The Hyperon and Nucleon Interactions. I

---Potentials---

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Two- and three-body \( Y-N \) potentials including the rescattering corrections are calculated by using Miyazawa's method. The \( Y-N \) systems are treated by the two-channel formalism in the case of \( T=1/2 \): The one is the channel of the \( \Sigma-N \) system and the other is of the \( A-N \) system. Then the potential is defined as the \( 2 \times 2 \) matrix form. The three-body force vanishes, in which no direct two-nucleon interaction is considered, if the \( \pi-A \) rescattering corrections are not taken into account. The other type of the three-body force in which one pion is exchanged between each pair of the particles is also investigated.

§ 1. Introduction

Since about 1950, successful results\(^1\) on the nuclear force problems have been obtained by many efforts based upon Taketani’s methodology. At present the main interests are in overcoming the frame of the static theory, that is, the attack on the region II. As for these subjects several attempts have already been proposed.\(^2\)

Now, it may be worthwhile to point out the possibility of another approach to the above subjects from the different point of view, though it is rather conservative. It is an approach through the investigation of the hyperon (\( Y \)) and nucleon (\( N \)) interactions, that is, the difference of the interaction being considered, the nuclear force problems can be reinvestigated. For this object it is most convenient to consider the \( A-N \) system. The investigation of the \( Y-N \) systems, however, is theoretically and experimentally more difficult than that of the two-nucleon systems; for example, the important freedoms such as the relative parity of a \( \Sigma \)- and \( A \)-hyperon, or the variety of the coupling constants are left undetermined. As for the coupling constants of the \( \pi-Y \) interactions, no direct \( \pi-Y \) scattering experiments can be made, contrary to the \( \pi-N \) scattering experiments, so we are forced to investigate the indirect processes such as \( K-N \) absorption, \( Y-N \) scattering and hyperfragments for the determination of the coupling constants. Therefore this is also the main aim of our research.

We summarize the difference of the \( Y-N \) interactions from the two-nucleon interactions as follows.

1°) In the case of \( T=1/2 \)(\( T \) means the total isospin quantum number), one must treat \( A \) and \( \Sigma \) as the two-channel system.

\(^1\) Also in \( T=0 \) system.

\(^2\) For example, for the \( K-N \) absorption, \( Y-N \) scattering and hyperfragments.
2°) As \( \Lambda \) is an iso-singlet, there exists no one-pion exchange potential. (In this case the charge independence is assumed.)

3°) \(^3\)H\(^2\) (or a \((\Lambda n)\) bound state) has not yet been found, so \(^3\)H\(^2\) must play the same role as the deuteron does in the two-nucleon system.

4°) Pauli’s exclusion principle does not act.

5°) The hyperons are the unstable particles which have very short lifetimes. Especially as for the point 2°), one must notice that the region II plays an essential role from the first, contrary to the two-nucleon force, so we can expect that the investigation of the \( \Lambda-N \) systems may throw light upon the dynamical properties of the potential in the region II. But unfortunately there have been but little \( \Lambda-N \) scattering data, and we know only the binding energies of the \( \Lambda \) hyperfragments as the phenomena in which \( \Lambda-N \) interactions participate. We therefore expect that the \( \Lambda-N \) scattering data will be accumulated in the near future.

The simplest hyperfragment which have up to now been found for sure is \(^4\)H\(^2\), in which \( \Lambda \)-binding energy is less than about 1 Mev. This means that \(^3\)H\(^2\) is more loosely bound than \(^3\)H or \( \text{He}^3 \), and the mean distance of any pair of constituent particles is about \( 3 \times 10^{-13} \text{cm} \) and then it is much larger than the force range. The force range of the two-body \( \Lambda-N \) force is \( 1/(2\mu) \) (\( \mu \) is a pion mass, \( \hbar = c = 1 \)), but three-body forces have the force range \( 1/\mu \) as will be discussed in § 4. Three-body force, therefore, may be more important in \(^4\)H\(^2\) than in \(^3\)H\(^2\) or \( \text{He}^3 \). In the latter case, the two-nucleon force of the classical region prevails over the three-body force.

Finally, we shall briefly argue on the kaon exchange potential. In the first step, we shall not consider even the one-kaon exchange potential (OKEP) for the following reasons.

a) As \( m_K > 3\mu \), OKEP may not have the definite meaning as OPEP (one pion exchange potential) has in the two-nucleon case, for OKEP acts in the region II or in the region III and the static calculation is not reliable.

b) OKEP is the exchange force which has the longest force range in the \( Y-N \) systems, but such a property as the exchange force, as well as its net effects, may give small effects for the hyperfragments in which the binding energy is very low and then the mean distance of any pair is larger than the force range.

c) As for the low energy scattering, it may also give little effects except for the very large angle scattering, for the exchange effects are largest for the backward scattering.

In the second step, we shall investigate this exchange effects by using the phenomenological potentials.

In § 2 the definition of the potentials in the two-channel formalism will be discussed. In § 3 we shall argue the two-body potentials up to the two-pion exchange ones and § 4 will be devoted to the calculation of the three-body potentials of the two different types.
§ 2. Two-channel formalism

In order to treat the $Y-N$ system as the two-channel one, we use the formalism\(^\text{3)}\) familiar in the theory of nuclear reactions. We first start with the following Schrödinger equation as the basic equation, i.e.

$$ (H^0 + V) \Psi = E \Psi, \tag{2·1} $$

$$ H^0 = \begin{pmatrix} H_i^0 & 0 \\ 0 & H_j^0 \end{pmatrix}, \tag{2·2} $$

$$ V = \begin{pmatrix} V_A & V_{A\Sigma} \\ V_{\Sigma A} & V_\Sigma \end{pmatrix}, \tag{2·3} $$

$$ \Psi = \begin{pmatrix} \psi_A \\ \psi_{\Sigma A} \\ \psi_{\Sigma} \end{pmatrix}, \tag{2·4} $$

where $H^0$ is the free Hamiltonian and includes the rest masses corresponding to the relevant systems, and $E$ is the initial energy including the rest masses, too. $\psi_A$ and $\psi_{\Sigma A}$ are the wave functions that correspond to the hyperon-exchange scattering such as $A+N\rightarrow \Sigma+N$.

