Determining dynamical parameters of the Milky Way Galaxy based on high-accuracy radio astrometry

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Abstract

In this paper we evaluate how the dynamical structure of the Galaxy can be constrained by high-accuracy VLBI (Very Long Baseline Interferometry) astrometry such as VERA (VLBI Exploration of Radio Astrometry). We generate simulated samples of maser sources which follow the gas motion caused by a spiral or bar potential, with their distribution similar to those currently observed with VERA and VLBA (Very Long Baseline Array). We apply the Markov chain Monte Carlo analyses to the simulated sample sources to determine the dynamical parameter of the models. We show that one can successfully determine the initial model parameters if astrometric results are obtained for a few hundred sources with currently achieved astrometric accuracy. If astrometric data are available from 500 sources, the expected accuracy of $R_0$ and $\Theta_0$ is $\sim 1\%$ or higher, and parameters related to the spiral structure can be constrained by an error of $10\%$ or with higher accuracy. We also show that the parameter determination accuracy is basically independent of the locations of resonances such as corotation and/or inner/outer Lindblad resonances. We also discuss the possibility of model selection based on the Bayesian information criterion (BIC), and demonstrate that BIC can be used to discriminate different dynamical models of the Galaxy.

Key words: astrometry — Galaxy: fundamental parameters — Galaxy: kinematics and dynamics

1 Introduction

The Galaxy is the only galaxy that can be studied in detail, and thus provides the best laboratory for studying the dynamics of gas and stars in galactic potential. Recent progress in VLBI (Very Long Baseline Interferometry) observations allows Galaxy-scale astrometry with accurate measurements of parallaxes and proper motions of maser sources in the Galactic star forming regions. For instance, VERA (VLBI Exploration of Radio Astrometry) and VLBA (Very Long Baseline Array) BeSSeL (Bar and Spiral Structure Legacy Survey) projects have already conducted astrometry of $\sim 100$ maser sources, which enables us to determine the fundamental parameters of the Galaxy (e.g., Honma et al. 2012; Reid et al. 2014; see also Reid & Honma 2014 for review of VLBI astrometry). VLBI astrometry is currently the only way to obtain the six-dimensional phase-space distributions of Galactic sources located at a distance of a few to $10$ kpc, and thus it provides a unique opportunity to explore the dynamical structure of the Galaxy.
The Galaxy is known to be a normal disk galaxy, and the motions of stars and gas in the Galaxy are quite often expressed as a sum of the circular rotation and noncircular perturbation by spiral arms and/or bar. The circular rotation of the Galaxy is described by the fundamental constants such as $R_0$ and $\Theta_0$, but their accurate determination is still at issue because they provide very important scales of the Galaxy (Reid et al. 2009, 2014; Bobylev & Bajkova 2010; McMillan & Binney 2010; Honma et al. 2012). Moreover, the Galaxy is known to possess asymmetric structure such as spiral arms and the bar in the central region, which makes the Galaxy the best target to test theories of galactic dynamics in detail. For instance, quasi-stationary density wave theory (e.g., Lin & Shu 1964) was proposed as the mechanism for maintaining galactic spiral arms. On the other hand, there also exists another type of spiral arm models in which spiral arms are basically transient and recurrent material arms [e.g., see Julian & Toomre (1966) and Toomre (1981) for early studies, and also see Fujii et al. (2011), Baba et al. (2013), and D’Onghia et al. (2013) for recent results based on numerical simulations].

To observationally discriminate such different dynamical models, comparison of six-dimensional astrometry observations with the models will be of great importance.

The aim of this paper is to investigate how well we can derive Galactic structure such as the fundamental constants and the structure and nature of asymmetric perturbations, such as spiral arms, based on currently ongoing VLBI astrometry. To achieve this, in the current paper we utilize the gas orbit model described by Piñol-Ferrer, Lindblad, and Fathi (2012) as an example to test if the Galaxy model parameters can be determined based on the Markov chain Monte Carlo (MCMC) analyses of simulated sample sources. Previously Sumi et al. (2009) has conducted a similar study as a preparation for the SIM (Space Interferometry Mission) project. The present paper is along the line of Sumi et al. (2009), but this paper targets star-forming regions, which are the main targets of VLBI astrometry. The main difference from Sumi et al. (2009) is that this study is based on the orbit model by Piñol-Ferrer, Lindblad, and Fathi (2012) where the collisional nature of gaseous component is taken into account. This is in contrast to Sumi et al. (2009), where they calculated stellar motions by treating them as collisionless particles.

