§1. Quantum-theoretical equation of motion

In Part II, we will first translate the results of Part I into the quantum-mechanical language, and then discuss the validity of our formalism, in connexion with both the classical and the quantum-mechanical limitations of theory. The passage from the classical to the quantum theory necessitates two steps: (i) establishment of commutation relations among the field quantities, and (ii) formulation of a quantum-mechanical equation of motion. In this paragraph we shall introduce the quantum-mechanical equation of motion by means of the theory of canonical transformations.

In accordance with Dirac's notation of representation, we shall express the state of a field by

$$[u'(\xi)],$$  \hspace{1cm} (1.1)

which is the representation making the canonical variables $u$ diagonal. There are as many $u'$s as the possible sets of values of the indices $a, \xi^1, \xi^2, \xi^3$. In order that such a representation may be possible, it is necessary that the $u'$s, with different $a$ and $\xi$, should be commutable and complete. Leaving this condition to a later discussion, let us now proceed to establish the equation of motion.

If we consider another representation diagonalizing a new set of canonical variables the two representation will be connected by a transformation functional:

$$[u'(\xi)|v'(\xi)].$$ \hspace{1cm} (1.2)

Now we are tempted to postulate the following principle for the passage from classical to quantum mechanics:

* Square brackets are used for the quantity which is a functional of $u'(\xi)$. 
"If a canonical transformation \( u, \pi \rightarrow v, \lambda \) is engendered, according to (4·2) Part I, by the generating functional:

\[
W[u, v; s]
\]

in the classical theory, the corresponding quantum-mechanical transformation functional will be given by

\[
[u' | v''] = \exp iW[u', v''; s],
\]

\( \hbar \) being put equal to unity."

This postulate, however, cannot be maintained as a general rule without abandoning the usually accepted conception about the quantum-mechanical transformation functions. For the classical and quantum-mechanical composition rules of transformations do not allow of the connection of the nature (1·4). If we are to generalize the quantum-mechanical composition rule to our case of continuum, we shall have to write

\[
[u' | v'''] = \int [u' | v''] dv'' [v'' | w'''],
\]

in which the integration, in reality, is an \( n \)-ple one, where \( n \) is continuously infinite. On the other hand, the composition rule of the classical generating functionals reads [see (5·1) Part I]

\[
W[u, w] = W[u, v] + W[v, w]
\]

That these two composition rules are incompatible with each other can be best seen by a simple illustration in point mechanics. If \( W(q_1, q_2) \) is the generating function for the transformation \( q_1 p_1 \rightarrow q_2 p_2 \), its inverse is given by

\[
W^{-1}(q_2, q_1) = -W(q_1, q_2)
\]

The composition of the quantum-mechanical transformation functions corresponding to \( W \) and \( W^{-1} \), when they are defined in accordance with (1·4), will give

\[
\int \exp i\{W(q_2, q_2) - W(q_3, q_3)\} dq_3,
\]

which is, in general, not equivalent to the identical transformation. An exception is the case where

\[
W = q_3 q_3
\]
i.e., the case where the transformation just means the exchange of the rôles of the coordinate and its conjugate momentum (with change of sign in one of them):

\[(q_i, q_j) = (q_j, q_i) = \exp (i\theta, \phi) \quad (1\cdot10)\]

It is clear that, in general, the group of quantum-mechanical transformation functions or functionals cannot be brought into isomorphism with the group of classical generating functions or functionals.

Now Dirac has shown,\(^{1,2}\) in point mechanics, that the two contradicting composition rules can harmonize, if (i) \(\hbar\) is infinitely small, and if (ii) there is an extremum principle about \(W\) such as the action principle about Hamilton’s principal function.

Our aim, however, lies primarily not in establishing an unobjectionable, self-consistent formalism, but rather in exploring the border-region of the legitimate theory surrounded by unknown possibilities, so that we wish to carry on provisionally with the tentative postulate (1\cdot4).

To the change with the observation parameter of a state in the classical theory corresponds, in the quantum theory, the change of the representation of a fixed state-vector, represented by the physical quantities \(u\{s\}\), which, the latter, change according to a certain dynamical law, with the observation parameter. Obviously we are here speaking in the language of the "Heisenberg picture."

