymmetry, since \( SL(2, H) = SO(5,1) \) to be compared with \( SL(2, C) \approx SO(3,1) \). As pointed out in a previous note,\(^{19}\) reduction from \( D=6 \) to \( D=4 \) naturally yields two-component Dirac spinor over \( H \). The spinor-quaternion (17) is a single-component field over \( Q \), obtained via diagonalization of \( SL(2, H) \) matrices, the set of which is isomorphic to \( SO(3,1) \). To summarize, the Dirac spinor is represented by either

4-component quantity over \( C \)

or

2-component quantity over \( H \),

or

single-component quantity over \( Q \).

It is obvious that only the last representation dispenses with the mixed use of matrix-vector notation. Quaternionic variational formalism takes advantage of this feature as we have seen in the previous sections.

Our use of quaternions is only formal in the sense that we have nothing to do with quaternionic quantum mechanics. Quantization is assumed to be carried out as usual. It is interesting, however, to realize\(^{13}\) that left-right Weinberg-Salam theory, QCD, Poincaré gauge theory and supergravity (without auxiliary fields) are all formulated over quaternions. Thus even a formal use of quaternions not only is convenient for the variational principle, but also would suggest a possible common origin of quantum numbers, flavor, color and spin.

Acknowledgements

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16) See, for instance, W. L. Bade and H. Jehle, Rev. Mod. Phys. 35 (1953), 714.