The structure of the present paper is as follows: in section 2, we present the Galaxy models which we use to simulate the spiral arms and/or the bar of the Galaxy, and also describe how to calculate the gas orbit. In section 3, we describe the simulation method including the creation of simulated maser sources as well as the MCMC technique to solve the Galaxy’s dynamical parameters. In section 4, we show the results of the Galactic parameter determinations for the spiral and bar cases with a fixed number of the sample sources ($N_{\text{obj}} = 500$), and in section 5, we present the dependence of the parameter determination accuracy on the number of sample sources. In section 6, we discuss the possibility of model selection based on the Bayesian information criterion, and also discuss the future prospect for the Gaia era.

## 2 Galaxy model

Here we consider a model galaxy which consists of axisymmetric and asymmetric parts, the latter of which corresponds to the Galactic bar and/or spiral arms. We use the Galactocentric coordinates defined by $(R, \theta, z)$, and here we describe the Galactic potential at the midplane ($z = 0$) by summing axisymmetric and non-axisymmetric potentials as

$$\Phi(R, \theta) = \Phi_0(R) + \Phi_1(R, \theta) = \Phi_0(R) - \sum_m \Psi_m(R) \cos m(\theta - \theta_m(R)).$$

where $\Phi_0$ and $\Phi_1$ correspond to the axisymmetric potential and bar/spiral potential, respectively. Here the perturbation term $\Phi_1$ is described in a frame corotating with the pattern such as spiral arms and/or a bar, with a constant pattern speed $\Omega_p$. Following the treatment by Piñol-Ferrer, Lindblad, and Fathi (2012), the centrifugal and Coriolis forces are not included in the potential above but rather appear in the equations of motions themselves.

Here we assume the perturbation term $\Phi_1(R, \theta)$ can be described as a sum of different modes $m$, but in this paper we mainly focus on $m = 2$ for the spiral arm (and also for the bar). We also note that we assume that the perturbation potential for each mode $m$ is written as multiplication of an $R$-dependent term $\Psi_m(R)$ and a sinusoidal term varying with $\theta$. Here $\theta_m(R)$ describes the phase of potential maximum, and this model can describe not only a bar but also a spiral (or a combination of both) by varying $\theta_m(R)$ with $R$.

### 2.1 Axisymmetric model

For the axisymmetric part, instead of using specific potential, here we use a power-law rotation curve to describe the rotation law in the Galaxy. For a power-law rotation curve, the potential and the circular rotation velocity are related to each other through the following relation,

$$\Theta(R) = \Theta_0 \left( \frac{R}{R_0} \right)^{\alpha}, \quad \Theta(R)^2 = R \frac{d\Phi_0}{dR}.$$  

Here $\Theta_0$ is the circular rotation velocity of the Galaxy at the local standard of rest (LSR), $R_0$ is the distance from the Sun to the Galactic center, and $\alpha$ is the rotation curve index.
which determines the shape of the rotation curve. There are three parameters which describe the circular rotation, namely, \((R_0, \Omega_0, \alpha)\).

The Galactic rotation velocity \(\Theta_0\) is linearly correlated with \(R_0\) through the relation \(\Theta_0 = R_0 \Omega_0\), where \(\Omega_0\) is the angular rotation velocity at the LSR. To avoid this correlation, in the present paper we use \((R_0, \Omega_0, \alpha)\) as the parameters describing the circular rotation instead of \(\Theta_0\), as was the case in Honma et al. (2012).