Before the calculation of the potentials, let us generally investigate the so-called repetition terms of the potential. From Eq. (2·1), one can construct the $S$-matrix by Lippmann-Schwinger's prescription,\(^\text{4)}\) i.e.

$$ S = -2\pi i \Theta (E_a - E_b) \langle \phi_b | V | \Psi_a^{(+)} \rangle, \tag{2·5} $$

$$ \Psi_a^{(+)} = \phi_a + \frac{1}{E - H^0 + i\epsilon} V \Psi_a^{(+)}, \tag{2·6} $$

where $\phi_a$ is the eigenstate of $H^0$ with the energy $E$ and has a form of the diagonal matrix, and $(\phi_b, V \Psi_a^{(+)})$ is the very $T$-matrix. The integral equation for the $T$-matrix is

$$ (\phi_b | T | \phi_a) = (\phi_b | V | \phi_a) + \sum_n \frac{(\phi_b | V | \phi_n) (\phi_n | T | \phi_a)}{E - E_a + i\epsilon} \tag{2·7} $$

and if we identify this $T$-matrix with the one calculated from the field theoretical prescriptions, we can regard Eq. (2·7) as the integral equation for the potential $V$. Hence the second term of the right-hand side of Eq. (2·7) gives the repetition terms. In the case of the static theory, $E$ and $E_a$ in Eq. (2·7) are merely the rest masses of the initial and intermediate states respectively, and we have

$$ (b | V | a) = (b | T | a) - \sum_n \frac{(b | V | n) (n | T | a)}{M_a - M_n + i\epsilon}, \tag{2·8} $$

where $|a\rangle$, $|b\rangle$ and $|n\rangle$ are the state vectors specifying the kind of the particles which contribute to the matrix elements. As for the two-pion exchange potentials, the repetition term is
where $V^{(1)}$ is the one-pion exchange potential. In the one channel formalism the repetition term vanishes for the $A$-$N$ potential, for the intermediate states $|n\rangle$ do not contain the $\Sigma$-$N$ state in this case. But the mass difference $\Delta M$ of $M_\Sigma$ and $M_A$ turns out to play an essential role in removing the divergency of the potential.

To see this circumstance more accurately, we reduce the Schrödinger equation (2.9) to the equation which contains only $\psi_A$, under the boundary condition in which $\psi_{2A}$ has only an out-going wave, then we have

$$ (E-H_A^{(0)})\psi_A(r) = V_A(r)\psi_A(r) + \int V_{dA}(r) \left( \frac{d^3r'}{E+i\epsilon-H_{dA}-V_{dA}} \right) V_{dA}(r')\psi_A(r'). $$

(2.10)

In the two-channel formalism, the effective "potential" becomes non-local and energy dependent. On the other hand, in the case of the one channel formalism there are of course no off-diagonal potentials, so the one channel potential is local and energy independent, and we can express the one channel potential by the two-channel ones if one restricts the potential up to the two-pion exchange as a whole, and we have

$$ (E-H_A^{(0)})\psi_A(r) = \left\{ V_A(r) - \frac{1}{\Delta M} V_{dA}(r) V_{dA}(r) \right\} \psi_A(r), $$

(2.11)

where $V_{dA}$ and $V_{dA}$ are restricted to the one-pion exchange. The second term has $\Delta M$ in the denominator, so $\Delta M \neq 0$ is essential.

A few years ago Ferrari and Fonda\(^5\) calculated the $A$-$N$ potential, which corresponds to our $V_A$ only, by Brueckner-Watson’s method.\(^6\) Then, from the standpoint of the two-channel formalism their Schrödinger equation corresponds to the one in which the non-local potential part of Eq. (2.10) is neglected, on the other hand from the standpoint of the one-channel formalism they omitted the second term of Eq. (2.11) which is larger than the first term. On the contrary, Lichtenberg and Reiss\(^7\) treated the $Y$-$N$ system by the two-channel formalism for the first time. But their application of the potential to the actual $Y$-$N$ system is less cautious; for example, their result on the binding of the $\Sigma^+$-$p$ system may not be reliable, because the form of the inner potential is the most important in order to bind two particles; or as the $S$-wave scattering, too, reflects the form of the inner potential, the situation is the same as in the case of the binding of two particles.

Moreover, they took account of the off-diagonal potential up to the two-pion exchange ones, but in such a case as the elastic components, $\psi_A$ or $\psi_{2A}$, of the wave function are considered, it is clear from Eq. (2.10) that this effectively introduces the three- or four-pion exchange potential into the Schrödinger equation, so if one restricts the potential up to the two-pion exchange as a whole, it is
sufficient to take only the one-pion exchange potential. On the contrary, for the
inelastic components, \( \phi_{\Delta x} \) or \( \phi'_{\Sigma A} \) of the wave function we have the following
equations,

\[
(E - H^0) \psi_{\Delta A}(r) = V_2(r) \psi_{\Delta A}(r) + \int V_{\Delta A}(r') \left| r \right> \frac{d^3r'}{E + i\varepsilon - H^0 - V_A} \left| r' \right> V_{\Delta A}(r') \psi_{\Delta A}(r)
\]

+ \( V_{\Delta A}(r) \phi_A(r) \),

(2·12)

where \( \phi_A \) is a solution of the equation

\[
(H^0_A + V_A(r)) \phi_A(r) = E \phi_A(r)
\]

(2·13)

and for \( \phi_{\Delta x} \) we have similar equations in which the suffices \( \Sigma \) and \( A \) are inter-
changed. Therefore due to the existence of the last term in Eq. (2·12) one must
consider the two-pion exchange potential for the off-diagonal ones, too, in order to
take account of the effects up to the two-pion exchange as a whole. Thus, whether
one needs to take the one-pion exchange potential or the one- plus two-pion ex-
change one depends on what process one will consider.

