A systematic approach is presented to prove that the charged sector of the Georgi-Glashow model contains an intrinsically CP-violating angle which shows up in the physical results in the leading order. By showing that this model has no instantons we proved that this angle has nothing to do with instantons, but is just a superselection label.

§ 1. Introduction

Recently, in an interesting paper, Witten\textsuperscript{1} has investigated the implications of introducing the CP violation into the Georgi-Glashow model on the question of dyon electric charge quantization.

In this paper, stimulated by Witten's work, we present a systematic approach to the problem of \( \theta \)-dependence of the physics in the dyon sectors\textsuperscript{2} of the Georgi-Glashow model. We use the Hamiltonian method of quantization, because we feel that this is the only systematic way of quantization. Of course for limited purposes the semiclassical quantization method of Bohr-Sommerfeld may suffice. The trick to use then is to introduce temporal periodicity into the theory, say, by the special gauge choice,\textsuperscript{3} \( A_\theta (r = \infty) = 0 \).

We witness the emergence of an arbitrary angle in the quantized theory, which breaks CP invariance. A careful study shows that this angle has its origin in the non-trivial topology of the gauge field, which in turn is due to the existence of charge. We demonstrate that this angle has nothing to do with instantons by showing that in spite of the fact that pure gauge theories have instantons, in the presence of Higgs fields, they disappear; they are replaced by finite energy configurations.

Gauge theories can be consistently canonically quantized in the \( A_\theta = 0 \) gauge. We have checked the consistency of this approach by showing that Schwinger algebra is undisturbed by this non-Lorentz invariant method of quantization and by the presence of charged sectors.\textsuperscript{4–6}

For completeness and logical continuity of this paper we include a systematic reproduction of Witten's proof of the \( \theta \) dependence of the dyon charge, thus

\textsuperscript{1} On leave of absence from Department of Physics, Ondokuzmayis University, Samsun, Turkey.

\textsuperscript{2} This should be compared with the opposite result in the two dimensions.\textsuperscript{4} This may be a good warning on how many of the results in two dimensions should be carried over to four dimensions.
proving that $\theta$ dependence is a leading order effect.

Finally we comment on the validity of the arguments presented in this work for realistic grand unified schemes like $SU(5)$, by demonstrating that it has the necessary topological structure.\footnote{Monopoles in the $SU(5)$ theory have recently been investigated in Ref. 5. We thank Dr. D. Storey for bringing this work to our attention.}

$\S$ 2. Dyon solutions in the Georgi-Glashow model

This is a theory with an $SO(3)$ gauge group spontaneously broken down to $U(1)$ by the Higgs sector. The bosonic fields are a triplet of gauge fields, $A_\mu^a$, and a triplet of Higgs scalars, $\phi^a$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a + \frac{1}{2} (D_\mu \phi)^2 - U(\phi^2)$$

with

$$U(\phi) = \frac{i}{4} (\phi^2 - \alpha^2)^2$$

and

$$(D_\mu \phi)^a = \partial_\mu \phi^a - ie^{\alpha a} A_\mu^a \phi^a.$$  

The equations of motion obtained from $\mathcal{L}$ read as follows:

$$(D_\mu F_{\nu}^a)^a = -e^{\alpha a} \phi^a (D_\mu \phi)^a,$$  

$$(D_\mu D^a \phi)^a = -i \phi^a (\phi^2 - \alpha^2)$$

to be accompanied by the Bianchi identity

$$D^\mu F_{\mu a}^a = 0.$$  

The energy density is given by

$$\theta_\alpha = \frac{1}{2} \left\{ (E_i^a)^2 + (B_i^a)^2 + (D_\mu \phi)^2 + (D_\mu \phi)^3 \right\} + U(\phi).$$

Here $E_i^a$ and $B_i^a$ are defined by

$$E_i^a = F_{ij}^a, \quad B_i^a = \frac{1}{2} e^{ijkl} F_{jk}^a.$$  

We immediately observe that the vacuum is defined by the configurations satisfying

$$F_{\mu\nu}^a = 0, \quad (D_\mu \phi)^a = 0, \quad U(\phi) = 0.$$
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A typical example of vacuum configuration satisfying (2·8) is

\[ \phi^a = a \phi^{\text{ss}}, \quad A_i^a = 0. \]  

(2·9)

Since \( \theta_{\infty} = 0 \) is gauge invariant, any gauge transformation of (2·8) is also a vacuum configuration. Imposing the finite energy condition enforces these conditions (2·8) asymptotically at the spatial boundary. In particular, this requires \( \phi \in M_9 \) (the set of \( \phi \) which minimizes \( U(\phi) \)). Here \( M_9 \) is a two-dimensional sphere of radius \( a \) in the internal space. The little group \( H \) of \( \phi \in M_9 \) is the group of rotations about the \( \hat{\phi} \) axis and is isomorphic to \( SO(2) \) (or equivalently to \( U(1) \)). Thus the original symmetry is broken down to \( G/H \) by Higgs fields, which can be identified with the electromagnetism.