2.2 Spiral arm

For the spiral model in the present paper, we consider a quasi-stationary density wave in the form of a logarithmic spiral, which is often used to approximate the spiral structures in galaxies (e.g., Binney & Tremaine 2008). For such a spiral arm, the phase angle of the potential maximum \(\theta_m(R)\) is written as a function of \(R\) as

\[
\theta_m(R) = \cot i \ln \left( \frac{R}{R_{\text{ref}}} \right) + \frac{\pi}{2}.
\]

where \(i\) is the pitch angle, which is assumed to be constant in the Galaxy. An additional term of \(\pi/2\) is included to adjust the definition of the spiral phase in equation (1) and the reference radius \(R_{\text{ref}}\). Here we note that in this definition \(R_{\text{ref}}\) provides the location of the potential peak along the \(y\)-axis in figure 1; the Sun is also assumed to be located on the \(y\)-axis, and the angle \(\theta\) is measured from the \(x\)-axis.

Concerning the strength of the spiral perturbation, we simply follow Sumi et al. (2009)’s method and use a constant strength,

\[
\psi_m(R) = c_{\text{sp}}^2,
\]

where \(c_{\text{sp}}\) is the amplitude of the spiral potential, having a dimension of velocity. We also note that the spiral mode \(m\) is fixed at 2 throughout the paper. In summary, there are four parameters that describe the spiral potential, namely, \(\Omega_{\text{sp}}, R_{\text{ref}}, i,\) and \(c_{\text{sp}}\).

2.3 Bar

For the bar model, we used a similar model to those used in Wada (1994) and Piñol-Ferrer, Lindblad, and Fathi (2012), in which the perturbing potential is written as

\[
\psi_m(R) = c_{\text{bar}}^2 \frac{R_{\text{bar}}^2 R^2}{(R^2 + R_{\text{bar}}^2)}.
\]

Here \(c_{\text{bar}}\) is a parameter that describes the strength of the bar in a dimension of velocity just like \(c_{\text{sp}}\), and \(R_{\text{bar}}\) is the bar length. Also, the bar orientation is described as

\[
\theta_m(R) = \phi_{\text{bar}},
\]

where the constant \(\phi_{\text{bar}}\) denotes the orientation angle of the bar with respect to the line-of-sight toward the Galaxy center. Similarly in the case of the spiral model, there are four parameters which describe the bar potential, namely, \(\Omega_{\text{sp}}, R_{\text{bar}}, \phi_{\text{bar}},\) and \(c_{\text{bar}}\).

2.4 Orbit calculation

In order to solve gas orbits in a given potential described above, in the present paper we utilize the analytic solution developed by Piñol-Ferrer, Lindblad, and Fathi (2012). This approach allows us to easily describe gas motion in non-axisymmetric potentials such as a bar and spiral arms. In Piñol-Ferrer, Lindblad, and Fathi (2012), the gas motion in the potential is approximated by introducing a damping term \(\lambda\) to the equations of motion, following the treatment proposed by Wada (1994) and Lindblad and Lindblad (1994) to account for the gas motion in a bar-galaxy potential (see also Sakamoto et al. 1999). The introduction of the damping term avoids divergence at the Lindblad resonances.

The treatment of gas motion with the damping term \(\lambda\) is in the regime of linear approximation of the perturbation. In the real spiral arms and/or bars, it is known that shock occurs in the density wave (e.g., Fujimoto 1968; Roberts 1969) and the gas motion may not be fully described by the approach introduced here, particularly in the shocked
regions. However, as discussed in Piñol-Ferrer, Lindblad, and Fathi (2012), the global gas motion in galaxies can be represented well by this treatment, and the model simplicity makes this type of treatment one of the easiest ways to analytically simulate the gas motion in galaxies.

A further important aspect of the model by Piñol-Ferrer, Lindblad, and Fathi (2012) is the introduction of the softening parameter $\epsilon$, which avoids divergence at corotation. While this softening parameter is rather artificial, the introduction of the parameter makes the model easier for practical use because the gas velocity never diverges at any point.

