A Model for Dipole Electromagnetic Form Factors

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A model of the electromagnetic form factor of the scalar \( \pi \) meson is discussed in which the \( \pi \) meson is a bound state of scalar nucleon-antinucleon system. Starting from an exact integral representation of the form factor, which is well defined in a space-like region of the momentum transfer squared, \( q^2 \), an analytic continuation of the representation is performed under some plausible assumptions on dynamics and a simple pole definition of a resonance in the \( \pi-N \) scattering amplitude. It is shown that the singularities of the form factor corresponding to resonances turn out to be dipoles and not simple poles contrary to the widely held belief. A generalization to the other hadron cases is discussed in the quark model scheme.

§ 1. Introduction

It is well known that the electromagnetic form factors of the nucleon have a simple form of a dipole, at least in the space-like region of \( q^2 \). More strikingly, the form factor of the \( \pi \) meson\(^1\) seems to have a similar simple structure.

Several years ago, Ball and Zachariasen\(^2\) and, independently, Amati et al.\(^3\) showed that the electromagnetic form factor of a bound state behaves like \( 1/q^4 \) for large \( q^2 \). The present author\(^4\) showed that the same conclusion can be obtained, even if one goes beyond the ladder approximation for Bethe-Salpeter equation, taking account of all possible graphs for the kernel.

However, there arise at least three important questions. First, the experimental data of the form factors of the nucleon as well as the \( \pi \) meson show that the dipole formula provides an excellent description for rather small \( q^2 \) and is consistent for large \( q^2 \). Secondly, the existence of the \( \rho \) meson suggests \( 1/q^2 \) behavior for large \( q^2 \), if the \( \rho \) meson contribution is described by a simple pole, since a naive consideration gives

\[
\frac{\langle 0 | J_\mu(0) | \rho \rangle \times \langle \rho | J_\nu(0) | \pi \rangle}{q^2 + m_\rho^2}
\]

(1)

for the \( \rho \) meson contribution to the \( \pi \) meson form factor, where \( J_\mu \) is the source of the \( \pi \) meson field and \( J_\nu \) is the electromagnetic current. Finally the discussions of Refs. 2) and 3) must be restricted to spinless cases and some difficulties concerning spins are pointed out in Ref. 4).

Recently, the problem of the form factor has been discussed\(^5\) in the light of the Veneziano formula for some scattering amplitudes. It has been shown that
the resulting formula describes the form factors rather well, although the formula
is not of the dipole form. This suggests that one or more of the characteristic
features of the Veneziano formula, like duality and Regge behavior, may give an
answer to the first question, in addition to the structure of the π meson or nucleon.
(Here we notice that, if they are elementary particles, the form factor should
behave at least as a constant for large $q^2$, due to the very existence of the lowest
order Feynman diagram.)

As to the second question, the formula (1) would be true only if the $\rho$ meson
were a stable particle. Being unstable, there is not such an intermediate state $|\rho\rangle$ in the usual "in" or "out" complete set of states. How can we obtain the
$\rho$-meson contribution to the form factor? This question is closely connected with
the definition of a resonance. The simplest way to define a resonance will be a
simple pole on the second sheet of certain amplitudes whose $s$-channels have a
definite set of quantum numbers. In this article we adopt this definition. Now
the question is, what singularity will appear at $\text{Re}(-q^2)=m_\rho^2$ in the form factor,
if we assume that there is a simple pole at $\text{Re}(s)=m_\rho^2$ in the amplitudes for the
related processes like $\pi+\pi\rightarrow\pi+\pi$, $N+N$, etc.

By one approach using elastic unitarity, the dispersion relation for the form
factor of the $\pi$-meson will be

$$ F(t) = \frac{1}{\pi} \int_{4m_\rho^2}^{\infty} \frac{F(t')\rho(t')T^*(t')}{t'-t} dt', $$

where $T$ is the $l=1$ partial wave $\pi$ meson-$\pi$ meson scattering amplitude and $F$
is the form factor. Suppose that we know completely the phase shifts of the $\pi$
meson-$\pi$ meson scattering, it may be rather simple to find the singularity of $F$ at
t=$m_\rho^2$, as we can use Omnes solution. But unfortunately, we know very little
about the phase shifts, and, even if we approximate $T$ in some appropriate way,
we must still solve the above integral equation in which the kernel must be very
complicated and we may obtain very unreliable solutions under some necessary
approximations.