Dyons are defined as those finite energy spatially localized configurations which carry electric as well as magnetic charge. They are defined originally with the following ansatz (which satisfies the gauge \( \partial_i A_i^a = 0 \)):

\[ e A_0^a = \tilde{x}^a J(r), \]
\[ e A_i^a = -\epsilon_{a ij} \tilde{x}_j 1/K(r), \]
\[ e \phi^a = \tilde{x}^a H(r). \]  

(2·10)

The energy finiteness condition imposes the following boundary conditions on \( K, J \) and \( H \):

\[ J \rightarrow \Theta r + b, \]
\[ K \rightarrow 0, \]
\[ H \rightarrow a r. \]  

(2·11)

First, we note a trivial observation that for \( J = 0 \) this ansatz reduces to the 't Hooft-Polaykov monopole ansatz, as it should. Second, rewriting the Lagrangian (2·1) in the form

\[ \mathcal{L} = \frac{1}{2} (E_i^a)^2 - \frac{1}{2} (B_i^a)^2 - \frac{1}{2} (D_\phi^a)^2 + \frac{1}{2} (D_\phi^a)^2 - U(\phi), \]  

(2·12)

we observe that due to the fact that \( D_\phi^a = 0 \) for static configurations \( (A_i^a \) and \( \phi^a \) are chosen to be parallel in the internal space in (2·10)) the Higgs fields and \( A_0^a \) do not directly influence each other. Again for static configurations \( E_i^a = -D_\phi^a A_i^a \). Thus we see that \( A_i^a \) acts very much like another isotriplet of Higgs field with negative metric \((E_i^a)^2 \) and \((D_\phi^a)^2 \) appear with opposite sign). To eliminate this potential source of ambiguity and to convert the theory into a form
which is suitable for canonical quantization we shall fix the gauge to be $A_0^\nu=0$, because the canonical momenta conjugate to $A_0^\nu$ do not exist. It should be noted that this is a stronger gauge condition than $A_0(x=\infty)=0$, which was sufficient for semiclassical quantization.

This gauge fixing is affected by the following gauge transformation:

$$U=\exp\left\{ \frac{i}{2} (\sigma \cdot \chi) \right\} J^a(r) \),$$

which yields the following new form for the dyon ansatz (note the general form of Witten's ansatz making its full appearance):

$$eA_0^\nu=0,$$

$$eA_\mu^\nu=-\varepsilon_{\nu\lambda J} \left[ 1 - \frac{K}{r} \cos \phi + \frac{2}{r^2} \sin \phi \right]$$

$$+ (\delta_{\mu\nu} - x_\nu \tilde{x}_\mu) \left( \frac{K}{r} \sin \phi + \tilde{x}_\mu \phi' \right),$$

$$e\phi^a=\tilde{x}^a \frac{H}{r}$$

with

$$\omega=\frac{-J}{r}$$

Note that as $r \to \infty$, $\omega \to -\vartheta t$ (also $\phi' \to 0$) yielding the results of Ref. 3). This complicated looking form is the correct form which should be used in quantizing the theory canonically.

Before getting involved with the question of quantization, let us comment on an important point: It is due to $A_\mu^\nu \neq 0(J \neq 0)$ that we have non-zero electric charge in the theory

$$Q_\nu=-8\pi \int_0^\infty dr \left( \frac{1}{2} J^\nu - \frac{1}{r^2} \right).$$

But as will be re-iterated shortly, this causes an ambiguity in the definition of the Lagrangian for the dyon sectors, which did not exist for the monopole sectors for instance. The quantity $\text{Tr}(FF)$ is non-zero for the dyon sectors

$$\text{Tr}(FF)=4\varepsilon_{\nu\lambda J} B^{\nu\lambda} \int_0^\infty \frac{dr}{r^2} \left[ \frac{1}{r} J^\nu - \frac{1}{r^2} \right].$$

Furthermore, it is a total divergence; thus we can freely add this term to the Lagrangian without affecting the equation of motion. The contribution to the energy for instance, by employing the boundary conditions (2.11), is proportional to
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\[
\int d^3x \text{Tr}(F \tilde{F}) = 4\pi \theta .
\] (2.17)

We shall see in the next section that it is this ambiguity which generates a superselection type structure in the dyon sectors of the Hilbert space.