With the galaxy models described above, we can solve the gas orbits based on equations (9) and (10) of Piñol-Ferrer, Lindblad, and Fathi (2012). Figure 1 shows the gas orbits in the spiral model. The model parameters are summarized in table 1, and here we show the case with $\Omega_p = 20$ km s$^{-1}$ kpc$^{-1}$. In figure 1 the locations of corotation and inner Lindblad resonance are also shown, but the outer Lindblad resonance is outside the plotted area. As seen in the figure, the gas motion occurs along closed orbits with elongated shapes and different position angles, and dense regions of the gas orbits constitute spiral patterns. At the central part of the galaxy in figure 1, the gas orbits cross each other, which makes the orbits unrealistic in that area. The occurrence of such an orbit crossing totally depends on the strength of the spiral potential. However, we note that in the current model such an orbit crossing occurs only in the central part, which is of no interest in the present paper.

Figure 2 shows orbits for the bar model, with a bar orientation angle of $\phi_{bar} = 45^\circ$. The model parameters are summarized in table 2, and here we show the case with $\Omega_p = 40$ km s$^{-1}$ kpc$^{-1}$. In figure 2, the locations of corotation and of inner/outer Lindblad resonances are also shown. Similarly in the spiral case, all the gas orbits are closed, with elongation and orientation varying with the orbit size, constituting spirallike patterns caused by the bar potential, though it is relatively weaker than in the spiral case. In the present paper, these two models provide a base for

| Table 1. MCMC results for the spiral cases (500 sources, $N_{try} = 10^6$). |
|---------------------------------|-----------------|-----------------|-----------------|
| Parameter                      | Input           | Output $\Omega_p = 20$ km s$^{-1}$ kpc$^{-1}$ | Output $\Omega_p = 30$ km s$^{-1}$ kpc$^{-1}$ | Output $\Omega_p = 40$ km s$^{-1}$ kpc$^{-1}$ |
| $R_0$ (kpc)                    | 8.0             | 7.87 ± 0.11     | 8.05 ± 0.12     | 8.02 ± 0.11     |
| $\Omega_0$ (km s$^{-1}$ kpc$^{-1}$) | 30.0           | 30.12 ± 0.20    | 30.28 ± 0.21    | 29.62 ± 0.20    |
| $\alpha$ (km s$^{-1}$)        | 0.05            | 0.0514 ± 0.0053 | 0.0496 ± 0.0062 | 0.0495 ± 0.0048 |
| $U_{i}$ (km s$^{-1}$)         | 5.0             | 5.51 ± 0.55     | 4.36 ± 0.56     | 5.66 ± 0.58     |
| $V_{i}$ (km s$^{-1}$)         | -10.0           | -9.96 ± 0.73    | -9.73 ± 0.55    | -9.86 ± 0.58    |
| $W_{i}$ (km s$^{-1}$)         | 0.0             | 0.10 ± 0.38     | 0.14 ± 0.38     | -0.20 ± 0.37    |
| $\Omega$ (km s$^{-1}$ kpc$^{-1}$) | 20/30/40      | 19.98 ± 0.67    | 30.68 ± 0.37    | 39.61 ± 0.40    |
| $R_{ref}$ (kpc)               | 10.0            | 9.83 ± 0.19     | 10.26 ± 0.13    | 9.98 ± 0.11     |
| $i$ (°)                       | 15.0            | 15.62 ± 0.32    | 15.10 ± 0.40    | 15.34 ± 0.39    |
| $\epsilon$ (km s$^{-1}$)      | 30.0            | 31.80 ± 1.58    | 33.13 ± 1.97    | 29.09 ± 0.85    |
| $\lambda$ (km s$^{-1}$ kpc$^{-1}$) | 5.0           | 7.06 ± 1.50     | 6.35 ± 0.77     | 4.45 ± 0.30     |
| $e$ (km s$^{-1}$ kpc$^{-1}$)   | 4.0             | 4.20 ± 0.54     | 5.27 ± 0.67     | 3.77 ± 0.44     |
| log posterior prob.           | —               | -3187           | -3200           | -3318           |
| $\chi^2$                      | —               | 4024            | 4341            | 5071            |
| DOF                           | —               | 1488            | 1488            | 1488            |

Fig. 2. Closed gas orbits in the face-on view of the model galaxy (bar case). Here $\Omega_p = 40$ km s$^{-1}$ kpc$^{-1}$ is adopted. For other parameters, see table 2. The direction of Galactic rotation is clockwise. Three red circles show the locations of the inner Lindblad resonance, the corotation, and the outer Lindblad resonance.
expanding whether we can determine the dynamical parameters of the Galaxy based on VLBI astrometry of maser sources.

