Applicability of Pauli’s Regulator to the $\gamma$-Decay of Neutrettos*

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§ 1. Possible origin of ambiguity involved in the field theory.

Although the present formulation of the field theory seems to be “proved” to satisfy the requirement of invariance with respect to the Lorentz and gauge transformations, puzzling facts are encountered in many problems that the results are non-invariant, such as the non-vanishing of the photon self-energy,\(^{(1)}\) the non-gauge-invariance of the matrix elements for the $\gamma$-decay of neutrettos,\(^{(2)}\) or for the radiative decay of heavy mesons. Detailed examination,\(^{(3)}\) however, shows that, in most of these cases, results depend on the method of calculation, so that the conclusions are not unique, but quite ambiguous.

Where on earth are involved the origin of these ambiguities in the formalism of the present field theory? To our opinion, the reason for this seems to lie in the fact that the consistent formulation of the relativistic commutation relations and of the generalized Schrödinger equation can be attained only by using a singular function \(d(x)\).

In the first place, \(d\) plays a role of Green’s function which expresses a wave field \(A(x')\) by its initial value given on a space-like surface \(\sigma\) by

\[
A(x') = \int_{\sigma} \, d\sigma_n A(x) \frac{\partial}{\partial x'_n} A(x - x').
\]

\(d\) is the solution of the wave equation that vanishes in the space-like region. From the condition that \(A(x')\) must approach to \(A(x)\) when \(x'\) approaches to a point \(x\) on \(\sigma\), one must have

\[
\lim_{x' \to x} \int_{\sigma} \, d\sigma_n \frac{\partial A(x)}{\partial x'_n} = 1,
\]

\(^{(1)}\)Although this study was, at first, made independently by the Tokyo and Tohoku groups, the results obtained were found to be almost identical when discussed together, and it was then approved to publish the paper jointly.
where $\sigma_4$ is an arbitrary space-like surface passing through the origin. This property of $\sigma$ would be inconsistent if it were finite at the origin. As $\sigma$ is an invariant, $\frac{\partial \sigma}{\partial x_{\mu}}$ should be a vector. However, on an arbitrary space-like plane through the origin its space-like components are zero and the time-like component has the value $i\delta(x)$. Thus, the value of $\frac{\partial \sigma}{\partial x_{\mu}}$ at the origin has no meaning, because it has only the time-like component in any reference system.

However, this singular behavior of the Green's function is not an essential difficulty of the field theory, but is common with any classical wave field. The difficulty gets worse by the fact that the same singular function $\sigma(x)$ also plays an important role in expressing relativistic commutation relations between field variables. The field quantities have thus infinite degrees of freedom, and fluctuation of various quantities became infinite. This is expressed by the more singular function $\sigma^{(0)}(x)$ that expresses the vacuum expectation value of a bilinear combination of the field quantity with its adjoint. The singularity of $\sigma$ is entangled with that of $\sigma^{(0)}$, and thus the essential divergence and ambiguity of the present theory result from this situation. In fact, the most integrals of products of several $\sigma$ and $\sigma^{(0)}$ become divergent. At times some integrals apparently converge, but they have ordinarily a form of $\infty - \infty$ or $\infty \times 0$. Thus the divergence affects even the finite terms and makes them ambiguous.

Since the consistency of the fundamental equations and the commutation relations could be formally proved, it is expected that reasonable results would be obtained by some careful procedure choosing suitable expressions out of the ambiguous ones. In fact, it is shown that we can obtain in this way gauge invariant results for the photon self-energy and the matrix elements for $\gamma$-decay of neutrettos. Pauli's regulator $\sigma$ seems to provide us an automatic procedure obtaining the reasonable results. If a set of conditions for the regulator were found in order to regulate reasonably all possible ambiguities it would become a powerful method filling up the gaps involved in the field theory.

At first sight, it would seem desirable to regulate $\sigma$ or $\sigma^{(0)}$ itself from the beginning, but replacement of $\sigma$ by some regular function may destroy correspondence of the interaction representation to the Heisenberg-Pauli theory. The field quantity will then no longer satisfy the wave equation, and, as a result of it, the current expression $i\overline{\varphi}\gamma_{\mu}\varphi$ will lose it meaning, because it no longer satisfies the continuity relation. The regularization must, therefore, be applied to the resulting matrix elements for individual processes.