If we regard the change with \(s\) of \(u\{s\}\) in the classical mechanics as a canonical transformation, the generating functional for it will be given by the Lagrangian integral (4\cdot8) Part I. The quantum-mechanical counterpart of this transformation will, according to our provisional postulate (1\cdot4), be given by

\[ [u'\{s\}, u''\{s_0\}] = \exp iW[u'\{s\}, u''\{s_0\}; s], \quad (1\cdot11) \]

where \(W\) should now be taken from (4\cdot8) Part I. This transformation is characterized by a unitary operator \(S\) which is defined by

\[ [u'\{s\}, S|u''\{s_0\}] = \exp iW[u'\{s\}, u''\{s_0\}; s], \quad (1\cdot11') \]

where \(u'\{s_0\}\) and \(u'\{s\}\) take the same numerical values.\(^{1,3}\)

This equation (1\cdot11) can already be regarded as a kind of equation of motion. For the left hand side bears, in a generalized sense, the mean-
ing of a probability amplitude, and the equation determines its dependence on the observation parameter. This way of formulating the change of state with \( s \) we want to call the Lagrange scheme, in distinction to the Hamilton scheme which uses the Hamiltonian functional.

In the classical field mechanics, \( u' (s) , \ u'' (s_0) \) have certain definite values for a definite field state. But in the quantum field mechanics, they must, even when the system is in a definite state, be regarded as capable of taking all the possible values. \( W \) is to be computed for all the possible boundary conditions expressed by \( u' (s) , \ u'' (s_0) \) and to be substituted in the right hand side of (1.11). (see §3 Part II—b.)

Corresponding to the unified expression (4.16) Part I, we can also introduce the quantum mechanical general transformation functional in a unified form. We divide the boundary of a four-dimensional volume in two parts, one of which we identify with a part of the surface \( s=s_0 \) and the other a part of the surface \( s=s \). If we now decrease the part occupied by the surface \( s=s_0 \) gradually, the transformation functional \( [u' (s) \vert u'' (s_0)] \) will become finally only dependent on \( u' (s) : \)

\[
    [u' (s)] . \tag{1.12}
\]

This corresponds to the classical generating functional (4.16) Part I, so we may put

\[
    [u' (s)] = \exp iW[u' (s);s]. \tag{1.13}
\]

Equation (1.13) belongs to the Lagrange scheme.

Now the transformation functional (1.11) depends on the observation parameter formally both through the explicit \( s \) and through \( u' (s) \). When we are interested, however, in the change of \( [u' (s) \vert u'' (s_0)] \) in quantum-mechanics, we fix the values of \( u' (s) \) and \( u'' (s_0) \) and investigate the dynamical change of this transformation function with \( s \). This change is, of course, caused by the explicit \( s \) in the right-hand side of (1.11). On differentiating with regard to \( s \), we get

\[
    \frac{d}{ds} [u' (s) \vert u'' (s_0)] = i \frac{\partial W[u' (s) \vert u'' (s_0)]}{\partial s} \cdot [u' (s) \vert u'' (s_0)]. \tag{1.14}
\]

Up to this equation, \( W \) is regarded as a \( c \)-number, but we want now to pass to the \( q \)-number theory. The \( u \)'s in \( W \) or \( \partial W/\partial s \) are, in general, to be regarded as non-commutable quantities, although we have not estab-
lished any commutation relation in their regard yet. We expand the functional $\frac{\partial W[u(s), u(s_0)]}{\partial s}$ in a power series of $u(s)$ and $u(s_0)$ according to our generalized Taylor theorem (2.10) Part I, and arrange the factors in each term of the series so that the power of $u(s)$ may stand to the left of that of $u(s_0)$. The operators in the following will be supposed to be written in this way. In the case of the unified formalism (1.13), difficulties of this kind disappear.