On the other hand, if we had an integral representation of the form factor,
which is valid in a certain region of $q^2$, our problem then becomes simply to
make analytic continuation of the integral representation, in order to find the
singularity at $q^2=-m_\rho^2$. Obviously in such a procedure, the definition of the
resonance we have adopted will play an important role. However, we should
stress that it is not clear whether the singularity in the form factor is a simple
pole or something else, before the analytic continuation is made, even though we
have a simple pole in the case of the scattering amplitudes.

In the next section we consider a field theoretical model in which scalar $\pi$
meson is a bound state of the scalar nucleon-antinucleon system. Then we write
down an integral representation for the form factor of the $\pi$ meson. The represen-
tation is exact and is valid in a certain space-like region of $q^2$. The integral
is then evaluated and analytically continued with respect to $q^2$ in the region where the original integral is singular. The assumptions we made at this step are

1. dispersion relation for the off-mass-shell $\pi$-$N$ scattering amplitude which is included in the integrand of the representation

and

2. the resonance dominance in the low energy region and the Regge behavior in the high energy region for the imaginary part of the $\pi$-$N$ scattering amplitude.

It will be shown that the singularities of the form factor $F(q^2)$ are only dipoles at $\alpha(q^2) = 1, 3, \ldots$ in the region $-q^2 < 4m^2$, where $\alpha$ is the $\rho$-Regge trajectory which is assumed to be linear and $m$ is the mass of the nucleon.

If we use the result of Refs. 2), 3) and 4) that $F(q^2)$ behaves as $1/q^4$ for large $q^2$, the result of § 2 shows that $F$ can be written as

$$F(q^2) = \sum_j \frac{R_j}{[\alpha(q^2) - 2j - 1]^2} + f(q^2),$$

where $f(q^2)$ is analytic in the region $-q^2 < 4m^2$.

In § 3, some discussions concerning our result will be given. The mathematical details of the analytic continuation will be given in the Appendix.

§ 2. Model for the form factor of a scalar $\pi$ meson

Consider a model in which scalar $\pi$ meson is a bound state of scalar nucleon-antinucleon system. In this model, only the nucleon may be a fundamental field and the electromagnetic interaction will be given by

$$i\phi^*(x) \overline{\psi}(x) A_{\mu}(x),$$

where $\phi$ and $A_{\mu}$ are the nucleon and the photon field, respectively. We consider all possible Feynman graphs which can contribute to the electromagnetic form factor of the bound state (scalar $\pi$-meson). The external photon must be attached to an internal nucleon line in any graph. Therefore, we can summarize all contributions to the form factor by a formula

$$i(p_\pi + p_\gamma) F(q^2) = \frac{-i}{(2\pi)^3} \int d^4k i(2k - q)_\rho T(\pi(\rho) + N(k)) \Delta_\rho(k) \Delta_\rho(k - q),$$

where $\Delta_\rho$ is the complete propagator of the scalar nucleon and $T$ is the off-mass-shell nucleon-$\pi$ meson scattering amplitude.

Equation (3) is exact and must be meaningful in a region of large $q^2$ where the integral is convergent. We notice that the $\pi$-$N$ scattering amplitude in the integrand of the right-hand side of Eq. (3) is the fixed $t(-q^2)$ amplitude and we can expect that the integral is convergent at least for large $q^2$. 

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Now we evaluate the integral in order to make the analytic continuation of the representation to the time-like region. First we approximate $Lh'$ by $JF$, the free propagator, and assume the fixed $t(=-q^2)$ dispersion relation for the $\pi-N$ scattering amplitude,

$$
T = \frac{g^2}{m^2 - s} - \frac{g^2}{m^2 - u} + \frac{1}{\pi} \int_{(m^2)^2}^{\infty} ds' \frac{\text{Im} T^s(s', -q^2)}{s' - s} \\
+ \frac{1}{\pi} \int_{(m^2)^2}^{\infty} du' \frac{\text{Im} T^u(u', -q^2)}{u' - u},
$$

(4)

where

$$
s = -(p + k)^2 = \mu^2 - 2p \cdot k - k^2
$$

and

$$
u = -(p' - k)^2 = \mu^2 + 2p' \cdot k - k^2
$$

$\text{Im} T^s$ and $\text{Im} T^u$ are the imaginary parts of the $s$-channel and $u$-channel reactions, respectively. We need no subtraction as long as $q^2 = -t$ is not too small and positive. Of course, there might be complex singularities in $T$ due to the off-shell external lines, but little is known about the singularities and we assume that their effects are small.\(^*)\) In other words, we have assumed that the off-shell effects are completely described by $k^2$ and $k \cdot p$ dependence of $s$ and $u$ in the denominator of the dispersion relation. This assumption seems to be good as far as Regge pole contributions are concerned, since it corresponds to the assumption that the Regge residue functions do not depend strongly on the off-shell effects, which has been used by Lovelace\(^6\) in successful application of the off-shell Veneziano formula.