\section{Canonical quantization}

In the \( A_3 = 0 \) gauge, \((A_1^a, E_1^a)\) and \((\phi^a, \dot{\phi}^a)\) are the canonically conjugate variables. To quantize the theory canonically, we impose the following standard equal-time commutation relation among the fields:

\[
\begin{align*}
[E_1^a(t, x), A_1^b(t, x')] &= -i \delta^{ab} \delta(t - t') \delta(x - x'), \\
[\dot{\phi}^a(t, x), \phi_b(t, x')] &= -i \delta^{ab} \delta(t - t') \delta(x - x')
\end{align*}
\] (3.1)

with all other commutators being zero.

Let us try to reproduce the equation of motion (2.4) in the Hamiltonian formalism: \( \partial_\tau A_1^a = i[H, A_1^a] \) reproduces the definition of the electric field (2.7). \( \partial_\tau E_1^a = i[H, E_1^a] \) reproduces the \( \mu = i \) component of Eq. (2.4a)

\[
\partial_\tau E_1^a = (D_x \phi)^a + e2\pi \phi \partial^0(D_0 \phi)^a.
\] (3.2)

But the \( \mu = 0 \) component of (2.4a), the Gauss law,

\[
G^a[A, \phi] = (D_1 E_1) + e2\pi \phi \partial^0 = 0
\] (3.3)

is not reproduced by the Hamilton equations.

Obviously, the Hamilton theory is larger than the Georgi-Glashow theory. It gives rise to an entirely consistent quantum mechanics, but to make it coincide with the Georgi-Glashow theory, i.e., to incorporate the Gauss law into the Hamiltonian quantum theory, we can demand that only those states of the Hilbert space which are annihilated by \( G^a \), regarded as an operator, are relevant to the Georgi-Glashow theory

\[
G^a[A] |\psi\rangle_{\text{phys}} = 0.
\] (3.4)

Let us recall that the unitary operator which implements the finite gauge transformations

\[
A_1 \rightarrow U^{-1} A_1 U + U^{-1} \dot{\phi} U,
\]

\[
U = \exp \left[ \frac{\sigma^a}{2i} \phi^a \right]
\] (3.5)

is given by

\[
\hat{G} = e^{iQ_{\omega_1}},
\]

\[
Q_{\omega_1} = \frac{1}{e} \int d^3 x E_1^a D_0^a \phi^a.
\] (3.6)
Let us perform a careful partial integration retaining the surface contribution

\[ Q_{\omega} = \frac{1}{e} \int d^3x \theta_i (E_i \phi^* \phi - \frac{1}{e} \int d^3x \phi^* \phi \phi^* C^a. \]  

(3.7)

Thus we see that for the conventional gauge transformations, with vanishing parameters, \( \omega^a \rightarrow 0 \), the first term vanishes and thus the Gauss law constraint would guarantee the gauge invariance of the physical states

\[ e^{iQ_{\omega}} |\text{phys}\rangle = \exp \left\{ - \frac{i}{e} \int d^3x \phi^* \phi \right\} |\text{phys}\rangle = |\text{phys}\rangle. \]  

(3.8)

For the case in hand, this asymptotic condition is not satisfied. For the finite rotation about the \( \hat{\phi} \) axis, \( \omega = (\omega / |\phi\rangle \langle \phi|) \phi^* \). Due to the finiteness of the energy, the Higgs vacuum is defined at the spatial boundary, thus \( \omega \rightarrow (\omega / |\phi\rangle \langle \phi|) \phi^* \). Thus the Gauss law constraint is not sufficient to enforce the gauge invariance on the physical states. For instance, for a full rotation around the \( \hat{\phi} \) axis, we get

\[ e^{iQ_{\omega}} |\psi\rangle = e^{i\pi(\hat{\phi} \cdot \phi^*)} |\psi\rangle, \]  