With the help of the so defined operator $\frac{\partial W}{\partial s}$, we can write (1.14)

$$\frac{d}{ds} [u'[s] | u''[s_0]] = i [u'[s] | \frac{\partial W}{\partial s} | u''[s_0]], \quad (1.15)$$

or, in virtue of (4.15) or (4.13) Part I,

$$\frac{d}{ds} [u'[s] | u''[s_0]] = -i [u'[s] | H | u''[s_0]], \quad (1.16)$$

or again

$$[u'[s + ds] | u''[s]] = [u'[s] | 1 - iHds | u''[s]], \quad (1.17)$$

where $u'[s + ds]$ and $u'[s]$ should take the same numerical values. The last equation shows that $1 - iHds$ is the operator of the unitary transformation connecting the representation by $u[s]$ and the representation by $u[s + ds]$. The repetition of this unitary transformation gives

$$[u'[s] | u''[s]] = [u'_0[s_0], | S | u''[s_0]], \quad (1.18)$$

with

$$S = \prod_{s_0} (1 - iHds), \quad (1.19)$$

$u'[s]$ and $u'[s_0]$ taking the same numerical values. The factors in $S$ should be arranged from right to left in the order of $s_0 \to s$. This multiplication symbol is again an awkward one, for it implies, in reality, a continuously infinite repetition of multiplications. The result (1.18) must not be expected to be equivalent to our original assumption (1.11). For (1.18) is obtained by assuming (1.11) for infinitesimal transformations and then by building up the total operator according to the quantum mechanical composition rule.

If $H$ is independent of $s$, one could write (1.19)

$$S = \exp (-iH(s - s_0)), \quad (1.20)$$
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if one were aware of its true implication. (1·19) can also be written, under certain precaution which we shall discuss later,

\[ S = \Pi(1 - iH(d\xi)) \).

(1·21)

We now multiply, from the right into (1·16), the representation \([u'' s] \) of a certain fixed state-vector (fixed in the sense of the Heisenberg picture), and write the result in the \( u' s \)-representation; so we get

\[ \frac{d}{ds} [u' s] = -i[u' s] [\Pi u'' s] [u'' s] ] .

(1·22)

This last equation is liable to interpretation also in the Schrödinger picture. We can now introduce the "moving" state-vector \( \mathcal{F} \), whose development with the observation parameter will be governed by the Schrödinger equation:

\[ i\frac{d\mathcal{F}}{ds} = H\mathcal{F}.

(1·23)

We shall say that the equations from (1·16) to (1·23) belong to the Hamilton scheme, in opposition to the Lagrange scheme of (1·11).

The representative \([u'] \) of the state-vector \( \mathcal{F} \) is a functional of \( u^a(\xi_1^2, \xi_3) \) \((a = 1, 2, \ldots)\) and contains \( s \) explicitly. The Hamiltonian \( H \) is an operatorial functional of \( u \) and \( \pi \), where \( u \) and \( \pi \) are, since we are now discussing in the Schrödinger picture, fixed operators.

The solution of (1·23) may be formally written

\[ \mathcal{F}(s) = S\mathcal{F}(s_0) .

(1·24)

If we want to go back to the Heisenberg picture, we must stop the motion of \( \mathcal{F} \) by multiplying

\[ S^{-1} = \Pi(1 + iHds)

(1·25)

into \( \mathcal{F} \). Note the change of sign before the Hamiltonian and the change of order of the factors under the multiplication symbol. At the same time, the fixed physical quantities, say \( Q_0 \), of the Schrödinger picture will start to change with \( s \) according to

\[ Q = S^{-1}Q_0S .

(1·26)
In §5 (iv) part I, where the Hamiltonian was composed of two parts:

\[ H = H_0 + H', \]  

(1·27)

we divided the change of physical quantities into two classes, which were governed by \( H_0 \) and \( H' \) respectively. We take as \( H_0 \) the Hamiltonian of independent fields and as \( H' \) that of perturbation or interaction. We can obtain a useful formalism in quantum mechanics, when we solve the motion of the first class by the Heisenberg picture and the second by the Schrödinger picture.