If we substitute Eq. (4) into Eq. (3) and perform the $k$ integration by Feynman techniques, we obtain

$$
i(p + p')_\mu F(q^2) = -\frac{i}{(2\pi)^3} \left\{ g^2 [I_{\pi s}(m^2) - I_{\pi p}(m^2)] \\
+ \frac{1}{\pi} \int_{(m^2)^2}^{\infty} ds' \left[ I_{\pi s}(s') \text{Im} T^s(s', -q^2) + I_{\pi p}(s') \text{Im} T^u(s', -q^2) \right] \right\},
$$

(5)

where

$$
I_{\pi s}(s') = \pi^2 \int_0^1 \prod_{i=1}^{3} dx_i \frac{-2x_3 p_\mu + (2x_2 - 1) q_\pi \delta (\sum_{j=1}^{3} x_j - 1)}{C(s')},
$$

$$
I_{\pi p}(s') = \pi^2 \int_0^1 \prod_{i=1}^{3} dx_i \frac{2x_3 p'_\mu + (2x_2 - 1) q_\pi \delta (\sum_{j=1}^{3} x_j - 1)}{C(s')}
$$

(6)

\(^*)\) Especially, in the case that one of the off-shell lines is time-like, we must have anomalous thresholds which depend on $k$, and there are many complications in analyticity which have not been clarified. However, there are normal thresholds certainly. We assume here that the singularities other than the normal branch points and poles are not so effective, for simplicity.
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We have changed the integration variable $u'$ to $s'$ in Eq. (5). If we write $p_\mu$ and $p_\nu'$ by $(p_\mu + p_\mu')/2 - q_\mu/2$ and $(p_\mu + p_\mu')/2 + q_\mu/2$ in Eq. (6), we can find easily that only $(p + p')_\mu$ term survives in Equation (5), which is the expected effect of gauge invariance. Then we obtain

$$ F(q^2) = \frac{1}{4\pi} \left\{ g^2 I(m^2) + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} ds'I(s') \left[ \text{Im} T^*(s', -q^2) - \text{Im} T^u(s', -q^2) \right] \right\}, $$

where

$$ I(s') = \prod_{i=1}^{3} \frac{dx_i}{C(s')} \delta \left( \sum_{j=1}^{3} x_j - 1 \right). $$

In order to perform the integration over $s'$, we divide the integral region $(m+\mu)^2 < s' < \infty$ into two parts, $(m+\mu)^2 < s' < s_0$ and $s_0 < s' < \infty$, so that in the region $s_0 < s'$, the Regge pole approximation is quite good. If we further approximate $\text{Im} T$s' by resonances in the region $(m+\mu)^2 < s' < s_0$, the contribution from this integral region to $F(q^2)$ is apparently analytic in the region $-q^2 < 4m^2$, since such resonance contributions to $F(q^2)$ are merely those of triangle graphs. Therefore, in order to find the singularities of $F(q^2)$ in $-q^2 < 4m^2$, we must analyze the part of the contribution to $F$ from the integration $s_0 < s' < \infty$.

$$ F_R(q^2) = \frac{1}{(2\pi)^3} \int_{s_0}^{\infty} ds'I(s') \left[ \text{Im} T^*(s', -q^2) - \text{Im} T^u(s', -q^2) \right]. $$

As discussed above, we can represent $\text{Im} T^*$ in this region by the Regge pole approximation

$$ \text{Im} T^*(s', -q^2) = \beta(-q^2) \left( \frac{s'}{s_1} \right)^{\alpha(q^2)} $$

and hence,

$$ \text{Im} T^u(s', -q^2) = \text{Im} \frac{1 - \exp(-i\pi\alpha)}{\sin \pi\alpha} \beta(-q^2) \left( -\frac{s'}{s_1} \right)^{\alpha} $$

$$ = [-1 + 2 \cos \pi\alpha] \beta(-q^2) \left( \frac{s'}{s_1} \right)^{\alpha}, $$

where $\alpha(-q^2)$ and $\beta(-q^2)$ are the trajectory and residue of the $p$-Regge pole, respectively, and $\alpha$ is assumed to be linear with respect to $-q^2$. If we substitute Eqs. (9) and (10) into Eq. (8), it becomes

$$ F_R(q^2) = \frac{1}{\pi^2} \beta(-q^2) \sin^2 \frac{\pi}{2} \alpha \int_{s_0}^{\infty} ds' \prod_{i=1}^{3} dx_i \frac{x_i(s'/s_1)^{\alpha}}{C(s')} \delta \left( \sum_{j=1}^{3} x_j - 1 \right), $$

\[ \text{(7)} \]
which can be rewritten as

$$F_R(q^2) = \frac{1}{\pi^2} \beta(-q^2) \sin^2 \frac{\pi}{2} \alpha \left( \frac{s_0}{s_1} \right) a J(q^2),$$