In this paper we want to examine applicability of the regulator in the case of $\gamma$-decay of neutrettos.

§ 2. $\gamma$-decay of neutrettos.

The general formulation and some solutions of the $\gamma$-decay problem were
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given by Fukuda and Miyamoto. Let us briefly summarize their results. The generalized Schrödinger equation describing the system of the neutrino $\nu(x)$, the proton $\varphi(x)$, and the electromagnetic field $A_\mu(x)$, is given by

$$i \frac{\partial \Psi}{\partial \sigma} = (H_c + H_f) \Psi.$$  \hspace{1cm} (3)

where $H = -i \bar{\varphi} \gamma^\mu \varphi A_\mu$ represents the interaction between the proton and the electromagnetic field, and $H_f$ that between the neutrino and the proton. The latter is given by

$$H_f = \int \bar{\varphi} \gamma^\nu \varphi 1^\nu$$ \hspace{1cm} (4)

for the scalar neutrino with the scalar interaction, and by

$$H_f = \int \bar{\varphi} \gamma^\nu \varphi \partial_\nu 1^\nu + 2\pi f^2 \bar{\varphi} \gamma^\nu \varphi \nu$$ \hspace{1cm} (5)

and

$$H_f = \int \bar{\varphi} \gamma^\nu \varphi V$$ \hspace{1cm} (6)

for the pseudoscalar neutrino with the pseudovector and pseudoscalar couplings respectively. In (5) $1^\nu$ means the normal component of $1$ on the surface $\sigma$.

The matrix element responsible to the $\gamma$-decay of the neutrino can be obtained by the following canonical transformation $\Psi' = U_\gamma \Psi$ with $U_\gamma$, determined by

$$i \frac{\partial U_\gamma \varphi}{\partial \sigma} = H U_\gamma \varphi.$$ \hspace{1cm} (7)

The original equation (3) is transformed into

$$i \frac{\partial \varphi}{\partial \sigma} = U_\gamma^{-1} H U_\gamma \varphi = < H_f > \varphi.$$ \hspace{1cm} (8)

The matrix element is obtained from this by evaluating the vacuum expectation value of $< \bar{\varphi} L \varphi >$ with respect to the proton field, where $L = 1$, $\bar{\varphi} \gamma^\nu \varphi$ and $\gamma_\nu$ for (4), (5) and (6) respectively.

Now the gauge invariance of $< \bar{\varphi} L \varphi >$ and the identity

$$\partial_\mu < \bar{\varphi} \gamma^\nu \gamma^\mu \varphi > = -2m \bar{\varphi} \gamma^\nu \varphi$$ \hspace{1cm} (9)

are evident without any proof, because $< \bar{\varphi} L \varphi >$ is regarded as the Heisenberg representation of $\bar{\varphi} L \varphi$ in the presence of electromagnetic field. But the direct proof shall be given here in order to make sure. For the transformation

$$A_\mu \rightarrow A_\mu + \partial_\mu 1,$$

$$\varphi \rightarrow \varphi, \quad \bar{\varphi} \rightarrow \bar{\varphi},$$

the equation (7) for $U_\gamma$ is invariant if $U_\gamma$ is transformed according to

$$U_\gamma \rightarrow e^{-i \gamma_5 (m) \gamma_5} U_\gamma.$$
The equation (7) is then transformed into

\[ e^{i\tau} i \frac{\partial}{\partial \sigma} e^{-i\tau} U_\varepsilon = e^{i\tau} (H_\varepsilon - i e^{-i\tau} \bar{\varphi} L \varphi \partial_\mu \varphi) e^{-i\tau}. \]

(12)

Now, as in the case of consistency proof of the generalized Schrödinger equation and the auxiliary condition, the commutability of \( \gamma \) with \( \bar{\varphi} L \varphi \) taken on the surface \( \sigma \) plays an important role for the following proof. The commutability is "proved" as follows:

\[ [\gamma(\sigma), \bar{\varphi} L \varphi] = -c \int_\sigma A(x') d\sigma' \bar{\varphi} \gamma_{\mu} S(x' - x') L \varphi - \bar{\varphi} L S(x - x') \bar{\varphi} \gamma_{\mu} \varphi \]

which, in the reference system whose time axis coincides with the normal of \( \sigma \) at the point \( x \), and using the property that \( \frac{\partial A}{\partial x_\mu} \) has only the time-like component \( i\delta^\tau(x) \) on the tangential plane of \( \sigma \) at \( x \), becomes

\[ [\gamma(\sigma), \bar{\varphi} L \varphi] = -c \int_\sigma A(x') d^2 x_\mu \delta^\mu(\bar{x}' - \bar{x}) (\bar{\varphi}' L \varphi - \bar{\varphi} L \varphi') = 0 \]