The authors mentioned above used Breuckner and Watson’s method for the
calculation of the potentials, but according to the analysis of the low energy two-
nucleon phenomena the most favourable potential is the one obtained by Konuma,
Miyazawa and Otsuki (KMO potential). Accordingly we shall use Miyazawa’s
method to calculate the \( Y-N \) potential in the following sections.

\section*{§ 3. Two-body potential}

Throughout this paper we assume that the charge independence of the \( \pi-Y \)
interactions holds and the members of a charge multiplet have the same intrinsic
parity and mass, the spin of the hyperon being also assumed 1/2. Therefore in
the static theory our vertex operator of the \( \pi-Y \) interactions are as follows,

\[
\Lambda \Sigma \pi \quad V_\Sigma = \left\{ \begin{array}{l}
\frac{(f_0^0/\mu)}{\mu} i\sigma \cdot k (t_k + t_k^\dagger) e^{ik \cdot x} \quad \text{for } P_{\Sigma A} = \text{even} \\
\frac{(d_0^0(t_k + t_k^\dagger))}{\mu} e^{ik \cdot x} \quad \text{for } P_{\Sigma A} = \text{odd}
\end{array} \right.
\]

(3·1)

\[
\Sigma \Sigma \pi \quad V_\Sigma = \frac{f_0^0}{\mu} i\sigma \cdot k n_{\Sigma A} e^{ik \cdot x}
\]

(3·2)

and

\[
NN\pi \quad V_N = \frac{f_0^0}{\mu} i\sigma \cdot k n_{\Sigma A} e^{ik \cdot x},
\]

(3·3)

where \( t_k \) is the \( 3 \times 1 \) matrix which picks up \( \Sigma_k \) from \( \Sigma = (\Sigma_1 \Sigma_2 \Sigma_3) \) and \( \dagger \) means
to take hermite conjugation, and \( \rho_k's \) are the \( 3 \times 3 \) matrix with \( \Sigma_k \rho_k = 2 \) and
hermite, \( f_0^0 \)'s being the unrenormalized and unrationalized coupling constants.

One can immediately construct the potentials up to the two-pion exchange
from the above assumptions.
3-1. **One-pion exchange potentials**

We write the results as follows, for the diagonal potential we have

\[
V_d(r) = 0
\]

\[
V_z(r) = -\frac{1}{4\pi} \left( \frac{f_{SN}}{\mu^2} \right) (t \cdot \tau) (\sigma \cdot \nabla) (\sigma_N \cdot \nabla) \frac{e^{-\rho r}}{r}
\]

(3.4)

and for the off-diagonal potentials

\[
V_{zd}(r) = \frac{1}{4\pi} \left( \frac{f_{SN}}{\mu^2} \right) (t \cdot \tau) (\sigma \cdot \nabla)(\sigma_N \cdot \nabla) \frac{e^{-\rho r}}{r}
\]

(3.5)

\[
V_{zd}(r) = -\frac{1}{4\pi} f_{SN} (t \cdot \tau)(\sigma_N \cdot \nabla) \frac{e^{-\rho r}}{r}
\]

for \(P_{zd} = \) even

(3.6)

where \(f\)'s are renormalized but unrationalized coupling constants. The eigenvalue of \((t \cdot \tau)\) is \(-\sqrt{3}\) for \(I=1/2\) and 0 for \(I=3/2\). As for the off-diagonal part, we neglected the mass difference \(M = M_z - M_d\) between the initial and final states.

3-2. **Two-pion exchange potentials**

We calculate at first the S-matrix of such a type as in Fig. 1, where the shaded area means \(\pi \cdot N\) and \(\pi \cdot Y\) rescattering corrections. In the case of the static theory one can use the \(\pi \cdot N\) and \(\pi \cdot Y\) S-matrix elements for the shaded area. Then one has

\[
S = \frac{1}{2} \int d^4 p d^4 q \frac{\langle p, i | S_N | q, j \rangle \langle -p, i | S_Y | -q, j \rangle}{(p^2 + \mu^2)(q^2 + \mu^2)}
\]

(3.7)

where the bar means that \(\mu^2\) has a small negative imaginary part \(-i\epsilon\) (\(\epsilon > 0\)), and \(p\) and \(q\) are four momenta of the exchanged pions. The factor \(1/2\) comes out from the fact that as \(\pi \cdot Y\) or \(\pi \cdot N\) scattering amplitude includes both crossed and uncrossed graphs, then one counts the topologically equivalent graphs twice. As for the scattering amplitudes \(\langle p, i | S_N | q, j \rangle\) and \(\langle p, i | S_Y | q, j \rangle\), we use the formula derived from the Low equation.

Now, we substitute the scattering amplitude thus obtained into Eq. (3.7) and integrate by \(p_0\) at first. The \(p_0\)-integration generally has the following form,

\[
\int_{-\infty}^{\infty} dp_0 \frac{f^{(\pm)}(p_0)}{(p_0^2 - \omega^2)(p_0^2 - \omega_q^2)}
\]

where \(f^{(\pm)}\) is essentially the scattering amplitude and

\[f^{(\pm)}(p_0) = [\lambda \pm p_0 - i\epsilon]^{-1} [\mu \pm p_0 - i\epsilon]^{-1}\]
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\[ f^{(+)}(p_0) = \left[ \frac{\lambda \pm p_0 - i\epsilon}{\mu \mp p_0 - i\epsilon} \right]^{-1}. \]

In the case of \( f^{(+)}(p_0) \), one can take the closed path in which no poles of \( f^{(+)}(p_0) \) exist. But in the case of \( f^{(-)}(p_0) \), one cannot do so, and either of the two poles of \( f^{(-)}(p_0) \) must be included in the closed path. When both factors of \( f^{(-)} \) come out from the Born term which make either \( \lambda \) or \( \mu \) equal to zero, it is convenient to take the closed path in which the pole at \( p_0 = 0 \) exists, for the residue of the pole at \( p_0 = 0 \) just gives the repetition term discussed in §2. Then one must subtract this term from Eq. (3·7) to obtain the two-pion exchange potential.