(3.9)

where \( Q_c \) is the electric charge operator. Thus we see the signs of emergence of an arbitrary angle exactly as in the instanton case. At first glance we get the impression that it may be a gauge artefact, because we are forced to introduce it to reinforce the gauge invariance of the physical states. This is not the case here, however, since the Hamiltonian is invariant under any gauge transformations, \( \mathcal{G} = e^{iQ_{\omega}} \) should commute with \( H \), \([H, \mathcal{G}] = 0\). Thus the physical dyon states should also be eigenstates of \( \mathcal{G} \):

\[ \mathcal{G} |\text{phys}\rangle = e^{-i\theta} |\text{phys}\rangle. \]  

(3.10)

Again, as in the instanton case, we have no a priori way of computing \( \theta \). However it can be shown by following the Jackiw's treatment of the instanton angle that it is not an artificial quantity arising from some peculiarity of our treatment of gauge invariance (thus cannot be eliminated). Here is a brief outline of this proof adapted to the dyon problem:

Suppose that there is no angle in (3.10), thus the states are invariant under large gauge transformations. But we must take account of the ambiguity in the definition of the Lagrangian

\[ \mathcal{L} \rightarrow \mathcal{L} - i \theta \frac{e^2}{32\pi^2} F_{\mu \nu} F^{\mu \nu}. \]  

(3.11)

where we have deliberately used the instanton normalization, for the same topological reasoning. The appearance of "i" in front of \( \theta \) is due to the fact that
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Self-dual fields in Minkowski space are complex, i.e., \( \tilde{E}_i = iB_i, \tilde{B}_i = iE_i \); thus \( \text{Tr} (\tilde{F}\tilde{F}) = 4iE_iB_i \). The Hamiltonian has the same form as earlier, but the definition of the canonical momentum conjugate to \( A_i \) has changed:

\[
H_i^\theta = \frac{\partial L}{\partial (\partial_\mu A_i^\theta)} = E_i^\theta + \frac{e^i}{8\pi^2} \theta B_i^\theta .
\] (3.12)

Thus the \( \theta \)-dependent energy density is

\[
\theta_{\text{tot}} = \frac{1}{2} \left( \left( H_i^\theta - \frac{e^i}{8\pi^2} \theta B_i^\theta \right)^2 + (B_i^\theta)^2 + (\pi^\theta)^2 + (D\phi^\theta)^2 \right) + U(\phi). \] (3.13)

We write a functional Schrödinger equation for the state functionals which are invariant under large gauge transformations (denoted with a prime), then employing the Jackiw transformations

\[
\mathcal{F}[A, \phi] = e^{-(i/\sqrt{2})W[A]} \mathcal{F}'[A, \phi] \]

with the functional \( W[A] \) satisfying the equation

\[
\frac{\partial W[A]}{\partial A_i^\theta} = \frac{e^i}{8\pi^2} B_i^\theta \]

we can eliminate the \( \theta \) dependence and get a \( \theta \)-independent Schrödinger equation for the new state functional \( \mathcal{F} \). Equation (3.15) may be integrated to yield

\[
W[A] = - \frac{e^i}{8\pi^2} \int d^4x \varepsilon_{ijk} \text{Tr} \left[ A_i \left( \partial_j A_k + \frac{2e}{3} A_j A_k \right) \right], \]

(3.16)

where the matrix notation \( A_i = iT^a A^a_i \) has been employed (\( T^a \) are the generators of the adjoint representation of the gauge group for \( SO(3) \), \( T^a \phi = -i\phi \) for instance). The quantity \( W[A] \) is recognized as the winding number of \( A_i \), \( \int d^4x K_a \), where \( K_a \) is the famous four-vector defined by

\[
\partial_\mu K^\mu = \frac{e^i}{16\pi^2} F^a_{\mu\nu} \varepsilon^{\mu\nu\rho\sigma} .
\]

