Equilibrium Condition in the Axisymmetric $N$-Reissner-Nordström Solution

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We solve the source-free Einstein-Maxwell equations in a static and axisymmetric spacetime by using the inverse scattering method that is one of the soliton techniques to solve the gravitational field equation. We obtain an exact solution that describes $n$ Reissner-Nordström black holes located along the symmetry axis. We define the mass and charge of each black hole and study the condition for static equilibrium of black holes. We show that the equilibrium is realized only when all the black holes are extremal and the spacetime becomes Majumdar-Papapetrou type.

§ 1. Introduction

Ever since the soliton techniques were first applied to solving the gravitational field equation we have been able to study the structure of many-body system in general relativity using exact solutions. Although the obtained solutions are static or stationary it may be important to study them as a first step toward considering a dynamical process such as black hole collision. The simplest example of many-body system is the axisymmetric $n$-Schwarzschild solution, which is obtained as a $2n$-soliton solution to the vacuum Einstein equation. This solution includes conical singularities between Schwarzschild black holes along the symmetry axis. These conical singularities, which we call struts, are necessary to uphold the gravitational attractive force between the black holes. Static balance without struts could be realized by introducing negative masses but naked singularities appear.

Another way of static balance is to introduce some fields with repulsive force. Scalar fields act attractively and also break event horizons of black holes, which brings about naked singularities again. Repulsive Coulomb force in electromagnetic field can balance attractive gravitational one and leads the system to static equilibrium. The solution given by Majumdar and Papapetrou is known as such a solution. This solution is an exact solution of the source-free Einstein-Maxwell equations in a conformastatic spacetime and describes a system of extremal black holes.

In a more general axisymmetric spacetime Ohta-Kimura studied the condition for equilibrium of two charged particles in the post-post-Newtonian approximation. From their analysis of the static potential they obtained the relation $m_i = |e_i|$, where $m_i$ and $e_i$ are the mass and charge of the $i$-th particle, respectively.

In an exact solution describing two Reissner-Nordström black holes, Tomimatsu first discussed their equilibrium. He used the Cosgrove solution that was obtained by means of the electrovac generalization of Neugebauer's transformation. In order to get the condition for equilibrium he also used a property of black hole that the surface gravity is constant over the horizon, because the solution was given to only
a part of metric. The obtained result was the same as that of Ohta-Kimura but it was a sufficient condition in this case.

In this paper we apply the inverse scattering method, which is one of the soliton techniques to obtain exact solutions to the gravitational field equation, to solving the static and source-free Einstein-Maxwell equations. We show that the solution in this method can be expressed in a compact form with "pole trajectories", which makes various calculations easier as we shall see later. We give the explicit expressions of all the metric functions and the electrostatic potential that correspond to the axisymmetric 2n-soliton solution. We show that the 2n-soliton solution describes n Reissner-Nordström black holes and struts between them located along the symmetry axis. We define the mass and charge of each black hole in terms of surface integrals and consider the condition for static equilibrium in the axisymmetric n-Reissner-Nordström solution. The word "static equilibrium" here means that there are no struts between black holes, and we have the same relation as those of Ohta-Kimura and Tomimatsu as a necessary and sufficient condition for equilibrium. This result indicates that each black hole should be extremal and the spacetime be Majumdar-Papapetrou type so that the static equilibrium may be realized.

In the next section we sketch how we solved the Einstein-Maxwell equations and present an exact solution that corresponds to the 2n-soliton solution. In §3 we study the structure of the solution obtained in the previous section and show that it can be interpreted as n-Reissner-Nordström black hole solution. We give the definitions and calculations of mass and charge of each black hole in §4. The last section is devoted to discussion on the condition for static equilibrium. We use the units \( c = G = 1 \) throughout this paper.

\section{2n-soliton solution to the Einstein-Maxwell equations}

In this section we apply the inverse scattering method to solving the source-free Einstein-Maxwell equations and give a static and axisymmetric exact solution corresponding to the 2n-soliton solution.