### 3 Simulation method

#### 3.1 Sample creation

Here we summarize how we create sample sources in our simulations. The basic idea is to generate a set of simulated sources which have distribution similar to the real sources so far observed by VERA and VLBA. To do this, we generate sample sources which obey the following probability distributions for the four parameters related to source locations and motions, namely, the galactic longitude \( l \), the source–Sun distance \( D \), the height from the Galactic midplane \( Z \), and the random motion velocity \( v_{\text{rand}} \). We assume that the four parameters follow the probability distributions described below.

\[
p(l) \propto 1 - \frac{l}{245^\circ} \quad \text{for } 0^\circ < l < 245^\circ, \tag{7}
\]

\[
p(D) \propto 1 - \frac{D}{10 \text{ kpc}} \quad \text{for } 0.1 < D < 10 \text{ kpc}, \tag{8}
\]

\[
p(Z) \propto \exp\left(-Z^2/2\sigma_Z^2\right), \tag{9}
\]

\[
p(v_{\text{rand}}) \propto \exp\left(-v_{\text{rand}}^2/2\sigma_{v_{\text{rand}}}^2\right). \tag{10}
\]

The above probability distributions are not physically motivated ones, but they are relatively simple and can reproduce basic trends in the real maser distributions as described below.

The first probability distribution of \( p(l) \) reflects the fact that the current observations are limited to sources observable from the northern hemisphere, and also the fact that there are more maser sources toward the inner Galaxy than the outer Galaxy. The second probability distribution \( p(D) \) describes the source distance distribution; as is usually expected, the favored observed sources are nearby ones as they are easier to observe. Also, this probability distribution sets the maximum measurable distance to 10 kpc, which is consistent with the real observations. For the third and fourth criteria, for the Galactic height \( Z \) and the random velocity \( v_{\text{rand}} \), we assume two components, i.e., normal disk source and “outlier”, the latter of which has larger random velocity and Galactic height distribution. The inclusion of outlier sources is to mimic the real observations, in which a few sources associated with supernova remnants and/or super-bubbles show relatively large deviation from the standard Galactic rotation (e.g., Sato et al. 2007; Sakai et al. 2014). We set the fraction of the normal sources to 90%, and outlier sources are 10% of the whole simulated sources, through the analyses by Honma et al. (2012). In this paper, for the normal disk sources we adopted a \( \sigma_Z \) of 0.4 kpc and \( \sigma_{v_{\text{rand}}} = 7 \text{ km s}^{-1} \) (e.g., McMillan & Binney 2010; Honma et al. 2012), and for the outliers we used a \( \sigma_Z \) of 0.16 kpc and \( \sigma_{v_{\text{rand}}} = 28 \text{ km s}^{-1} \), which are four times as large as than those of nonoutliers.

We generated sample sources by Monte Carlo simulations according to the above probability distributions. First we calculated the source location \((l, D, Z)\) based on the first three probability distributions in equations (7)–(9).
As described above, 10% of the sources are assumed to have larger scale heights as outliers. In figure 3, we show an example of simulated sources (the number of sources, N_{obj}, is 500). For comparison, we also plot the distribution of 52 observed sources listed in Honma et al. (2012). From figure 3 one can see that the distribution of simulated sources resembles that of the real observations.

Once the source positions are given based on the probability distributions described above, the motion of the sources can be calculated by the dynamical model of the Galaxy described in the previous section. As such, we can obtain the three-dimensional velocity (i.e., V_{LSR} and the proper motion) as a combination of the Galactic rotation and noncircular motion by the spiral arms and/or bar. In order to take the source random motion into account, we add the random motion generated with the probability distribution in equation (10). Equation (10) only determined the amplitude of the random motion, and its orientation is assumed to be isotropic.