That the gauge fields of the Georgi-Glashow model may have non-trivial topology is one of the novel aspects of this work and may come as a surprise to those who are familiar with the monopole sectors of the theory. Indeed, we recall that in the monopole sectors the gauge and the Higgs fields do not have simultaneous topology. Usually the preference there is given to the gauge where

\[ \theta_{\text{tot}} \rightarrow \theta_{\text{tot}} + \theta \left( E_i^\theta + \frac{e^i}{8\pi^2} B_i^\theta \right) A_i^\theta . \]
Higgs fields have the non-trivial topology, whereas gauge fields are topologically trivial. (In that case the compactification, \( g \rightarrow I \), is not possible due to the slow decrease of the monopole field asymptotically. Thus the topology of the gauge field is defined by \( \pi_1(G) \) which vanishes. It is not an obstacle to using the Cartan theorem that \( G = SO(3) \) is doubly connected. Because when the matters like existence of spin is not in question, we have the freedom of working in the universal covering space.) The Higgs field topology is described by the first homotopy group of the little group \( H \), thanks to Cartan's celebrated theorem\(^6\)

\[
\pi_1(G/H) \approx \pi_1(H) = \mathbb{Z}.
\]

Charged sectors of the Hilbert space are due to the existence of a second Higgs-like object (the temporal component of the gauge field) that is how we depart from the monopole physics. It can be shown\(^{10}\) that the topology of the \( A_0 \) is transferred on \( A_i \) due to our gauge-fixing transformation. The new form of \( A_i \) in the \( A_0 = 0 \) gauge, contains the \( A_0 \) and its topology in disguise and for the charged sectors of the Hilbert space the gauge fields and Higgs fields have equivalent topology, in the sense of \( \text{Woo,}^{10} \) which is labelled by the same integer, \( W[A] \) given in (3-16). (This equivalence has very important consequences because, for example, this implies that the topology of gauge fields is non-trivial if and only if the same is true for the Higgs fields.) If this had not turned out to be the case, say the gauge fields still had a trivial topology as in the monopole case, we would have eliminated the \( \theta \) angle from the theory altogether, even though we had to include it in the Lagrangian explicitly due to the ambiguity in its definition (due to \( \text{Tr} (FF) \) being a non-zero quantity for charged states), or had it appeared as a phase ambiguity in the physical dyon states, associated with the gauge invariance under large gauge transformations (as can be seen from the following discussion, especially Eqs. (3-17) and (3-18)).

The transformation law of \( \mathcal{F}[A, \phi] \) which satisfies a \( \theta \)-independent functional Schrödinger equation is determined by the behaviour of \( W[A] \). Under large gauge transformations with winding number \( n \), \( W[A] \) would transform like\(^9\)

\[
U(n): W \rightarrow W + n
\]  

thus inducing for \( \mathcal{F} \) the following transformation:

\[
U(n): \mathcal{F}[A, \phi] \rightarrow e^{-i(n+\theta)\phi} \mathcal{F}[A, \phi],
\]

which shows that the states \( \mathcal{F}[A, \phi] \) which satisfy a \( \theta \)-independent functional Schrödinger equation are only phase-invariant under large gauge transformations, a result which we wanted to evade in the first place.

\section*{§ 4. No instantons in non-Abelian Higgs models}

To erase the slightest doubt, that the \( \theta \) angle might be related to the instanton
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...effects, we shall now show that the non-Abelian Higgs models do not have instantons anyway (thus there is no ambiguity of these two different sources of $\theta$ effects competing against each other) and the emergence of $\theta$ in the GG model is in accord with the axioms of relativistic quantum field theory due to the existence of superselection (charged) sectors.\textsuperscript{10}

Let us recall that the pure Yang-Mills theory has instantons only in four dimensions, a result which can best be seen, for instance, from the elegant relation of Deser.\textsuperscript{13}

\begin{equation}
\int d^d x \theta_l^i = 0 ,
\end{equation}

where the superscript $d$ denotes the number of space dimension.

The trace of the space part of the energy-momentum tensor is easily obtained as follows:

\begin{equation}
\theta^i = \frac{1}{2} (d-2) (E_i^n)^2 + \frac{1}{2} (4-d) (B_i^n)^2 - \frac{1}{2} (d-2) (D_i \phi^n)^2 - dU(\phi).
\end{equation}

When $\phi$ is absent, (4·1) and (4·2) yield, as they should, the familiar result that the instantons exist only in $d+1=4$ dimensions. Since instantons in $M_i$ can be thought of as the static finite energy configuration in $M$, let us first specify the restriction imposed by the staticity condition. For static configuration, $\mathcal{H} = -\mathcal{F}$. Substituting (3·9) and (3·12) in this equation yields

\begin{equation}
(E_i^n)^2 + (\pi^n)^2 = 0 ,
\end{equation}

which is satisfied only for

\begin{equation}
E_i^n = 0 , \quad \pi^n = D_i \phi^n = 0 .
\end{equation}

The first condition in (4·4) is the same as in the pure Yang-Mills theory, i.e., that all static configurations should be of magnetic type. The second states that $A_i^n$ is parallel to $\phi^n$. Thus we see that the static $E<\infty$ configuration should satisfy

\begin{equation}
\int d^d x \left[ \frac{1}{2} (4-d) (B_i^n)^2 - \frac{1}{2} (d-2) (D_i \phi^n)^2 - dU(\phi) \right] = 0 .
\end{equation}