Let us consider the metric in the canonical cylindrical coordinates:

\[ ds^2 = f^{-1} [Q(dp^2 + dz^2) + \rho^2 d\phi^2] - f dt^2, \tag{2.1} \]

where \( f \) and \( Q \) are functions of \( \rho \) and \( z \). The source-free Einstein-Maxwell equations are given by

\[ R^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu}, \tag{2.2} \]

\[ F^{\mu\nu} = 0 \tag{2.3} \]

with

\[ T^{\mu}_{\nu} = \frac{1}{4\pi} \left( F^{\mu\sigma}F_{\sigma\nu} - \frac{1}{4} \delta^{\mu\sigma}F^{\alpha\beta}F_{\alpha\beta} \right), \tag{2.4} \]

\[ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{2.5} \]

In the metric (2.1) Eqs. (2.2) and (2.3) are written as
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\[
(\ln f)_{,\phi} + (\ln f)_{,\phi} + \rho^{-1}(\ln f)_{,\phi} = 2f^{-1}(\chi^2_{,\phi} + \chi^2_{,z}),
\]

\[
(\ln Q)_{,\phi} = \frac{\rho}{2}[(\ln f)_{,\phi} - (\ln f)_{,z}^2] - 2\rho f^{-1}(\chi^2_{,\phi} - \chi^2_{,z}),
\]

\[
(\ln Q)_{,z} = \rho(\ln f)_{,\phi}(\ln f)_{,z} - 4\rho f^{-1}\chi_{,\phi}\chi_{,z},
\]

\[
\chi_{,\phi} + \chi_{,z\phi} + \rho^{-1}\chi_{,\phi} = \chi_{,\phi}(\ln f)_{,\phi} + \chi_{,z}(\ln f)_{,z},
\]

where we have written the electrostatic potential as \( A_0 = -\chi \). To solve these coupled differential equations we adopt an assumption that the solution should be the Reissner-Nordström type, that is, the first non-trivial solution given by setting \( n = 1 \) should be the Reissner-Nordström solution. This assumption implies that the metric function \( f \) is a function of \( \chi \) and this leads to the relation

\[
f = 1 - 2c\chi + \chi^2,
\]

where \( c \) is an arbitrary constant. This reduces the differential equations that we should solve to

\[
\chi_{,\phi} + \chi_{,z\phi} + \rho^{-1}\chi_{,\phi} = 2f^{-1}(\chi - c)(\chi^2_{,\phi} + \chi^2_{,z}),
\]

\[
(\ln Q)_{,\phi} = 2\rho f^{-1}(c^2 - 1)(\chi^2_{,\phi} - \chi^2_{,z}),
\]

\[
(\ln Q)_{,z} = 4\rho f^{-1}(c^2 - 1)\chi_{,\phi}\chi_{,z}.
\]

We now introduce a new function \( R(\rho, z) \) with constants \( e \) and \( m \) by

\[
\chi = \frac{e}{R + m},
\]

where \( m \) is related to \( e \) through the constant \( c \) as \( m = ce \). By using the function \( R \) we can write \( f \) as

\[
f = \frac{R^2 - d^2}{(R + m)^2},
\]

and Eq. (2.11) as

\[
R_{,\phi} + R_{,z\phi} + \rho^{-1}R_{,\phi} = 2R(R^2 - d^2)^{-1}(R^2_{,\phi} + R^2_{,z}),
\]

where \( d^2 = m^2 - e^2 \). We further introduce the function \( \bar{f} \) by

\[
R = d^{1/2} \frac{1 + \tilde{f}}{1 - \tilde{f}},
\]

which transforms Eq. (2.16) into the linear differential equation

\[
(\ln \bar{f})_{,\phi} + (\ln \bar{f})_{,z\phi} + \rho^{-1}(\ln \bar{f})_{,\phi} = 0.
\]

We can also rewrite Eqs. (2.12) and (2.13) by using the function \( \bar{f} \) as

\[
(\ln Q)_{,\phi} = \frac{\rho}{2}[(\ln \bar{f})_{,z} - (\ln \bar{f})_{,z}],
\]

\[
(\ln Q)_{,z} = \rho(\ln \bar{f})_{,\phi}(\ln \bar{f})_{,z}.
\]
These expressions (2·18)-(2·20) are exactly the same as those for the vacuum Einstein equation in the metric (2·1) with \( f \) instead of \( f \). As the soliton solution to Eq. (2·18) and the corresponding solution to Eqs. (2·19) and (2·20) by means of the inverse scattering method have already been given in Ref. 1, we can construct a solution to the Einstein-Maxwell equations by using them. The constructed solution through the transformation (2·17) may describe a many-body system composed of the Einstein-Maxwell field. We shall study the structure of solution in the next section.