Let \( Q_0 \) and \( \varphi_0 \) first be a physical quantity and the state-vector in the pure Schrödinger picture—pure in the sense that the total Hamiltonian is supposed to be used in the Schrödinger equation. We then pass to a mixed picture, in which \( H_0 \) is treated by the Heisenberg method and \( H' \) by the Schrödinger method. In this mixed picture, \( Q_s \) and \( \varphi_s \) will be transformed into

\[ Q = S_0^{-1} Q_0 S_0 \]
\[ \varphi = S_0^{-1} \varphi_0 \]  

(1·28)

with

\[ S_0 = \frac{\mathcal{H}}{S_0} \]
\[ 1 - i H_0 ds \]  

(1·29)

This transformed state-vector will change with \( s \) according to the Schrödinger equation:

\[ i \frac{d \varphi}{ds} = H \varphi \]  

(1·30)

in correspondence with (5·17) Part I. The transformed physical quantities, such as \( u \) and \( \pi \), change with \( s \) according to (1·28), i.e. they satisfy the "vacuum" wave equation. \( H' \) is written in terms of such \( Q \)'s, and so depends necessarily on \( s \). The solution of (1·30) can be written in the form:

\[ \varphi(s) = S' \varphi(s_0) \]  

(1·31)

with

\[ S' = \frac{\mathcal{H}}{S_0} \]
\[ 1 - i H ds \]  

(1·32)
One of the advantages of this formalism lies in the fact that the $Q$'s satisfy the vacuum equation, so that it is easy to establish the commutation rule for them. Another advantage is that the operator $S'$ can be written in a form whose relativistic covariance is easy to see:

$$S' = iH(1 - iH'(d\xi)^4)$$  \hspace{1cm} (1.33)

$$= iH(1 - i\tilde{\xi}'(dx)^4),$$  \hspace{1cm} (1.34)

for $\tilde{\xi}'$, in ordinary cases, is a relativistic invariant, the $x$'s being rectangular coordinates. The passage from (1.32) to (1.33), which is of the same kind as that from (1.19) to (1.21), necessitates a specific discussion; see § 4.

When $s$ coincides with the time axis of a Minkowski frame, the operator (1.34) will become identical with that which Heisenberg introduced when he discussed the limitation of the current relativistic field theory.\(^{(4)}\) And when the $s$-surface is a curved but everywhere spatial one, our formalism (in Hamilton's scheme) has practically the same reach as the "super-manytime theory" of Tomonaga.\(^{(5)}\) We do not, from the beginning, limit ourselves to the case of a temporal $s$ (i.e. a spatial $s$-surface), and want to clarify in what respect the non-temporal $s$ interferes with the limitation of the current field theory.

\section{2. Quantization of field quantities

1) \textit{Hamilton Scheme}

In this paragraph we will examine to what extent the ordinary method of quantization can be adapted, without interfering with the established principles of quantum mechanics, to the method of observation parameter in its Hamilton scheme of formulation. In the cases where there are interactions among field variables, we have to use the mixed standpoint which was introduced in connexion with the perturbation theory. The field quantities then satisfy the vacuum wave equations again.

On differentiating (1.26) or the first equation of (1.28) with regard to $s$, we obtain

$$\frac{dQ}{ds} = i(HQ - QH).$$ \hspace{1cm} (2.1)
This equation, applied to the $u$'s and $\pi$'s can be brought into accord with the Hamiltonian wave equation (3·19) Part I, if we assume
\begin{equation}
\begin{aligned}
&\left\{ \begin{array}{l}
\pi_\alpha(\xi)u^\beta(\xi') - u^\beta(\xi')u_\alpha(\xi) = 0 \\
\pi_\alpha(\xi)\pi_\beta(\xi') - \pi_\beta(\xi')\pi_\alpha(\xi) = 0 \\
i[\pi_\alpha(\xi)u^\beta(\xi') - u^\beta(\xi')\pi_\alpha(\xi)] = \delta_{\alpha\beta}\delta(\xi - \xi')
\end{array} \right. \\
(2·2)
\end{aligned}
\end{equation}
referring to a fixed value of $s$, for we can prove, in virtue of the expansion theorem (2·10) Part I,
\begin{equation}
\begin{aligned}
&\frac{du^\alpha(\xi, s)}{ds} = i\left\{ \frac{\partial H}{\partial u^\beta(\xi', s)} [u^\beta(\xi' - u^\alpha(\xi, s)u^\alpha(\xi', s)](d\xi')^3 \\
&+ i\left\{ \frac{\partial H}{\partial \pi_\beta(\xi', s)} [\pi_\beta(\xi') - u^\alpha(\xi, s)\pi_\beta(\xi', s)](d\xi')^3 \\
&- \frac{\partial H}{\partial \pi_\alpha(\xi', s)}
\right. \right. \\
(2·3)
\end{aligned}
\end{equation}
and a similar equation for $d\pi_\alpha/ds$. With a certain restriction on the form of $H$, we can also assume the Fermi-type of commutation rule instead of (2·2), but we will, for sake of simplicity, limit ourselves to the Bose-type.\(^{(6)}\)

It is admissible, in view of the deduction so far developed, to assume these commutation (2·2) rules in the general case of Bose-type, but they lose relativistic invariance when the observation parameter $s$ is not time-like, i.e. when the $s$-surface is not space-like.