(13)

The conclusion (13) is true only under the assumption that, in the first place, the relation (2) holds when \( \sigma \) is equated with \( \sigma_0 \) from the beginning, and in the second place, \( \bar{\varphi} L \varphi \) approaches \( \bar{\varphi} L \varphi \) uniformly when \( x' \) approaches \( x \). As we remarked in the preceding paragraph, the last assumption is quite uncertain, because, for example, the vacuum expectation value of \( \bar{\varphi} L \varphi \) has just the singularity of \( S^{(\nu)}(x' - x) \) and (13) is indeterminate in the strict sense.

Nevertheless, admitting (13) for the moment, we shall show that (12) is identical with (7). This is easily seen from the relation:

\[ e^{i\tau} i \frac{\partial}{\partial \sigma} e^{-i\tau} U_\varepsilon = i \frac{\partial U_\varepsilon}{\partial \sigma} - i e^{-i\tau} \bar{\varphi} \gamma_{\mu} \varphi \cdot \partial_\mu A \cdot U_\varepsilon. \]

The invariance of \( \langle \bar{\varphi} L \varphi \rangle \) with respect to the gauge transformation is also trivial.

The proof of the identity (9) runs similarly. First we notice that \( \langle \bar{\varphi} L \varphi \rangle \) is a point function independent from the form of \( \sigma \), and we suppose that \( \sigma \) is a space-like plane. \( \partial_\mu \langle \bar{\varphi} \gamma_{\mu} \varphi \rangle = \partial_\mu (U_\varepsilon^{-1} \bar{\varphi} \gamma_{\mu} \varphi U_\varepsilon) \) is divided into the differentiation of \( \bar{\varphi} \gamma_{\mu} \varphi \) and that of \( U_\varepsilon^{-1} \) and \( U_\varepsilon \). The former gives \( -2m \langle \bar{\varphi} \gamma_{\mu} \varphi \rangle \). In the latter case, the differentiation in the space-like direction is zero because \( U_\varepsilon \) is unchanged for the displacement in the plane. For the time-like differentiation we utilize the differential equation for \( U_\varepsilon \) and \( U_\varepsilon^{-1} : \)

\[ i \partial_\mu U_\varepsilon = \int d^3 x' H_\varepsilon(x') \cdot U_\varepsilon \quad \text{and} \quad -i \partial_\mu U_\varepsilon^{-1} = U_\varepsilon^{-1} \int d^3 x' H_\varepsilon(x'). \]

Consequently we have
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$i < \left[ \int d^3 x' H_\gamma(x'), \bar{\psi} \gamma_\mu \psi \right] >$

for the latter, which vanishes similarly.

Now, according to the calculation of Fukuda and Miyamoto (2), the term $< \bar{\psi} L \psi >_2$ of order $\epsilon^2$ in $< \bar{\psi} L \psi >$ becomes

$$f < \bar{\psi} \gamma_\mu \psi >_2 = \frac{f m}{8 \pi^2} \epsilon^2 \int_1^\infty du \int_1^u \epsilon^2 \left[ \frac{1}{u^2} A^2_{\mu} - \frac{1}{u^2} \left( 1 - \frac{1}{u^2} \right) [u^2]^{-1} \frac{1}{2} F_{\mu \nu} \right], \quad (14)$$

$$f < \bar{\psi} \gamma_{\mu \nu} \psi >_2 = \frac{f}{4 \pi^2} \epsilon^2 \int_1^\infty du \int_1^u \epsilon^2 \left[ \frac{1}{u^2} \sum_A \lambda \sum_F_{\mu \nu} \gamma_{\mu \nu} \right]$$

$$+ \frac{1}{u^2} \left( 1 - \frac{1}{u^2} \right) [u^2]^{-1} \sum_A \lambda \sum_F_{\mu \nu} \gamma_{\mu \nu} \right], \quad (15)$$