We here give the expressions of 2\( n \)-soliton solution for \( \tilde{f} \) and the corresponding solution for \( Q \):

\[
\tilde{f} = \prod_{k} (i\mu_{k}/\rho)^{q_{k}}, \quad (2·21)
\]

\[
Q = \frac{\rho^{D/2} \prod_{k=1} \left( \mu_{k} - \mu_{j} \right)^{2q_{k}q_{j}}}{\prod_{k} \left( \mu_{k}^{2} + \rho^{2} \right)^{q_{k}} \prod_{l} \mu_{l}^{(D-2q_{l})q_{l}} C^{(2n)}}. \quad (2·22)
\]

In these solutions \( \mu_{k}'s \) are the pole trajectories

\[
\mu_{k} = w_{k} - z + (-1)^{k-1} \sqrt{(w_{k} - z)^{2} + \rho^{2}}, \quad (k=1, 2, \ldots, 2n) \quad (2·23)
\]

where \( w_{k}'s \) are constants. The positive constants \( q_{k} \) appearing in Eqs. (2·21) and (2·22) are related to the distortion parameters \( \delta_{i} \) by \( q_{2i-1} = q_{2i} = \delta_{i} (i=1, 2, \ldots, n) \), and \( D = \sum_{k}^{2n} q_{k} \). The constant \( C^{(2n)} \) in Eq. (2·22) is given by

\[
C^{(2n)} = \prod_{i>j} [2^{2} (w_{2i}-w_{2j})(w_{2i-1}-w_{2j-1})]^{2\delta_{ij}}, \quad (2·24)
\]

which has been determined so that the solution may satisfy the asymptotic flatness condition. In order to see that the constructed solution includes the Reissner-Nordström solution we consider the 2-soliton \( (n=1) \) case:

\[
\tilde{f} = (-\mu_{1}\mu_{2}/\rho^{2})^{z_{1}}, \quad (2·25)
\]

\[
Q = \left[ \frac{\rho^{2}(\mu_{1}-\mu_{2})^{2}}{\left( \mu_{1}^{2} + \rho^{2} \right) \left( \mu_{2}^{2} + \rho^{2} \right)} \right]^{\delta_{1}^{2}}, \quad (2·26)
\]

where

\[
\begin{align*}
\mu_{1} &= z_{1} - z - d + \sqrt{(z_{1} - z - d)^{2} + \rho^{2}}, \\
\mu_{2} &= z_{1} - z + d - \sqrt{(z_{1} - z + d)^{2} + \rho^{2}}.
\end{align*} \quad (2·27)
\]

Here we have written the constants in the pole trajectories as \( w_{1} = z_{1} - d \) and \( w_{2} = z_{1} + d \). When \( \delta_{1} = 1 \), we introduce the spherical coordinates \( (r, \theta) \) by

\[
\rho = \sqrt{(r-m)^{2} - d^{2}} \sin \theta, \quad z - z_{1} = (r-m) \cos \theta, \quad (2·28)
\]

and obtain

\[
R = r - m, \quad (2·29)
\]

which leads to the solution.
This shows that the solution given by Eqs. (2.25) and (2.26) becomes the Reissner-Nordström solution in this case. When \( \delta_1 \neq 1 \), the solution is not spherically symmetric and we may call it the charged-Weyl solution.

\section*{§ 3. Structure of 2n-soliton solution}

In this section we study the structure of the solution given by Eqs. (2.21) and (2.22) and show that it describes a system of \( n \) Reissner-Nordström black holes or \( n \) charged Weyls. In the solution under consideration the electrostatic potential \( \chi \) and metric function \( f \) are given through the transformation (2.17) by Eq. (2.14) and Eq. (2.15), respectively. The metric function \( Q \) is given by Eq. (2.22). We shall consider the situation that all the black holes or Weyls are separated and have positive masses. For this purpose we assume that the constants \( w_k \) in the pole trajectories (2.23) satisfy

\[
w_1 < w_2 < \cdots < w_{2n-1} < w_{2n},
\]

and parameterize them by using the positive constants \( d_i \) as

\[
w_{2i-1} = z_i - d_i, \quad w_{2i} = z_i + d_i. \quad (i = 1, 2, \cdots, n)
\]