For $d=4$ (which is the relevant dimension for instantons) this reduces to

\begin{equation}
\int d^d x \left[ (D_i \phi^n)^2 + 4U(\phi) \right] = 0 ,
\end{equation}
which is satisfied only for

\[ D \phi^a = 0, \quad (4.7a) \]

\[ U(\phi) = 0. \quad (4.7b) \]

This states that all the space is filled by the Higgs vacuum. \( (4.7b) \) implies \( \partial_\nu \phi^a = 0 \), which, when combined with \((4.7b)\), yields

\[ D \phi^a = e^{abc} A_\mu \phi^b = 0. \quad (4.8) \]

Thus \( A_\nu \) is parallel to \( \phi^a \). Since in the Higgs vacuum \( \hat{\phi} \) specifies a constant direction in the isospin space, \( A_\nu \phi^a \) essentially defines an Abelian theory, along \( \hat{\phi} \). Note that for static \( E < \infty \) configurations, the equation of motion \((3.2)\) is reduced to (defining the direction specified by \( \hat{\phi} \) to be the third direction),

\[ \partial_\nu F_{\nu\zeta} = 0, \quad (4.9) \]

from which we conclude that \( B_{\alpha-\beta} = \frac{1}{2} \delta_{\alpha\beta} F_{\mu=0} = 0 \) together with the other components \( B_{\zeta}^{\alpha-1,0} \). This clearly proves that no static \( E < \infty \) configurations exist in 4+1 dimensions, thus there are no instantons in 3+1 dimensions in the non-Abelian Higgs models, although they existed in the pure Yang-Mills theory. Thus we see a very interesting thing happening: The Higgs fields cause the instantons to disappear, replacing them with finite energy sectors—the monopole and the dyon sectors. Indeed, our theorem for \( d = 3 \) yields

\[ \int d^4x \left[ \frac{1}{2} (B_\mu)^2 - \frac{1}{2} (D \phi^a)^2 - 3U(\phi) \right] = 0. \quad (4.10) \]

The integrand is not a positive definite quantity. Thus this equation can be satisfied by any non-zero values of \( B_\mu \) and \( \phi^a \) satisfying the equation

\[ \frac{1}{2} (B_\mu)^2 - \frac{1}{2} (D \phi^a)^2 = 3U(\phi). \quad (4.11) \]

Thus static \( E < \infty \) configurations may exist in the Georgi-Glashow model. The fact that we cannot make a stronger statement than this, is due to the weakness of the theorem \((4.1)\). It serves more effectively as a no-go theorem. When it comes to positive statements, it says that those configurations may exist. Proof would be explicit construction, which we happen to have for the GG model, the monopoles and the dyons.

§ 5. Lorentz invariance

The obstacle to canonically quantizing the theory was eliminated by choosing the gauge \( A_\nu = 0 \). But we are aware of the fact that this desire to clarify the quantum nature of the dyon sector is not without an expense. In spite of the fact that the

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superselection sectors are now exhibited by the Lorentz invariant term, $\text{Tr}(FF)$, manifest Lorentz invariance is given up in the Hamiltonian formalism.

A sufficient condition for showing that Lorentz invariance is maintained is that the famous Schwinger algebra is satisfied:

\[ i[\theta_\alpha(t, x), \theta_\beta(t, x')] = [\theta_\alpha(t, x) + \theta_\beta(t, x')] \partial_\gamma (x-x'), \quad (5.1) \]

which is supplemented with the condition that the $\theta_\alpha$ should not have an explicit dependence on the spatial co-ordinate. The equivalence of this commutation to the inhomogeneous Lorentz algebra can easily be seen from the integrated form

\[ \frac{1}{2} \int d^3x d^3y (x-y) [\theta_\alpha(t, x), \theta_\beta(t, y)] = -iP_\gamma, \quad (5.2) \]

where $P_\gamma = f d^3x \partial_\gamma$ is the field momentum operator, which is given explicitly in terms of the fields as

\[ P_\gamma = \int d^3x \{ i \epsilon \tau_{
abla} \hat{B}_i^\gamma(x) + \Pi_i^\gamma D_j^\gamma A_i^\beta + \pi^\gamma D_j^\gamma \partial_i^\beta - e \varepsilon^{ijk} A_i^\gamma \pi^\rho \phi^\rho \}. \quad (5.3) \]