To simulate the observations, we also add observational errors to the calculated motion as well as the source parallax. To match with the real observations, here we assume a parallax error of 10% for any distance, a proper motion measurement error \sigma = 3 \text{ km s}^{-1} for any distance, and a radial velocity measurement error \sigma = 3 \text{ km s}^{-1}, where the error bars correspond to the 1\sigma error. The parallax and proper motion errors might be distance-dependent, but here we assume that they are independent of the source distance; for instance, in VLBI maser astrometry, there is a significant contribution of maser structure to the parallax error. The maser structure always has the same fractional effect in the parallax, independently of the source distance (e.g., Honma et al. 2010).

### 3.2 MCMC analyses

In order to estimate the best Galactic parameters from the simulated sample sources, we utilize an MCMC method. The basic method is similar to what we have done in Honma et al. (2012), but it is extended to include more parameters for nonaxisymmetric structure such as the spiral arm and the bar.

In the case of the spiral model, the model parameter vector M is given as

$$ M = (R_0, \Omega_0, \alpha, U_s, V_s, W_s, \Omega_p, R_{ref}, i, c_{sp}, \lambda, \epsilon), \quad (11) $$

and for the bar model, it is written as

$$ M = (R_0, \Omega_0, \alpha, U_s, V_s, W_s, \Omega_p, R_{bar}, \phi_{bar}, c_{bar}, \lambda, \epsilon). \quad (12) $$

The parameter sets are similar in both the spiral and the bar models, with the same number of parameters: 12 in total (including the axisymmetric part). The three parameters \((U_s, V_s, W_s)\) describe the mean deviation of the maser sources from the circular rotation curve. These parameters were originally introduced by Reid et al. (2009) to take
into account the possible systematic motion of Galactic star-forming regions.

On the other hand, the observations are described as

\[ O_i = (V_{\text{LSR},i}, \mu_{l,i}, \mu_{b,i}), \quad (13) \]

which describes the three-dimensional motions of the source for a given source location \((l, b, D)\). The subscript \(i\) denotes the source index, which runs from 1 to \(N_{\text{obj}}\).

The posterior probability for a model parameter vector \(M\) is obtained through the Bayes theorem as

\[ P(M|O) = \frac{P(O|M)P(M)}{P(O)} = \frac{P(O|M)P(M)}{\int P(O|M)P(M)dM}. \quad (14) \]

Here \(P(M)\) is the prior probability for a model \(M\), and \(P(O|M)\) is the likelihood of observations \(O\) when a model \(M\) is given. On the left-hand side of equation (14), \(P(M|O)\) shows the posterior probability of a model \(M\) for given observations \(O\). In this paper the parameter set which maximizes the posterior probability \(P(M|O)\) is regarded as the best estimate of the parameters. We also note that \(P(M)\) is the prior probability of having the parameter vector \(M\). The denominator on the right-hand side of equation (14), \(P(O) \equiv \int P(O|M)P(M)dM\), can be regarded as an uninteresting constant. Also, \(P(O|M) = L(O|M)\) is the likelihood, and it can be written as

\[ L(O|M) = \prod_i P_i(O_i|M). \quad (15) \]

Here the probability \(P_i\) is described by using a Gaussian error as detailed in Honma et al. (2012), with a standard deviation determined by considering the observational error and the effect of source random motion. The parallax errors in the simulated sample sources are also considered in the same manner as in Honma et al. (2012), where \(P_i\) is obtained by integrating over a possible range of parallax indicated by its error bar.

In order to sample the posterior probability distribution \(P(M|O)\), here we run a MCMC simulation in which the model vector \(M\) randomly walks in the multidimensional space. We utilized the Metropolis algorithm following Honma et al. (2012). Basically we use flat prior probability, i.e., \(P(M) = \text{const.}\), except for parameters \(\lambda\) and \(\epsilon\), and we additionally introduced a positivity constraint, i.e., \(P(M) = 0\), for a negative value of \(\lambda\) or \(\epsilon\).