$$f < \bar{\psi} \gamma_\mu \psi >_2 = \frac{f}{4 \pi^2} \epsilon^2 \int_1^\infty du \int_1^u \epsilon^2 \left[ \frac{1}{u^2} + \frac{1 - \tau^2}{4 m^2 u^2} D^2 [u^2]^{-1} \right]$$

$$\times (F_{\mu \nu} F_{\mu \nu} + F_{\mu \nu} F_{\mu \nu} + F_{\mu \nu} F_{\mu \nu}) \quad (16)$$

respectively, where $\gamma_{\mu \nu} = \gamma_\mu \gamma_\nu$ and $(\omega_{\tau})$ assumes $(\tau 24), (134), (214), (123)$ as $\mu$ runs from 1 to 4, and $[u^2]$ is written for $u^2 - \frac{1 - \tau^2}{4 m^2} [\tau^2]$. That the non-gauge-invariant terms $A^2_{\mu}$ and $\sum A_{\mu} F_{\mu \nu}$ still survive in (14) and (15) is the serious contradiction that $< \bar{\psi} L \psi >_2$ should, in general, be gauge invariant. Besides, the identity (9) is not satisfied. In fact, we have from (15)

$$\partial_\mu < \bar{\psi} \gamma_{\mu \nu} \psi >_2 = - \frac{1}{2 \pi^2} \epsilon^2 \int_1^\infty du \int_1^u \epsilon^2 \left[ \frac{1}{u^2} + \frac{1 - \tau^2}{4 m^2 u^2} D^2 [u^2]^{-1} \right] (F_{\mu \nu} F_{\mu \nu} + \ldots) \quad (17)$$

the first term of which differs from that of $-2m < \bar{\psi} \gamma_\mu \psi >_2$, as $\int_1^\infty du/2u^2 = 1/2$ and $\int_1^\infty du/2u^4 = 1/6$. It is worthwhile to note that the gauge invariance and identity hold for all terms of higher order in expansion of $D^2/u^2$.

The results from (14) to (16) were calculated by applying Schwinger's method of integration. It can, however, be shown that the coefficients of $A^2$ and $\sum A F^2$ in (14) and (15) are only conditionally convergent, and these ambiguous terms are hoped to be able to be dropped by Pauli's regulator. In Schwinger's method of integration, there seems to be no ambiguity of the first term in the bracket of (16), but Schwinger's method itself is quite a special one and we arrive at a different result when different representation are used for $J$ and $J^a$. This situation is clearly seen in the fact that the value of $D^{-1}$ at the origin is infinite in the Fourier representation but becomes zero in the Schwinger one.
§ 3. Discussions of the results.

First of all, we think it appropriate to drop $\Sigma AF$ term in (15) for the following reasons:

(1) It might be, at first sight, urged that the presence of the non-gauge-invariant term $\Sigma AF$ does not destroy the gauge invariance of the real processes, because $\Sigma A \nu F_{at} \partial_\mu \nu$ can be written as $\partial_\mu (\Sigma A \nu F_{at} V) + 2 V (F_{at} F_{14} + ...)$, and the last term is clearly gauge invariant while the first term has no contribution to the real processes on account of conservation of energy and momentum. We suppose, however, the gauge invariance of $\langle \bar{\psi} \gamma_5 \gamma \rho \psi \rangle$ itself should be required because the presence of the non-gauge-invariant term $\Sigma AF$ in $\langle \bar{\psi} \gamma_5 \gamma \rho \psi \rangle$ constitutes a difficulty in the case of pseudovector neutron $U_\nu$ interacting with the nucleon by pseudovector coupling. In this case the corresponding matrix element has the form $\langle \bar{\psi} \gamma_5 \gamma \rho \psi \rangle U_\nu$ and the existence of the problematic term in $\langle \bar{\psi} \gamma_5 \gamma \rho \psi \rangle$ gives rise to $\Sigma U_\nu A \nu F_{at}$ which is no longer gauge invariant even for the real processes. Indeed, the life time evaluated from it is infinite in the case of the neutron at rest, but not so for moving one, whence the life time of the pseudovector neutron does not transform correctly under Lorentz transformation if one retains this term.

(2) As the coefficient of $\Sigma AF$ is independent from the mass $m$, this term is dropped off by the regulator. That is to say, if we suppose that there are auxiliary Fermi particles of mass $m_1$ interacting with the neutron by the coupling constant $f_1$, which satisfy the condition

$$\sum f_1 = 0,$$

then $\Sigma AF$ terms disappear, and the remaining terms retain their original value by making $m_1$ infinitely large.