Note that $P_\gamma$ does not have any explicit $\theta$ dependence. Thus to show that (5.4) is satisfied it would be sufficient to see that all the $\theta$ dependent contributions to the left-hand side vanish. We shall not show all this complicated algebra here, but only present the computation of the most dangerous term as far as Lorentz invariance is concerned. Given the fact that the $\theta$ labelled superselection sectors are boundary sustained sectors, then the most dangerous terms would be gauge Higgs cross terms. Local terms clearly would give vanishing contributions, thanks to the antisymmetric $(x-y)$ term in the front

\[ X = -\theta \frac{e^2}{16\pi^2} \left[ \Pi_i^\gamma(x) B_i^\gamma(x), (D_j^\gamma(y)) \right]^2. \quad (5.4) \]

To compute this, we first observe that

\[ [\Pi_i^\gamma(t, x), B_j^\gamma(t, y)] = i\epsilon \delta_{ij} D_k^\gamma \partial^\gamma (x-y) \quad (5.5) \]

and

\[ [\Pi_i^\gamma(t, x), D_j^\gamma \phi^\rho(t, y)] = -i\epsilon \delta_{ij} \varepsilon^{\rho \gamma \delta} \delta^\gamma (x-y). \quad (5.6) \]

Using (5.4) and (5.5), we can now easily compute $X$:

\[ X = i\theta \frac{e^2}{16\pi^2} \varepsilon^{\rho \gamma \delta} B_i^\gamma (D_j^\gamma \phi^\rho) \delta^\gamma (x-y). \quad (5.7) \]

This also has a vanishing contribution, thanks to the antisymmetric factor $(x-y)$. Thus we have checked the consistency of the Hamiltonian method. It is gratifying that the existence of charged states in non-Abelian Higgs models does not get into
§ 6. Dyons of charge $e(\theta/2\pi)$

In § 3 we have observed\(^\text{4)}\) the striking property that even the large gauge transformations are topologically trivial, a result reflecting the presence of $\theta$ dependence in leading order. We shall reproduce Witten's proof of $\theta$ dependence of dyon charge for completeness and logical continuity, within our systematic framework.

In the representation where all the $\theta$ dependence is contained in the Lagrangian, so that physical states can be chosen invariant under large gauge transformation (since even the large gauge transformations can be obtained by iterating the infinitesimal ones from now on we can easily drop the word large), the Noether charge generating a $2\pi$ rotation around $\phi$ is given by ($\omega_\alpha=2\pi$)

$$Q = \frac{1}{e\omega_\alpha} \int d^3 x \Pi_\phi D_\phi \omega^\phi. \quad (6\cdot1)$$

Recalling that $\omega^\phi = \phi^\phi (\omega/|\phi|)$ and using the definition (3.12) of $\Pi_\phi$, we can perform a partial integration and obtain,

$$Q = \frac{1}{e\alpha} \int d^3 x \partial_\alpha [\Pi_\phi^\phi^{\phi^\phi}] - \frac{1}{e\omega_\alpha} \int d^3 x \omega^\phi D_\phi \Pi_\phi^\phi. \quad (6\cdot2)$$

Using equations of motion (3.4a), we get for the second term

$$\omega^\phi D_\phi \Pi_\phi^\phi = -e\omega_\alpha \alpha \frac{\phi^{\phi^\phi}}{|\phi|} + \theta \frac{\phi^{\phi^\phi}}{8\pi^2} \frac{\omega}{|\phi|} D_\phi \Pi_\phi^\phi, \quad (6\cdot3)$$

which clearly vanishes, the first term due to symmetry and the second due to the Bianchi identity. Thus on the physical dyon states the operator

$$e^{\pm \theta} = \text{exp} \left\{ \frac{\pm 2\pi}{e} \frac{1}{\alpha} \int d^3 x \partial_\alpha (\Pi_\phi^\phi^{\phi^\phi}) \right\} \quad (6\cdot4)$$

acts like a unit operator:

$$e^{\pm \theta} |\text{phys}\rangle = |\text{phys}\rangle, \quad Q = \frac{1}{e} \left( Q_\alpha + \theta \frac{\phi^{\phi^\phi}}{8\pi^2} Q_\phi \right). \quad (6\cdot5)$$