3-3. The Born terms of the two-pion exchange potentials

At first we shall discuss the Born terms of the potentials in the case of \( \Delta M = 0 \). In the case of \( P_{34} = \text{even} \), as for \( V_4 \), we have (hereafter we use \( \mu = 1 \) unit)

\[ V_4^A(r) = 0 \]

\[ V_4^A(r) = \left( \frac{f_2^2}{4\pi} \right) \left( \frac{f_x^2}{4\pi} \right) \frac{8}{\pi} \left[ \frac{3}{r^3} K_0(2r) + \left( \frac{2}{r^2} + \frac{3}{r^4} \right) K_1(2r) \right] \quad (3·8) \]
\[ V^A_r(r) = -\left( \frac{f_s^2}{4\pi} \right) \left( \frac{f_s^2}{4\pi} \right) \frac{8}{\pi} \left[ \frac{3}{r^3} K_0(2r) + \left( \frac{1}{r^2} + \frac{15}{4r^4} \right) K_1(2r) \right], \]

where \( K_i \)'s are the modified Bessel function. This potential is identical with the two-nucleon potential derived by Taketani, Machida and Ohnuma\(^{10} \) (TMO potential), provided one replaces \( f_s^2 \) into \( f_s^2 f_\Lambda^2 \). Fukuda, Sawada and Taketani\(^{11} \) discussed the relation between the TMO potential and the BW (Brueckner and Watson\(^e \)) potential and obtained the FST potential, that is given as follows,

\[
V_{FSF} = \frac{\langle H' \frac{1-\gamma}{-H_0} H' \rangle + \langle H' \frac{1-\gamma}{-H_0} H' \frac{1-\gamma}{-H_0} H' \rangle}{1 + \langle H' \frac{1-\gamma}{(-H_0)^2} H' \rangle},
\]

(3.9)

where \( \langle \ \rangle \) denotes the meson vacuum expectation value and \( \gamma \) is the projection operator on the meson vacuum state and \( H_0 \) and \( H' \) are the free and interaction Hamiltonian, respectively. If we expand the denominator by \( H' \) up to the second order, then we obtain the TMO potential, say,

\[
V_{TM0} = \langle H' \frac{1-\gamma}{-H_0} H' \rangle + \langle H' \frac{1-\gamma}{-H_0} H' \frac{1-\gamma}{-H_0} H' \rangle - \langle H' \frac{1-\gamma}{-H_0} H' \rangle \langle H' \frac{1-\gamma}{(-H_0)^2} H' \rangle.
\]

(3.10)

If this is the case, as for the \( \Lambda-N \) potential the TMO potential has to coincide with the BW potential, for \( \langle H'(1-\gamma)/(\gamma H_0)^2 \rangle \) in the denominator of (3.9) vanishes owing to \( \Lambda \) being an isosinglet. But if we consider TMO expression (3.10) as a base and permit the existence of such a matrix element as \( \langle H'(1-\gamma)/(\gamma H_0)^2 \rangle \), that is to say, if we use the two channel formalism, then not only the \( \Lambda-N \) but the other two-pion exchange potentials thus obtained are identical with the two-nucleon potential, provided the appropriate replacement of the coupling constants or the iso-spin operators is made. This prescription, however, cannot be applied to the FST method. On the other hand, the method of canonical transformation given by Nishijima\(^{12} \) is valid in these multi-channel cases.

As for the \( \Sigma-N \) potential we have the same expression as Eq. (3.8) for the spin dependent and tensor forces, provided \( f_\Lambda^2 \) be replaced by \( (2f_\Lambda^2 + f_s^2) \), and for \( V_e \) we have

\[
V_e^s(r) = -\left( \rho \cdot \tau \right) \left( \frac{f_s^2}{4\pi} \right) \frac{1}{4\pi} \left( \frac{f_s^2 + f_\Lambda^2}{2} \right) \frac{8}{\pi} \times \left[ \left( \frac{1}{r} + \frac{23}{4r^3} \right) K_0(2r) + \left( \frac{3}{r^2} + \frac{23}{4r^4} \right) K_1(2r) \right].
\]

(3.11)
Similarly we have

\[ V_{o}^{s,t}(r) = -(t \cdot \tau) \left( \frac{f_{s,t}^{2}}{4\pi} \right) \left( \frac{f_{s,t}^{2}}{4\pi} \right) \frac{8}{\pi} \times \left[ \left( \frac{1}{r} + \frac{23}{4r^3} \right) K_0(2r) + \left( \frac{3}{r^2} + \frac{23}{4r^4} \right) K_1(2r) \right], \tag{3.12} \]

for the off-diagonal potential for which we have no spin dependent and tensor forces.

When \( P_{s,t} = \text{odd} \), as for the \( \Lambda-N \) diagonal potential we have no Born terms. This is due to the fact that the \( S \)-wave \( \pi-\Lambda \) scattering amplitude has neither spin nor isospin dependency, so the contributions from the Born terms of the scattering amplitude cancel each other when we substitute the amplitude into Eq. (3.7).