The current method of relativistic quantization starts from the so-called invariant delta-function:
\begin{equation}
D(x^1, x^2, x^3, x^4) = \frac{1}{16\pi^3} \left\{ \exp(i\sum k^i x^i + ik^ix^i) - \exp(i\sum k^i x^i - ik^ix^i) \right\} (dk)^3,
\end{equation}
where the $x$'s are Lorentz coordinates and $\vec{k}$ is the wave-number vector whose time-component is, in its absolute value, given by
\begin{equation}
k^0 = \sqrt{\sum (k^i)^2 + m^2}
\end{equation}
\begin{equation}
(2·5)
\end{equation}
This $D$-function is, of course, applicable only to the fields which satisfy the second order wave equations:
\begin{equation}
\left( \triangle - \frac{\partial^2}{\partial x^m} - m^2 \right) u = 0.
\end{equation}
\begin{equation}
(2·6)
\end{equation}
It has the properties:

\[
\left( \frac{\partial D}{\partial x^0} \right)_{x^0=0} = \partial(x') \partial(x^3) \delta(x^4)
\]
\[
\left( \frac{\partial D}{\partial x^i} \right)_{x^i=0} = 0, \quad i=1, 2, 3 \tag{2.7}
\]

Therefore, if we are to assume, for \( \partial L/\partial u_{x^0} \),

\[
i \left[ \frac{\partial L(x)}{\partial u_{x^0}} u^0(x') - u^0(x') \frac{\partial L(x)}{\partial u_{x^0}} \right] - \delta_{\alpha \beta} \partial(x^\alpha - x') \partial(x^3 - x^0),
\]

then on the ground of relativistic consistency, we must put, for \( \partial L/\partial u_{x^i} \),

\[
\frac{\partial L(x)}{\partial u_{x^i}} u^i(x') - u^i(x') \frac{\partial L(x)}{\partial u_{x^i}} = 0, \quad i=1, 2, 3. \tag{2.9}
\]

In the latter relation, physical quantities refer to a fixed value of \( x^i \), and the symbol \( x \) (as well as \( x' \)) without index stands for the \( x \)'s except \( x^i \).

The momentum in the method of observation parameter is given by

\[
\pi_\alpha = \frac{\partial L}{\partial u_{x^\alpha}} = \frac{\partial L}{\partial u_{x^\alpha}} \frac{\partial x^\alpha}{\partial x^\alpha} \frac{\partial (x)}{\partial (\xi)} \tag{2.10}
\]

If \( s \) is time-like (space-like), it will always be possible to choose a Lorentz frame so that the \( x^i \)-plane (\( x' \)-plane) osculates the \( s=\text{const} \) surface at the considered point. In the case of a time-like \( s \), we can therefore write, using orthogonal \( \xi \)-coordinates, and supposing \( \delta_{\xi \xi} = 1 \),

\[
\pi_\alpha = \frac{\partial L}{\partial u_{x^\alpha}} \frac{\partial (x^i x^j x^k)}{\partial (\xi, \xi, \xi)}, \tag{2.12}
\]

and in the case of a space-like \( s \),

\[
\pi_\alpha = \frac{\partial L}{\partial u_{x^\alpha}} \frac{\partial (x^i x^j x^k)}{\partial (\xi, \xi, \xi)} \tag{2.13}
\]

In the former case, we may put

\[
i [\pi_\alpha(\xi) u^\alpha(\xi') - u^\alpha(\xi') \pi_\alpha(\xi)] = \delta_{\alpha \beta} \partial(\xi^\alpha - \xi') \partial(\xi^3 - \xi^0) \partial(\xi^3 ... \xi^0), \tag{2.14}
\]

but in the latter, we have to put
The last relation is in direct contradiction to the relation (2·2).