It is known\(^\text{4)}\) for this model the smallest value for the magnetic charge is $Q_m^{(\text{min})} = 4\pi/e$. This is due to the fact that one can introduce isospin particles of charge $e/2$ in this theory. Thus the eigenvalues of the operator $Q$ on the dyon states with fixed magnetic charge $4\pi/e$ is given by

$$Q = ne - \frac{\theta}{2\pi} e. \quad (6\cdot6a)$$

\(^4\) Recently there has been an interesting claim that preparation of charged states causes spontaneous violation of Lorentz invariance in QED.\(^\text{4)}\)
Electric Charge as the Source of CP Violation

Had we chosen to work in the representation in which all the $\theta$ dependence is contained in the phase invariance property of the dyon states (and no explicit dependence on $\theta$ in the Lagrangian) then $Q$ would be just the electric charge operator. And comparing (3.9) and (3.10) we would get for the eigenvalue of $Q$ on the dyon states

$$\frac{Q_0}{e} = -\frac{\theta}{2\pi} \quad (\text{mod} \ 2\pi) \quad (6.6b)$$

It is amusing to note that $\theta \to 0$, as $Q \to 0$ (or vice versa); thus CP violation and the existence of (electrically) charged states go hand in hand.

§ 7. Concluding remarks

The arguments presented above can easily be generalized to the grand unified $SU(5)$ theory.\textsuperscript{10} Let us sketch the essentials of this generalization briefly:

The $SU(5)$ gauge symmetry is broken down spontaneously to $SU(3)^f \times U(1)$ into two steps. The minimal scheme to arrange this symmetry breaking is to introduce two Higgs multiplets

$$SU(5) \rightarrow SU(3)^f \times SU(2)_L \times U(1)_Y \rightarrow SU(3)^f \times U(1). \quad (7.1)$$

The first Higgs transforms according to the 24-dimensional (real) adjoint representation of $SU(5)$ (it is written as a $5 \times 5$ hermitian traceless matrix). The second transforms according to five-dimensional (complex) fundamental representation (a column vector).

The little group of $\langle \phi_1 \rangle$ is $H_1 = SU(3)^f \times SU(3) \times U(1)/Z_6$, whilst that of the pair $\langle \phi_1, \phi_2 \rangle$ together is $H_{1,2} = SU(3)^f \times U(1)$. The vacuum manifold for the first step breaking, and for the first and second steps combined together is

$$M_{10} = SU(5)/H_1,$$
$$M_{11,12} = SU(5)/H_{1,2}. \quad (7.2)$$

$G = SU(5)$ being simply connected, the topology is determined by $H_1$ and $H_{1,2}$

$$\pi_1(G/H_1) = \pi_1(H_1),$$
$$\pi_1(G/H_{1,2}) = \pi_1(H_{1,2}). \quad (7.3)$$

Thus the relevant homotopy groups for the existence of static finite energy configurations are $\pi_1(H_1)$ and $\pi_1(H_{1,2})$. These can be computed using the homotopy sequence of a bundle,\textsuperscript{11} easily as

$$\pi_1(H_1) = Z,$$
$$\pi_1(H_{1,2}) = Z. \quad (7.4)$$

Thus as far as the topological structure is concerned, $SU(5)$ shares all the nice...
features of $SO(3)$. This result is not as trivial as it sounds, given the fact that $H_i$ and $H_{1,2}$ are semisimple groups (containing three and two simple factors, respectively). That one needs one integer to label each homotopy class of $\pi_1(H_i)$ and $\pi_1(H_{1,2})$ is quite an impressive result. 10

Thus the charged $E<\infty$ sectors of the theory would show up as super-selection sectors and would be labelled by a (intrinsically) CP violating angle which is present to leading order. Again this angle has nothing to do with the instantons, because of our proof in §4 that the non-Abelian Higgs model has no instantons but only finite energy (static) configurations.

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Note added: After the submission of this paper for publication it was brought to our attention that N. Christ and R. Jackiw (Phys. Letters 91B (1980), 228), also investigated the connection between pontryagin index and the magnetic charge strength for the dyon sectors, which partially overlaps with some of our topological considerations in §3.