In our simulation, we usually generated a Markov chain with a trial number \(N_{\text{try}}\) of \(10^5\) to \(10^6\), and the first 10% of the data are discarded as “burn-in”. After the random walk process converges, we obtained the best parameters and their uncertainties, calculating the expectations and standard deviations of the parameters based on the resultant posterior probability distributions.

\[ \text{Fig. 4.} \quad \text{Distribution of simulated sources and their motions relative to pure circular motions for the spiral case with} \, \Omega_p = 20 \, \text{km s}^{-1} \text{kpc}^{-1} \, \text{(the same as the model galaxy shown in figure 1). Vectors in red are those for outliers, which have larger velocity dispersions.} \]

\[ \text{4 Results I: cases for 500 sources} \]

First we present simulation results for a fixed \(N_{\text{obj}}\), with \(N_{\text{obj}} = 500\) for the spiral and bar cases.

\[ \text{4.1 Spiral case} \]

Figure 4 shows the distributions and motions of 500 simulated sources in the Galaxy (\(\Omega_p = 20 \, \text{km s}^{-1} \text{kpc}^{-1}\)). For the detail of the parameter, see table 1. In this figure, the circular rotation is subtracted, and source motions relative to the circular rotation are shown as vectors. The spiral potential peak is also indicated with a thick curve for reference. As described in the previous section, the sources are mainly located in the Galactic disk visible from the northern hemisphere, and also cluster around the Sun due to the observational bias toward nearby sources. We limited the effective sample of the spiral case to those with \(R > 4 \, \text{kpc}\), because within 4 kpc the two spiral arms are too close to each other and thus the velocity field becomes unrealistic near the Galaxy center.

In figure 4, a systematic deviation from the circular motion is noticeable near the spiral arm, which is certainly due to the perturbation by the spiral potential. Figure 5 is a plot of source velocities along the direction of the Galactic rotation (i.e., a rotation curve plot). A perturbation with \(\pm 30 \, \text{km s}^{-1}\) at maximum is seen in the rotation curve, showing the effect of the spiral potential. This figure shows that while our sample has a bias in the distribution over the Galaxy (as described in subsection 3.1), one can trace the
Galactic rotation curve between 4 and 17 kpc, covering a fairly large fraction of the Galactic disk.

With the simulated observational data for the 500 sources, we have run an MCMC simulation to explore the posterior probability distribution of the parameters ($M$) described in equation (11). The number of trials, $N_{\text{try}}$, was set to $10^6$. Figure 6 shows the posterior probability distributions for the 12 parameters of the spiral model. As seen in the figure, all the parameters have a posterior probability distribution with a single-peaked symmetric structure, which ensures that the current MCMC simulations provide reasonable estimates of the parameters. The best parameters and their uncertainties are presented in table 1, with the initial values of the parameters.

As summarized in table 1, the MCMC simulation successfully determined all the 12 parameters. The deviations of the best parameters from the initial values are basically within statistical uncertainty. Although the source distribution is quite biased (e.g., figure 3), no systematic offset is seen in the estimated parameters. This fact ensures that even with astrometric data obtained with the northern arrays, one can retrieve dynamical parameters of the Galaxy, as long as the parameters can be regarded as constant on the Galactic scale.

Table 1 also shows that the fundamental constants such as $R_0$ and $\Omega_0$ have uncertainties of only $\sim 1\%$ if 500 sources are observed. Also, the mean peculiar motion ($U_v$, $V_v$, $W_v$) is determined at a level of 0.5 km s$^{-1}$ or higher. As for the parameters related to the spiral perturbation (such as $\Omega_p$, $R_{\text{ref}}$, $i$, $c_{sp}$), the uncertainties are slightly larger than the circular parameters, but they are still at a few percent. The parameters related to the gas orbit approximation ($\epsilon$ and $\lambda$) have a relatively larger uncertainty ($\sim 20\%$–$30\%$). However, these are not as of much interest as the other physical parameters because the dependence of the other parameters on $\epsilon$ and $\lambda$ is relatively small.

In table 1, we show the results with three different pattern speeds, $\Omega_p$, of 20, 30, and 40 km s$^{-1}$ kpc$^{-1}$. This is to test if the location of the resonance could affect the accuracy of the Galactic parameter determinations. The three values of $\Omega_p$ approximately correspond to corotation radii of 12, 8, and 6 kpc (outside, near, and inside the solar circle), respectively. From table 1, one can see that the resultant parameter estimates and the error bar sizes are basically independent of the location of the resonance. In fact, in table 1, we also show the degree-of-freedom (DOF) as well as $\chi^2$. With these values, one obtains a reduced $\chi^2$ of $\sim 3$ for all the three cases, indicating that the modeling with the best parameters is reasonable.