In the case of \( \Sigma-N \) potential, as for the part to which the \( P \)-wave \( \pi-\Sigma \) scattering amplitude contributes we have the same expressions as \( P_{s,t} = \text{even} \) provided that \( f_{s,t}^{2} \) be of course set equal to zero, and as for the part to which the \( S \)-wave \( \pi-\Sigma \) scattering amplitude contributes, we have

\[ V_{o}(r) = (\rho \cdot \tau) \left( \frac{g_{s,t}^{2}}{4\pi} \right) \left( \frac{f_{s,t}^{2}}{4\pi} \right) \frac{2}{\pi} \left[ \frac{2}{r} K_0(2r) + \frac{5}{r^2} K_1(2r) \right]. \tag{3.13} \]

This has the opposite sign to the central potential which comes out from \( P \)-wave \( \pi-\Sigma \) scattering. The potential in the case of \( P_{s,t} = \text{odd} \) is made rather weak, though it depends on the value of \( g_{s,t}^{2} \).

3–4. **Qualitative properties of the potentials**

First, we shall consider the effect of the mass difference of \( \Sigma \) and \( \Lambda \). It has been concluded in the works\(^{(a)}\) so far presented that this effect is almost 10% and then the mass difference can be neglected. But generally the effect of the mass difference depends on the inter-particle distance; indeed in the case of \( V_{s} \), expressing the ratio of \(|\text{Born}(\Delta M=0) - \text{Born}(\Delta M \neq 0)|\) to the Born term where \( \Delta M=0 \) in percentage, it is less than 20% when \( r<1 \), but it becomes about 40% when \( r \geq 2 \), thus the mass difference is more important in the outer region than in the inner region. As for \( V_{s,t} \), however, the situation is rather different from the case of \( V_{s,t} \) that is, if \( f_{s,t} \) is not zero, \( V_{s,t} \) has the one-pion exchange potential and moreover in the two-pion exchange potential there is the part which does not include \( A \) in the intermediate states, so by the very reasons this effect of the mass difference is not so important for \( V_{s,t} \). As for the off-diagonal potentials the mass difference between the initial and final state must be set equal to zero, because the definition of the potentials in the static theory includes \( \phi(0) \).

Next, we shall discuss the rescattering corrections. In Figs. 2 to 4 only the \( \pi-N \) rescattering corrections are taken into account, for we cannot know of the \( \pi-Y \) scattering cross sections at all. But there may be no room to have a doubt about the importance of the rescattering corrections, especially for the central potential of \( V_{o} \). The rescattering corrections to the tensor force is not so large, i.e. it
is less than 20% when \( r > 1 \) and becomes less and less as \( r \) increases, but when \( r \leq 0.7 \) it is about 30–40%. Thus these effects are, contrary to the effect of the mass difference, more important in the inner region than in the outer region. The \( \pi-Y \) rescattering corrections generally participate additively with the \( \pi-N \) rescattering corrections in the results. Moreover, it has been pointed out\(^{(10)}\) that the \( \pi-Y \) resonances corresponding to the 3-3 resonances in the \( \pi-N \) scattering may occur near a few hundred Mev in the various states. Then the uncertainty will not be avoidable.

In Figs. 3 we have drawn the curves of \( \alpha^2 = 0, 1/5 \) and 1 when \( \alpha = f_\pi/f_A \) and \( \alpha \) is taken to be positive, and moreover \( f_N/f_A \) is taken to be one. The reason why we take the ratio of \( f_\pi \) to \( f_A \) is unfounded, but it is considered in a somewhat interesting philosophy given by Miyazawa, Nakano and Umezawa.\(^{(15)}\) They have insisted that all the pion or the kaon coupling constants are respectively one or zero in the certain units, and conjectured that \( |f_\pi| \) is one and \( |f_\pi| \) is zero. If \( f_A \) has the sign opposite to \( f_N \), the sign of the potentials in the region I is reversed, but in the region where the two-pion exchange potentials exceed the one-pion exchange potentials the sign is naturally invariant.

\[ V_x \text{ Singlet state with } T = \frac{1}{2}. \]

\[ V_x \text{ Triplet state with } T = \frac{1}{2}. \]

\( V_x \)'s divided by \( f_A^2 \) are plotted. The dashed lines are for \( \alpha^2 = 0 \), the broken lines for \( \alpha^2 = 1/5 \), and the real lines for \( \alpha^2 = 1 \), where \( \alpha = f_\pi/f_A \) and is taken to be positive. Moreover \( f_A \) is taken to be equal to \( f_N \) in these figures. The units are the same as in Fig. 2.
The meaning of the lines is the same as in Fig. 3a and 3b. The $V_{T}$'s with $T=3/2$ resemble to the two-nucleon potentials of the triplet odd and the singlet even state. The units are the same as in Fig. 2.

On the contrary, the off-diagonal potentials are not sensitive to the value of $\alpha$, because there appears $f_{s}$ instead of $f_{s}$ in the one-pion exchange potential; they depends on the value of $f_{N}/f_{x}$.

The $\Sigma$-$N$ potential in the outer region is not so sensitive to the relative parity, provided $f_{s}$ is not zero, but as for $V_{d}$ the relative parity is more important than in $V_{s}$, for $V_{d}$ is constructed through the interaction $\Sigma \pi$. When $P_{2d}=$ odd, not only the spin dependence but also the depth of $V_{d}$ are much weakened, though they depend on the value of $g_{A}^{2}$. The off-diagonal potentials, of course, depend sensitively on the relative parity and include the factor $(\sigma \cdot r)$ which makes the orbital angular momentum of the final state different by one from that of the initial state and at the same time makes the spin of either of two baryons flipped. The potential generally has the following form,

$$V(r) = (\sigma \cdot r) V_{N}(r) + (\sigma \cdot r) V_{Y}(r),$$

(3.14)

where $V_{N}(r)$ has the one-pion exchange tail but $V_{Y}$ has not. The lowest order graph of $V_{Y}$ is the two-pion exchange one and then the Born term of $V_{Y}$ has the following form,
§ 4. **Three-body forces**

The necessity for the three-body force is not so strong in the ordinary nuclei, because the two-nucleon force has the one-pion exchange part whose force range is $1/\mu$. On the contrary, it may be expected that the three-body force is more important in the $\Lambda$-hypernuclei than in the ordinary nuclei.