Let us first consider the asymptotic behavior of solution at spatial infinity. When \( \sqrt{\rho^2 + z^2} \to \infty \), \( \bar{f} \) behaves as

\[
\bar{f} \sim 1 - \frac{2}{\sqrt{\rho^2 + z^2}} \sum_{i=1}^{n} \delta_i d_i.
\]

This gives

\[
R \sim \frac{d}{\sum \delta_i d_i} \sqrt{\rho^2 + z^2},
\]

and the asymptotic behaviors of \( \chi \) and \( f(= -g_{tt}) \):

\[
\chi \sim \frac{e}{d} \frac{\sum \delta_i d_i}{\sqrt{\rho^2 + z^2}},
\]

\[
f \sim 1 - \frac{2m}{d} \frac{\sum \delta_i d_i}{\sqrt{\rho^2 + z^2}}.
\]

From these behaviors we obtain the total charge and mass of the system described by the solution:

\[
e_{\text{total}} = \frac{e}{d} \sum_{i=1}^{n} \delta_i d_i,
\]

\[
m_{\text{total}} = \frac{m}{d} \sum_{i=1}^{n} \delta_i d_i.
\]
Alternatively we find that these total charge and mass are given in terms of the surface integrals at spatial infinity

\[ e_{\text{total}} = \frac{1}{4\pi} \lim_{r \to \infty} \int F_{\mu\nu} \sqrt{-g} \, d\sigma, \quad (3.9) \]

\[ m_{\text{total}} = \frac{1}{4\pi} \lim_{r \to \infty} \int \xi^\mu \sqrt{-g} \, d\sigma, \quad (3.10) \]

where \( \xi^\mu \) is the timelike Killing vector defined by \( \xi^\mu = \delta_0^\mu \).

We next consider the behavior of solution around the z-axis defined by \( \rho = 0 \). In the regions \( w_{2i-1} < z < w_{2i} \) (where \( i = 1, 2, \ldots, n \)) along the z-axis \( \bar{f} \) behaves as \( \bar{f} \sim \rho^{2\delta_i} \), which gives

\[ f \sim \rho^{2\delta_i} \quad (3.11) \]

and

\[ Q \sim \rho^{2\delta_i}. \quad (3.12) \]

When \( \delta_i = 1 \) we have

\[ f(= -g_{00}) \sim \rho^2, \quad f^{-1}Q(= g_{11} = g_{22}) \sim \rho^0. \quad (3.13) \]

These behaviors of metric functions are the same as those on the horizon in the Reissner-Nordström solution. Therefore, if all the \( \delta_i \)'s are equal to 1, we may interpret that the solution describes \( n \) Reissner-Nordström black holes located in the regions \( w_{2i-1} < z < w_{2i} \) (where \( i = 1, 2, \ldots, n \)) along the z-axis (Fig. 1). If \( \delta_i \)'s are not equal to 1, \( f^{-1}Q \) becomes 0 or infinity at \( \rho = 0 \). This means that these regions are naked singularities and the solution describes \( n \) charged Weyls in this case. In the outer regions \( z < w_1 \) and \( z > w_{2n} \) along the z-axis \( Q = 1 \), which ensures asymptotic flatness of the solution. However, in the regions \( w_{2i} < z < w_{2i+1} \) (where \( i = 1, 2, \ldots, n - 1 \)) along the z-axis, which represent the regions between black holes or Weyls, \( Q \neq 1 \) in general. This implies that the z-axis does not have Euclidean nature in these regions and the so-called struts exist there. These struts may be unnecessary if the balance between the gravitational and Coulomb force is realized. We consider the condition for this balance in §5.

\section*{4. Definitions of charge and mass}

In the previous section we found that the solution describes a system of \( n \) Reissner-Nordström black holes or charged Weyls aligned along the z-axis and obtained the total charge and mass of the system. However, we do not know the charge and mass of each black hole or charged Weyl yet. Therefore we give their definitions in this section.
Let us first consider the definition of charge. The charge $e_i$ of the $i$-th black hole or charged Weyl can be evaluated from the flux integral over a surface $S_i$:

$$e_i = -\frac{1}{4\pi} \oint_{S_i} \mathcal{F} \sqrt{-g} \, d\sigma,$$  

(4.1)

where $S_i$ should be the horizon in the case of black hole or a surface around the singular region on the $z$-axis in the case of charged Weyl. In the metric (2.1) we can write the integral explicitly as

$$e_i = -\frac{1}{4\pi} \oint_{S_i} \left[ \rho \epsilon^{-1} \chi_{1,\rho} d\sigma_{\rho} + \rho \epsilon^{-1} \chi_{1,\sigma} d\sigma_{\sigma} \right],$$  