(11)

where

$$J(q^2) = \int_0^1 dt \int_0^\infty dx_1 t^{-a-1} \left\{ \sum_{j=1}^3 \frac{x_1 + x_2 (m^2 - x_3 t)}{x_3} + \frac{x_1 x_2 q^2}{x_3} \right\}.$$

(12)

The Regge residue function, $\beta(-q^2)$, must be analytic without any singularities since there is no dynamical reason to produce them, and our problem is then to find the singularity of $J(q^2)$. In the Appendix, the analytic continuation of $J(q^2)$ is performed, and it is shown that $J(q^2)$ has a set of dipoles at $\alpha(-q^2) = 1, 2, 3, \ldots$ and a simple pole at $\alpha = 0$ in the region $-q^2 < 4m^2$. Therefore, $F(q^2)$ has a set of dipoles at $\alpha(-q^2) = 1, 3, \ldots$ in the region $-q^2 < 4m^2$, because the factor $\sin^2 (\pi/2) \alpha$ eliminates the dipoles at $\alpha = 2, 4, \ldots$ and the simple pole at $\alpha = 0$.

Using that our $F(\infty)$ drops rapidly as $q^{-4}$, $F(q^2)$ can therefore be rewritten as

$$F(q^2) = \sum_{0<j-1<n(m^2)} \frac{R_j}{[\alpha(-q^2) - 2j - 1]^2} + f(q^2),$$

(13)

where $f(q^2)$ is holomorphic in the region $-q^2 < 4m^2$.

§ 3. Discussion

It is very interesting that $s_0$ does not depend on the nature of the singularities in Eq. (13) and that the dipoles come from the contribution of the Regge pole term in $T$. There may exist such an $s_0$ that the amplitude $T$ above $s_0$ can be approximated well by the $\rho$-Regge pole amplitude. If this is the case, we can conclude that the $\rho$ singularity of the form factor $F(q^2)$ is a dipole and not a simple pole contrary to the widely held belief.

Moreover, if we use a quark model instead of considering $\pi$ meson as a $N\bar{N}$ bound state, we can apply the above logic to the isovector nucleon form factor as well as to the $\pi$ meson form factor, and even to any other form factors, to which isovector photon contributes. Then we may have a completely equivalent expression as Eq. (13). We anticipate that problems will be encountered from the effects of the off-shell amplitudes when we include the real spin of the quark. The most serious problem will be the existence of the contributions of the negative energy states of the quarks for such quark-hadron scattering amplitudes in which quarks are off-mass shell. We have no good reason for assuming that there exists such an $s_0$ above which Regge pole behavior is dominant for such amplitudes. However, if this is the case, we may obtain the result that all isovector electromagnetic form factors have the same structure as Eq. (13). Moreover, the axial
vector form factors will also have the same structure as the simple replacement of \( \alpha \) by an appropriate Regge trajectory, \( \alpha_{A_1} \).

Here we add a remark concerning the approximation we have made for \( \text{Im} \, T \). The resonance approximation for \( s' < s_0 \) and Regge pole approximation for \( s' > s_0 \) may be reasonable. But there remains a question as to whether the analytic continuation of the approximated quantity is reliable in the continued \( q^2 \) region. We must say that this is an open question at this moment, but we notice that Eq. (13) should be equal to the original integral representation in the region of \( \alpha(-q^2) < 0 \). In other words, we can say at least that, looking at \( F(q^2) \) in the region \( \alpha(-q^2) < 0 \), the effect of the singularity at \( -q^2 = m_s^2 \) to \( F(q^2) \) in this region, \( \alpha < 0 \), is that of a dipole, and that the questions (1) and (2) we have mentioned in the introduction are said to have been solved. Namely, we conclude that the dipole form of the form factor for wide range of \( q^2 \) comes from the Regge pole behavior as well as the crossing symmetry of the amplitude \( T \) in the exact representation of \( F \) (Eq. (13)), and that the rapid fall of \( F(q^2) \) for \( q^2 \to \infty \) is due to the bound state structure of the hadrons as studied in Refs. 2), 3) and 4).