We shall calculate the three-body forces of two different types in this section. In the case of the three-body problem which includes a $\Lambda$-hyperon, it is also necessary to use the two-channel formalism, i.e. we must solve the Schrödinger equation like Eq. (2.1) or Eq. (2.10). Then the three-body potential is defined

$$V_V(r) = -\left(\frac{g_A f_A}{4\pi}\right) \left(\frac{f^2}{4\pi}\right) \frac{2}{\pi} (t \cdot t) \left[\frac{8}{r^2} K_0(2r) + \left(\frac{4}{r^2} + \frac{9}{r^3}\right) K_1(2r)\right].$$

On the other hand, $V_N$ has not the two-pion exchange part. Thus the spin-flip of a nucleon is more liable to occur than that of a hyperon in the outer region, but in the inner region the situation is reversed.

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The meaning of the lines is the same as in Fig. 3. The units are the same as in Fig. 2. These potentials convert $\Lambda$ into $\Sigma$ and vice versa. The $\alpha$-dependency is small in the outer region but become large in the inner region.
as the $2 \times 2$ matrix form similar to Eq. (2·3) but we calculate only the diagonal part in the case of $P_{x_A} = \text{even}$.

We write the three-body potential as follows,

$$V(x, y, z) = V^{(1)}(x, y, z) + V^{(2)}(x, y, z)$$

where $x$, $y$ and $z$ are the distances of three pairs and the suffices 1, 2 and 3 denote the number of exchanged pions. Then $V^{(1)}$ can be decomposed as in Fig. 6,

$$V^{(1)}(x, y, z) = V_1^{(1)}(x) + V_2^{(1)}(y) + V_3^{(1)}(z).$$

Though $V^{(2)}$ generally includes such potentials as two pions are exchanged between the same pair, it is possible to make them included in $V^{(1)}$ and then we do not consider them, but we deal with the potentials where only one pion is exchanged between the same pair. Thus we can decompose $V^{(2)}$ as in Fig. 7,

$$V^{(2)}(x, y, z) = V_1^{(2)}(y) + V_2^{(2)}(z, x) + V_3^{(2)}(x, y).$$

$V^{(3)}$ corresponds to Fig. 8 provided our consideration is restricted to the diagonal potential and we do not consider the potentials where more than one pion are exchanged between the same pair.

Now, we can separate the repetition terms from the $S$-matrix by the same prescription as discussed in § 2:
\[ (b | V_i^{(3)} | a) = (b | T_i^{(2)} | a) - \left\{ \frac{(b | V_i^{(1)} | n) (n | V_i^{(1)} | a)}{M_0 - M_n + i\epsilon} \right\} \]
\[ + \frac{(b | V_i^{(1)} | n) (n | V_i^{(1)} | a)}{M_0 - M_n + i\epsilon} \] (i, j, k cyclic.) \hspace{1cm} (4.4)

and for \( V^{(4)} \) we have
\[ (b | V_i^{(4)} | a) = (b | T_i^{(2)} | a) - \sum_i \left\{ -\frac{(b | V_i^{(1)} | n) (n | V_i^{(1)} | a)}{M_0 - M_n + i\epsilon} \right\} \]
\[ + \frac{(b | V_i^{(1)} | n) (n | V_i^{(1)} | a)}{M_0 - M_n + i\epsilon} \]
\[ - \sum_{i=0}^{N} \frac{(b | V_i^{(4)} | n) (n | V_i^{(4)} | m) (m | V_i^{(1)} | a)}{(M_0 - M_n + i\epsilon)(M_0 - M_m + i\epsilon)} \] \hspace{1cm} (4.5)

where \( \sum_i \) means to take all permutation of \( i, j \) and \( k \). We have not taken the summation on \( |n\) and \( |m\) in the above, for \( |n\) or \( |m\) is uniquely determined by the suffices \( i, \) etc.

4-1. Calculation of \( V^{(2)} \)

The \( S \)-matrix for Fig. 6 is given as
\[ S = \frac{1}{(2\pi)^6} \int d^4p d^4q \tilde{\phi}(p_0) \tilde{\phi}(q_0) \cdot f_{j}^{(1)} \tau_j^{(2)} \frac{(\sigma^{(1)} \cdot p)}{(p^2 + m^2)} \frac{(\sigma^{(2)} \cdot q)}{(q^2 + m^2)} e^{i(p_{x_1} - q_{y_1})} \]
\[ \hspace{1cm} \times \frac{1}{x y} \] \hspace{1cm} (4.6)

where suffices 1 \( \) and \( 2 \) refer to the nucleons 1 \( \) and \( 2 \). Subtracting the repetition terms from Eq. (4.6), we have
\[ V^{(2)}(x, y) = -\left( \frac{f_{j}^{(1)}}{4\pi} \right) (\tilde{A}_d(0) / 4\pi) (\tau_1^{(1)} \cdot \tau_3^{(3)}) [(\sigma^{(1)} \cdot \nabla_x \sigma^{(0)} \cdot \nabla_x, \sigma^{(2)} \cdot \nabla_y \sigma^{(0)} \cdot \nabla_y)] \frac{e^{-i(x+y)}}{x y} \] \hspace{1cm} (4.7)

where suffix 0 refers to \( A \) and \( \tilde{A}_d \) is the \( \pi \)-\( A \) scattering amplitude subtracted by the Born term and then
\[ \tilde{A}_d(0) = \frac{1}{\pi} \int dk \frac{\sigma_{\pi A}(k)}{\sigma^2} \] \hspace{1cm} (4.8)