(4.2)

and by using the function $\mathcal{F}$ we have the expression

$$e_i = \frac{1}{8\pi} \oint_{S_i} \left[ \rho \epsilon^{-1} \chi_{1,\rho} d\sigma_{\rho} + \rho \epsilon^{-1} \chi_{1,\sigma} d\sigma_{\sigma} \right].$$  

(4.3)

The surface integral appearing in Eq. (4.3) can be calculated over any arbitrary surface $S$ of the cylinder that is specified by a lower base at $z = z_a$, upper base at $z = z_b$ and side at $\rho = \rho_c$ (Fig. 2):

$$\oint_{S} \left[ \rho \epsilon^{-1} \chi_{1,\rho} d\sigma_{\rho} + \rho \epsilon^{-1} \chi_{1,\sigma} d\sigma_{\sigma} \right]$$

$$= 2\pi \left[ \int_{z_a}^{z_b} \rho \epsilon^{-1} \chi_{1,\rho} d\sigma_{\rho} + \int_{\rho_c}^{\rho} \rho \epsilon^{-1} \chi_{1,\sigma} d\sigma_{\sigma} \right]$$

$$= 2\pi \delta_i \left( |w_{2i-1} - z_b| - |w_{2i} - z_b| - |w_{2i-1} - z_a| + |w_{2i} - z_a| \right)$$

$$= 2\pi I(z_a, z_b).$$  

(4.4)

In this calculation we have used the following relations that the pole trajectories $\mu_k$ satisfy:

$$\mu_{k,\rho} = -\frac{2\rho \mu_k}{\mu_k^2 + \rho^2}, \quad \mu_{k,\sigma} = -\frac{2\mu_k}{\mu_k^2 + \rho^2}. \quad (k = 1, 2, \ldots, 2n)$$  

(4.5)

As is seen in Eq. (4.4) the surface integral does not depend on $\rho_c$, and therefore we have written it as $2\pi I(z_a, z_b)$. For the $i$-th charge we set $z_a = w_{2i-1}$ and $z_b = w_{2i}$ and obtain

$$e_i = \frac{e}{4\pi d} I(w_{2i-1}, w_{2i}) = \frac{e}{d} \delta_i d_i.$$

(4.6)
As for the mass $m_i$ of the $i$-th black hole or charged Weyl, we can also define it by the surface integral of the covariant derivative of the timelike Killing vector $\xi^\nu$ over $S_i$. Taking account of the contribution of electrostatic field to the surface integral, we have the definition

$$m_i = \frac{1}{4\pi} \int_{S_i} (\xi_0^0 + \xi F^0) \sqrt{-g} \, d\sigma_i.$$  \hspace{1cm} (4.7)

The integral in Eq. (4.7) is again reduced to the surface integral in Eq. (4.4):

$$m_i = \frac{1}{8\pi} \int_{S_i} \left[ \rho f^{-1}(f-\chi^2)_\rho d\sigma_\rho + \rho f^{-1}(f-\chi^2)_\varphi d\sigma_\varphi \right]$$

$$= \frac{1}{8\pi} \frac{m}{d} \int_{S_i} \left[ \rho (\ln f)_\rho d\sigma_\rho + \rho (\ln f)_\varphi d\sigma_\varphi \right].$$  \hspace{1cm} (4.8)

Therefore, for the $i$-th mass we obtain

$$m_i = \frac{m}{4d} I(w_{2i-1}, w_{2i}) = \frac{m}{d} \delta_i d_i.$$  \hspace{1cm} (4.9)

We note that the surface integral in the definition of mass does not depend on whether $S_i$ is the horizon of black hole or a surface around singularity. This means that we have a definition of singularity mass in the present solution. The mass of each Reissner-Nordström black hole can be obtained if only we set $\delta_i=1$ in Eq. (4.9). We also note that we can obtain the mass of strut if we put the surface $S_i$ around a strut:

$$m_{\text{strut}} = \frac{m}{4d} I(w_{2i}, w_{2i+1}) = 0.$$  \hspace{1cm} (4.10)