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**Appendix**

In this appendix we perform analytic continuation of the function \( J(q^2) \) defined by Eq. (12). \( J(q^2) \) can be rewritten for \(-q^2 < 4m^2\) as

\[
J(q^2) = \int_0^1 dt \int_0^n \prod_{i=1}^n dy_t \frac{L^{n-1}}{(\sum_{j=1}^n y_j)^2} \times \exp \left[ - \frac{y_t \sum_{j=1}^n y_j + t (y_1 + y_2) \{ m_s^2 \sum_{j=1}^n y_j - \mu^2 y_3 / y_3 \} / s_0 + ty_1 y_2 (q^2 / s_0) }{y_3} \right].
\]

(A1)

By introducing the variables \( \rho \) and \( x_t \) represented by Ref. 7)

\[
y_t = \rho x_t \quad \text{with} \quad \sum_{i=1}^n x_t = 1 \quad \text{and} \quad 0 \leq x_t \leq 1,
\]

it can be easily shown that Eq. (A1) reduces to the original form, Eq. (12), for \(-q^2 < 4m^2\).

First of all, we notice that the right-hand side is convergent if \( \alpha < 0 \). This can be seen by noticing that the integral with respect to the \( y_i \)'s for large \( y_i \) are convergent and that the finite \( y \) contribution is bounded by

\[
\int_0^1 dt \int_0^n \prod_{i=1}^n dy_t \frac{L^{n-1}}{(\sum_{j=1}^n y_j)^2} < \infty \quad \text{for} \quad \alpha < 0.
\]
Therefore $J(q^2)$ is analytic in the region $\alpha (-q^2) < 0$.

In order to make analytic continuation of $J(q^2)$ in the region $\alpha > 0$, we rewrite $t$ integral in (A1) as

$$
\int_0^\infty dt' = \int_0^\infty dt - \int_1^\infty dt'.
$$

Then, it is clear that the contribution of the second integral is convergent for $\alpha > 0$ and $-q^2 < 4m^2$, because the integral is bounded by

$$
\int_1^\infty dt \int_0^\infty \prod_{i=1}^{3} dy_i \frac{t^{-a-1}}{(\sum_{i=1}^{3} y_i)^2} \exp \left[ -\sum_{j=1}^{3} y_j \right]
$$

in the case of $-q^2 < 4m^2$, and this is finite for $\alpha > 0$. (This second term has a simple pole at $\alpha = 0$, as can be easily checked.) Therefore all the singularities of $J(q^2)$ in the region $\alpha > 0$ and $-q^2 < 4m^2$ are involved in the first integral $\int_0^\infty dt$ term, whose $t$ integration can be carried out to give

$$
J_1(q^2) = \Gamma(-\alpha) \int_0^\infty \prod_{i=1}^{3} dy_i \frac{1}{(\sum_{i=1}^{3} y_i)^2} \times \left( (y_1 + y_2) (m^2 \sum_{i=1}^{3} y_i - \mu^2 y_3) / s_0 + y_1 y_2 (q^2 / s_0) \right) \exp \left[ -\sum_{j=1}^{3} y_j \right].
$$

(A2)

It is clear that, apart from the factor $\Gamma(-\alpha)$, the integral above is analytic in the region $-1 < \alpha < 1$, and that there exists such a region of $q^2$ as $\alpha > 0$ and $-q^2 < 4m^2$ in the case of $\rho$-trajectory.

In order to make the analytic continuation to the region $\alpha > 1$, we integrate partially with respect to $y_3$, drop the surface contribution and get

$$
J_1^s(q^2) = \Gamma(-\alpha) \int_0^\infty \prod_{i=1}^{3} dy_i (y_3)^{-a+1} \frac{\partial}{\partial y_3} \times \left\{ \left( (y_1 + y_2) (m^2 \sum_{i=1}^{3} y_i - \mu^2 y_3) / s_0 + y_1 y_2 (q^2 / s_0) \right)^s \exp \left[ -\sum_{j=1}^{3} y_j \right] \right\},
$$

which is well defined in $1 < \alpha < 2$ and has a dipole at $\alpha = 1$. Continuing the same procedure, we can see that the analytically continued function $J_1^s(q^2)$ for $\alpha > 0$ has a set of dipoles at

$$
\alpha (-q^2) = 1, 2, 3, \ldots
$$

and a simple pole at $\alpha (-q^2) = 0$. Hence $J(q^2)$ has only dipoles at $\alpha = 1, 2, 3, \ldots$ and a simple pole at $\alpha = 0$ in the region $-q^2 < 4m^2$.

References

2) J. S. Ball and F. Zachariasen, Phys. Rev. 170 (1968), 1541.
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