After the differentiation, Eq. (4.7) turns out to be as follows,
\[ V^{(3)}(x, y) = -\left( \frac{f_{j}^{(1)}}{4\pi} \right) (\tilde{A}_d(0) / 4\pi) (\tau_1^{(1)} \cdot \tau_3^{(3)}) [V_1^{(1)}(x), V_2^{(1)}(y)]_+, \] \hspace{1cm} (4.9)

where
\[ V_1^{(1)}(x) = (\sigma^{(1)} \cdot \sigma^{(0)}) \frac{e^{-x}}{3x} + S_{10} \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{e^{-x}}{3x} \] \hspace{1cm} (4.10)

and \( \{ \} \) denotes anticommutator. Thus \( V^{(3)} \) has a very simple structure and it
can be clearly understood that the force range of \( V^{(3)} \) is just \( 1/\mu \). In this two-channel formalism the Born terms do not exist, contrary to the one channel formalism.\(^{16}\) \( \tilde{A}_4(0) \) is, however, not zero but may be rather larger than the coupling constant \( f_4^2 \), for the resonance may occur near a few hundred Mev in the state with the total angular momentum \( 3/2 \). If this resonance has the same structure as the \( 3-3 \) resonance, that is, in the case of the universal pion coupling,\(^{14}\) \( \tilde{A}_4(0) \) is nearly equal to 0.4 and 5 times as large as the coupling constant \( f_4^2 = f_3^2 \). If this is not the case, \( \tilde{A}_4(0) \) may at least be comparable to \( f_4^2 \).

The "central" potential is simple and

\[
V^{(3)}(x, y) = -2 \left( \frac{f_3^2}{4\pi} \right) \left( \frac{\tilde{A}_4(0)}{4\pi} \right) \left( \tau^{(1)} \cdot \tau^{(2)} \right) \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) e^{-(x+y)} \frac{1}{9x y}. \tag{4·11}
\]

Then \( V^{(3)} \) is repulsive when two nucleons are both in the singlet even and triplet even state, and is attractive both in the singlet odd and triplet odd state.

**4·2. Calculation of \( V^{(3)} \)**

The \( S \)-matrix for Fig. 8 is given as

\[
S = \frac{i}{(2\pi)^{12}} \int d^4 p d^4 q d^4 k \frac{\langle \phi_1 | \hat{S}_1 | k, l \rangle \langle k, l | S_k | q, j \rangle \langle q, j | S_q | p_i \rangle}{(p^2 + \mu^2)(q^2 + \mu^2)(k^2 + \mu^2)}. \tag{4·12}
\]

Although one can take into account the rescattering corrections in this case, too, if one spares no trouble, we do not consider them here and only investigate the qualitative feature of the Born terms.

Subtracting the contributions of the pole at \( p_0 = 0 \) from Eq. (4·12), we have

\[
V^{(3)}(x, y, z) = \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) \left[ \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_y \right) \left( \sigma^{(2)} \cdot \nabla_z \sigma^{(2)} \cdot \nabla_z \right) \right.
\]

\[
\times \left( \sigma^{(3)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_z \right) + \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(3)} \cdot \nabla_z \right) \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right)
\]

\[
+ \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right) \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(3)} \cdot \nabla_z \right)
\]

\[
+ \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right) \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_z \right) \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(3)} \cdot \nabla_x \right) \right] F_{MM, 0, 0}(x, y, z)
\]

\[
- \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) \left( \tau^{(1)} \cdot \tau^{(2)} \right) \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right)
\]

\[
\times \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(2)} \cdot \nabla_z \right) + \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(3)} \cdot \nabla_z \right) \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right)
\]

\[
+ \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right) \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(3)} \cdot \nabla_z \right)
\]

\[
+ \left( \sigma^{(2)} \cdot \nabla_y \sigma^{(2)} \cdot \nabla_y \right) \left( \sigma^{(1)} \cdot \nabla_x \sigma^{(1)} \cdot \nabla_x \right) \left( \sigma^{(3)} \cdot \nabla_z \sigma^{(3)} \cdot \nabla_z \right) \right] F_{MM, 0, 0}(x, y, z),
\tag{4·13}
\]

where

\[
F_{\lambda, \sigma, \alpha}(x, y, z) = \frac{1}{\pi} \int \frac{dp dq}{(xyz)^2} \sin p(x + y + z) \sin p(x) \sin p(y) \sin p(z)
\]

\[
\frac{1}{\omega (\lambda + \omega)(\mu + \omega)(\nu + \omega)}. \tag{4·14}
\]
We neglect the mass difference in Eq. (4·13) and \( F_{0,0,0}(x, y, z) \) is
\[
F_{0,0,0}(x, y, z) = 3G(x, y, z) - [G(-x, y, z) + G(x, -y, z) + G(x, y, -z)]
\]
and
\[
G(x, y, z) = \frac{1}{16} \frac{x+y+z}{xyz} \exp[-(x+y+z)],
\]
where the triangle inequality appears explicitly in brackets, i.e. \( x+y-z > 0 \), etc. From this the potential with such structure as \( f(x, y, z) \exp[-(x+y-z)] \) appears, where \( f(x, y, z) \) is a rational function. Such structure does not exist in the ordinary potentials we have so far considered, and this gives the longest force range.