This coincides with the statement that the Weyl strut has no gravitational mass.\(^9\)

As we know the charge and mass of each black hole or charged Weyl, we now have a clear interpretation that the present solution describes $n$ Reissner-Nordström black holes or charged Weyls aligned along the $z$-axis. The total charge and mass given in Eqs. (3.7) and (3.8) precisely coincide with the summations of $n$ charges and masses, respectively:

$$e_{\text{total}} = \sum_{i=1}^{n} e_i,$$  \hspace{1cm} (4.11)

$$m_{\text{total}} = \sum_{i=1}^{n} m_i.$$  \hspace{1cm} (4.12)

§ 5. Condition for static equilibrium

In the preceding sections we have seen that the present solution describes $n$ Reissner-Nordström black holes or charged Weyls aligned along the $z$-axis but in general there are struts to support them. The struts have no gravitational masses but cause conical singularities, that is, the axes between black holes or charged Weyls are not locally Euclidean. If there are no conical singularities and all the $\delta_i$'s are equal to 1, the solution describes $n$ Reissner-Nordström black holes in a static equilibrium
in which Coulomb force precisely balances gravitational force without struts. In this section we study the condition for such an equilibrium on the charges and masses defined in the previous section.

In order to measure a deviation from the spatially Euclidean metric we evaluate the quantity

$$P_0^2 = \lim_{\rho \to 0} \left( \frac{g_{\rho \rho}}{\rho^2 g_{\rho \rho}} \right) = \lim_{\rho \to 0} Q^{-1}. \quad (5.1)$$

It is easy to see that $P_0 = 1$ for a spatially Euclidean axis. We now concentrate on the region $w_{2i} < z < w_{2i+1}$ on the $z$-axis which denotes the region between the $i$-th and $(i+1)$-th black holes. In the present solution, $P_0^2$ in this region is given by

$$P_0^2 = \prod_{j=1}^{i-1} \left[ \frac{(z_j - z_{i+1})^2 - (d_j - d_{i+1})^2}{(z_j - z_i)^2 - (d_j + d_{i+1})^2} \right] \prod_{j=i+1}^{n} \left[ \frac{(z_j - z_i)^2 - (d_j - d_{i+1})^2}{(z_j - z_{i+1})^2 - (d_j + d_i)^2} \right]. \quad (5.2)$$

From the condition $P_0 = 1$ on Eq. (5.2) with the assumption (3.1) assuring that black holes are separated we obtain

$$\begin{cases} d_{i+1} = 0, & (j=1, \ldots, i-1) \\ d_i = 0, & (j=i+1, \ldots, n) \end{cases} \quad (5.3)$$

Because $m_j m_{i+1} \neq 0$ or $e_i e_{i+1} \neq 0$, the relation in Eq. (5.3) with the definition (4.6) or (4.9) of charge or mass of each black hole leads to

$$d = 0 \quad \text{or} \quad m = |e|. \quad (5.4)$$

From this and the relation $m_i/e_i = m/e$ we have the following condition on each charge and mass:

$$m_i = |e_i|, \quad (i=1, 2, \ldots, n) \quad (5.5)$$

Conversely when this condition holds we have $d = 0$ and all the $d_i$'s should be zero. Therefore $P_0 = 1$ in every region between black holes, which means that there is no strut at all and the static equilibrium of Reissner-Nordström black holes is realized. Thus we have the relation (5.5) as a necessary and sufficient condition for static equilibrium. In this equilibrium every Reissner-Nordström black hole is extremal as is seen from Eq. (5.5), and the global structure of spacetime is the same type as that in the Majumdar-Papapetrou solution.

In the above derivation of the relation (5.5) we imposed the condition that the axis should be locally Euclidean only on one region between black holes. However, the obtained result is relating to the charges and masses of all the black holes and to the global structure of spacetime. Furthermore, the static equilibrium is realized only when all the Reissner-Nordström black holes are extremal and therefore each black hole has a single horizon. Why does a condition on a local structure determine the global structure of spacetime? What relation exists between the static equilibrium of black holes and the degeneracy of two horizons of each black hole? We do not have any answer to these questions yet and they are future problems.
References

3) S. D. Majumdar, Phys. Rev. 72 (1947), 390.