The conclusion we can draw from this situation is that the current method of relativistic quantization is adaptable to the Hamilton scheme of our formalism only when \( s \) is time-like.

If we want, on the contrary, to maintain the standpoint expressed by (2·2) even for a space-like \( s \), we have to expect results which are specifically dependent on the choice of the observation parameter, at the expense of relativistic objectivity.

We can satisfy the commutation relation (2·2), which refers to the same value of \( s \), by the operator:

\[
\pi_a(\xi)u^\alpha(\xi') - u^\alpha(\xi')\pi_a(\xi) = 0.
\]  

(2·15)

in the representation which makes \( u \) diagonal and write the Schrödinger equation (1·22) formally

\[
\frac{d}{ds}[u'(\xi)] = \frac{d}{ds}[u'(\xi)].
\]  

(2·17)

which has the form of a "wave" equation in the functional space.

2) Conservation Law

The possibility of the current method of quantization depends on the existence of a conservation law of energy-momentum tensor. The generalized energy-momentum tensor in our case may be defined by

\[
T_{\mu}^{\nu} = u_{\xi\mu} \frac{\partial L}{\partial u^{\xi\nu}} - \partial_{\mu} L,
\]  

(2·18)

which, in virtue of the wave equation, obeys

\[
\frac{\partial T_{\mu}^{\nu}}{\partial \xi^\nu} = \frac{\partial L}{\partial \xi^\mu}.
\]  

(2·19)

It will be well that the differentiation on the left-hand side includes the differentiation through the field quantities, whereas, on the right-hand side, the differentiation is purely partial. (2·19) shows that, in general, it is impossible to assume a corpuscular structure other than energy-momentum quanta.
In the case where the conservation law holds in the rectangular coordinates, we have

\[
\frac{\partial \tilde{T}^{\mu \nu}}{\partial x^\nu} = \frac{\partial (\xi)}{\partial (x)} \frac{\partial}{\partial \xi^\nu} \left( \frac{\partial \xi^\mu}{\partial x^\nu} T^\nu_{\cdot \nu} \right) = 0, \tag{2.20}
\]

where \( \tilde{T}^{\mu \nu} \) is the energy-momentum tensor in the ordinary sense:

\[
\tilde{T}^{\mu \nu} = \frac{\partial (\xi)}{\partial (x)} \frac{\partial x^\nu}{\partial \xi^\mu} T^\nu_{\cdot \nu}.
\]

The conservation law introduced in Part I, which is valid when the Jacobian can be adjusted so that the \( \xi \)'s in equation (1.11) are all \( c \)-numbers:

\[
T^{\mu \nu} = \frac{\partial \xi^\mu}{\partial x^\nu} T^\nu_{\cdot \nu} = 0. \tag{2.23}
\]

3) Lagrange Scheme

In the Lagrange scheme, it would seem, at first glance, as though these waves need not be concerned over the commutation relations of the field quantities, because the \( \xi \)'s in equation (1.11) are all \( c \)-numbers. However, the whole problem here is concentrated on the possibility of the representation by the eigen-values of the \( \xi \)'s:

\[
[u', s] [ ] \tag{2.24}
\]

Within the limitation of current theory, the \( \xi \)'s with different values of \( a, \xi^i, \xi^j \) and \( \xi^s \) on the \( s \)-surface may be considered as commutable, only if the surface is spacelike, i.e., if \( s \) is time-like. The situation is therefore the same as in the Hamilton scheme.

However, if we should expect the future theory to retain certain of the current formula with a new interpretation, we could give more chance of survival—or at least more chance of being modified and adapted to the new way of attack—to the Lagrange scheme than to the Hamilton scheme.
scheme, for, in the former, the actual limitation of applicability of the theory is less intricately woven into the theoretical structure. See § 5 Part II—b.

References

(1) Read on Nov. 19, 1944 at the Symposium on the theory of elementary particles at the Second Faculty of Technology, Tokyo University, Chiba.
(3) See, e.g., p. 112, Dirac: The Principles of Quantum Mechanics, 2nd ed.
(6) W. Heisenberg & W. Pauli: ZS. f. Phys. 56 (1923), 1, especially p. 29.