The "central" potential of \( V^{(3)} \) has the following structure,
\[
V^{(3)}_c(x, y, z) = (3+2\sigma^{(1)} \cdot \sigma^{(2)}) V^{(3)}_r(x, y, z) + (3-2\sigma^{(1)} \cdot \sigma^{(2)}) V^{(3)}_s(x, y, z)
\]
and
\[
V^{(3)}_r(x, y, z) = 8 \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) \frac{1}{9} \left( x y z \frac{\partial^3 f}{\partial x \partial y \partial z} - \left[ x \frac{\partial^3 f}{\partial x \partial y \partial z} + y \frac{\partial^3 f}{\partial y \partial z} + z \frac{\partial^3 f}{\partial z} \right] + 3f \right),
\]
\[
V^{(3)}_s(x, y, z) = -8 \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) \frac{1}{3} \left( x y z \frac{\partial^3 f}{\partial x \partial y \partial z} + \left[ x \frac{\partial^3 f}{\partial x \partial y \partial z} + y \frac{\partial^3 f}{\partial y \partial z} + z \frac{\partial^3 f}{\partial z} \right] + 3f \right),
\]
where \( i=x/x, j=y/y \) and \( f=1/(xyz) \). Thus this "central" potential depends on the angles of the triangle which \( A \) and \( 2N \) construct, and has no simple structure as \( V^{(3)} \). Then we shall consider the asymptotic forms where \( x, y \) and \( z \) are large, i.e.
\[
V^{(3)}_c(x, y, z) \to \frac{1}{2} \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) (i \cdot j) (j \cdot k) (k \cdot i) A(x, y, z)
\]
and
\[
V^{(3)}_s(x, y, z) \to -\frac{1}{18} \left( \frac{f_3^2}{4\pi} \right)^2 \left( \frac{f_4^2}{4\pi} \right) A(x, y, z),
\]
where
\[
A(x, y, z) = \frac{1}{xyz} \left[ 3(x+y+z) e^{-(x+y+z)} + (x+y+z) e^{-(x+y+z)} \right]
\]
Moreover if we add the condition that $A$ and $2N$ construct the regular triangle the asymptotic forms become

\[ V_{t}^{(3)} \rightarrow -\frac{3}{16} \left( \frac{f_{N}^{3}}{4\pi} \right)^{2} \left( \frac{f_{d}^{3}}{4\pi} \right) \left[ \frac{e^{-3r}}{r^{2}} + \frac{e^{-r}}{r^{2}} \right] \]  

(4.21)

and

\[ V_{p}^{(3)} \rightarrow -\frac{1}{6} \left( \frac{f_{N}^{3}}{4\pi} \right)^{2} \left( \frac{f_{d}^{3}}{4\pi} \right) \left[ \frac{e^{-3r}}{r^{2}} + \frac{e^{-r}}{r^{2}} \right] \]  

(4.22)

where $r$ is a side of the regular triangle. Thus the “force range” is rather longer than $V^{(2)}$ when $x$ is taken to be equal to $y$ in $V^{(3)}$.

The tensor part of $V^{(3)}$ is more and more complicated and we no longer discuss it here.

§ 5. Discussions

It could be understood that by the use of the two-channel formalism one can treat both the $Y-N$ potentials of the two-body system and those of the three-body system in the parallel fashion as the two-nucleon potential. However, we have so far calculated the potentials in the frame of the static theory, in spite of the existence of the attempts\textsuperscript{2} to overcome the static theory. In addition, it has recently been pointed out\textsuperscript{7} that the $\pi-\pi$ interactions are very important in the pion physics, but we have not taken into account the $\pi-\pi$ interactions at all. In the case of the two-nucleon system the potential which is constructed by the $S$-wave scattering amplitudes was very small because of the smallness of the $S$-wave $\pi-N$ scattering, but this may not be the case in the $Y-N$ system. Thus in the various senses our approach to the problems is conservative, but considering the success in the two-nucleon potentials it will be significant at the first step to investigate the $Y-N$ systems by the use of the static potentials as in the two-nucleon systems. The system which corresponds just to the two-nucleon system is the one with $T=3/2$ such as $\Sigma^{+}p$ or $\Sigma^{-}n$; in this case there is not the $A-N$ channel, so it is able to treat the system by using the same method as in the two-nucleon system with $T=1$. In this sense it will be most suitable to determine the coupling constant $f_{\Sigma}$ and the sign, though these experiments are difficult because of the short lifetime of $\Sigma^{+}$ which is almost half of the lifetime of $\Sigma^{-}$ and of the impossibility of the neutron target in the case of $\Sigma^{-}n$ system. Therefore the existence of the bound state of $\Sigma^{-}n$ or $\Sigma^{+}p$ is very interesting, but the theoretical prediction on the existence of these bound states is not significant from the meson theoretical point of view. This is indeed evident from the fact that even the binding of the deuteron has never been accomplished by the meson theory. The experimental data\textsuperscript{8} which exist up to now are on the low energy $\Sigma^{-}p$ reactions. It may make us confused to consider the various reaction modes as in the same level; for example,
in the hyperon exchange scattering such as $\Sigma^- + p \rightarrow A + n$ the final energy is larger than the mass difference of $\Sigma$ and $A$, that is, almost 80 Mev, and then it is already a high energy phenomenon in the sense of the nuclear force.\(^{19}\) The hyperon exchange scattering is, however, advantageous to test the off-diagonal potential which has the one-pion exchange tail and then to determine the type of the coupling and the coupling constant $f_A$ or $g_A$. There are the experimental results on the $Ap$ system,\(^{20}\) but they as well as the experimental results on the $\Sigma^- p$ system only indicate the order of the cross section. In the case of the $Ap$ scattering, the elastic scattering below the threshold energy of a $\Sigma$-production will be interesting. But even in the energy region below the threshold energy of a $\Sigma$-production it will be rather difficult to separate the outer region and the inner region due to the existence of the $\Sigma$-$N$ channel. This situation will also be suggested by considering Eq. (2·10), i.e. the non-local potential part includes the behaviour of the $\Sigma$-$N$ system in the inner region.

Now, we proceed to the problems of the hyperfragments which have been analysed phenomenologically by Dalitz and Downs.\(^{20}\) The analysis made so far, however, is in the one-channel formalism. As pointed out in § 4, it is impossible to ignore the $\Sigma$-component in the analysis of the hyperfragments, even if the analysis is phenomenological. Moreover, the three-body potentials have long force ranges, so the three-body correlation becomes important.

We are now in a position to proceed with the investigations of the above subjects and the succeeding paper will be devoted to them.

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