Next, we consider the case of pseudoscalar coupling. All terms contained in (16) are Lorentz and gauge invariant, so that we have no reason to drop the first term in the bracket. But, if we drop the $\Sigma AF$ term in the case of pseudovector coupling, this first term must also be dropped in order to preserve the equivalence between the pseudoscalar and pseudovector couplings.

In this situation we have two alternatives either to abandon the equivalence or to admit to drop the first term even if it seems convergent and invariant. We are not sure which of these alternatives should be taken, but we remember that the equivalence has been "proved" and the latter alternative does not seem altogether impossible. In order that the pseudovector coupling $if_\nu < \bar{\psi} \gamma_5 \gamma \rho \psi > \partial_\mu V$ is equivalent to the pseudoscalar one $if_\nu < \bar{\psi} \gamma_5 \rho > V$, we must have the relation

$$\frac{1}{2m} f_\nu = f_\rho$$

(19)

If we use the regulator in the case of pseudoscalar coupling after we put (19)
into (16), we find that the first term \( \xi_m \xi_n + \ldots \) in the bracket of (16) really disappears by the condition similar to (18):

\[
\sum_i (f_{\mu i})_i = 0. \tag{20}
\]

This procedure can be interpreted either as mixing auxiliary fields with different masses maintaining Pauli's condition (18) but, on the other hand, assuming the coupling constants to depend on the masses, \( f_{\mu i} \propto m \), or as mixing the fields with the condition

\[
\sum_i \frac{1}{m} (f_{\mu i})_i = 0, \tag{21}
\]

which differs from Pauli's one.

Similar situation is encountered in the case of scalar interaction (14). In this case there would be no reason to retain the non-gauge-invariant term \( \xi^2 \) because this term can be shown only conditionally convergent. Pauli's condition is, however, not sufficient to drop this term, and a more strict condition

\[
\sum_i m_i (f_{\infty i})_i = 0 \tag{22}
\]

must be required. As there is in this case no identity such as (9), we have no reason to consider the coupling constant \( f_{\infty} \) to be inversely proportional to the mass. (22) must be taken as a new condition for the regulator.

There are two alternatives interpreting the regulator method, either as a mixed fields theory or as a formal procedure. If we take the first point of view, the regulator must satisfy the conditions

\[
\sum_i f_i = 0, \quad \sum_i m_i f_i = 0 \quad \text{and} \quad \sum_i \frac{1}{m_i} f_i = 0. \tag{23}
\]

If these conditions were to be universally applied, the third condition would bring a serious change of the life time of the scalar neutrino, because then the first term of expansion in \( \xi^2/m^2 \) of the second term of (14) would also be dropped.

If we take the second point of view and require that the regulator must always be applied only to the function of even power in \( m \), we must first separate a factor of an odd power of \( m \) (\( m \) and \( 1/m \) in the cases of (14) and (16) respectively) and afterwards apply the regulator to the remaining terms. In this case the second term of (14) is unchanged, in contrast to the first point of view. But then there remains ambiguity in separating an odd power of \( m \), because it is also possible to separate \( m^2 \) in the case of (14).

In the case of vector neutrino, two-\( \gamma \) decay is forbidden, and the matrix element for three-\( \gamma \) decay turns out to

*If the third condition be true, the anomalous magnetic moment of electron will vanish, which contradicts with the experimental fact.
\[ \langle \bar{\psi} \gamma_\mu \psi \rangle_s = -\frac{1}{24\pi^2} e^2 A^2 A_\mu + \ldots, \]

the first term of which can be made to vanish by the regulator. We are thus inclined to believe that the present theory does not teach us which of various alternatives we should take. We have hoped that Pauli's regulator could give decision at this point, but it seems that there remains still some ambiguity how to use the regulator. In any case, we think that our problem will present a severe test to the regulation method and in this situation we think it desirable that some experiment which could detect the \( \gamma \)-decay of neutratto will answer this problem and provide us some clue to the correct future theory.

References.

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(5) J. Steinberger, Private communication to Fukuda and Miyamoto. According to his latest manuscript, he arrived at the same conclusion as ours to abandon this non-invariant term.