In figure 7, we also show the correlations between the parameters. All the plots are mostly single-peaked, and thus they are well-behaved. Relatively strong correlations, with a correlation coefficient larger than 0.5, are seen between some parameters, such as parameter pairs of $R_0$ and $R_{\text{ref}}$, $\alpha$ and $R_{\text{ref}}$, $\alpha$ and $c_{sp}$, $\Omega_p$ and $V_v$, and $\epsilon$. The first pair simply indicates a scaling relation with the Galaxy size (e.g., $R_0$), but the last two pairs imply a correlation between circular rotation and spiral perturbation, for which we may need to use special care when interpreting the results.

4.2 Bar case

The bar sample creation and MCMC simulations are conducted in the same manner as in the spiral case, with the only difference being in the input model. Figure 8 shows the distributions of the simulated maser sources ($N_{\text{obj}} = 500$, and $\Omega_p = 40$ km s$^{-1}$ kpc$^{-1}$). For the details of the parameter, please refer to table 2. The orientation of the bar is indicated with a thick line in the figure. The same as in the case for the spiral model, again the source distribution is biased to the solar vicinity as well as to the northern hemisphere. In the bar case, the residual motion with respect to the circular Galactic rotation is larger toward the Galactic center, as the effect of the bar is stronger in the central part of the Galaxy. Figure 9 shows the rotation curve plot in the bar case. Here the Galactic rotation curve is basically smooth in the range of from $R = 5$ to 17 kpc, but within a distance of 5 kpc from the Galaxy center the deviation is significant, being $\pm 70$ km s$^{-1}$.

In the same manner as carried out in the spiral case, we ran an MCMC simulation to obtain the best parameter estimates for the bar model. The trial number in MCMC is
Fig. 6. Posterior probability distribution for the Galactic parameters (spiral case). From top-left to right-bottom, each panel shows a plot for $R_0$, $\Omega_0$, $\alpha$, $U_s$, $V_s$, $W_s$, $\Omega_p$, $L$, $c_{sp}$, $\lambda$, and $\epsilon$. The number in the brackets indicates the initial value of the parameters.

also set to $N_{\text{try}} = 10^6$. In the bar case, we do not show the figure for the posterior probability distributions, but they are similar to those in the spiral case shown in figure 6, exhibiting a smooth profile with a single peak. The resultant best parameters are summarized in table 2. Again the estimated parameters are basically consistent with the initial parameters within statistical uncertainties. The Galactic constants are determined at an $\sim 1\%$ level, and the perturbation parameters are estimated at a few percent level, similar to the spiral case. For the correlation of the parameters, we do not show the figure for the bar, but relative large correlations with a correlation coefficient larger than 0.5 are found for parameter pairs of $\Omega_0$ and $U_s$, $\alpha$ and $c_{\text{bar}}$, $\alpha$ and $R_{\text{bar}}$, $\alpha$ and $\phi_{\text{bar}}$, $c_{\text{bar}}$ and $\lambda$, $c_{\text{bar}}$ and $\epsilon$, and $\Omega_p$ and $\phi_{\text{bar}}$. Similarly in the spiral cases, in table 2 we show the results with three different pattern speeds, $\Omega_p$, of 20, 30, and 40 km s$^{-1}$ kpc$^{-1}$, to check the effect of the resonance location. The values of reduced $\chi^2$ in the three cases are comparable to those in the spiral case, being 3–5. Also, all the parameters are determined well within the statistical uncertainty independent of the pattern speed. Thus, from table 2, one can again see that the resultant parameter estimates and the error bar sizes are basically independent of the location of the resonance.

The results obtained above for both the spiral and the bar cases indicate that while the source distributions are biased due to the observational effect, the Galactic parameters can be determined reasonably well based on VLBI astrometry observations of $\sim 500$ sources, as far as a
Fig. 7. Correlations between the parameters in the spiral case. In most cases, correlations between parameters are weak, but relatively strong correlations are found for parameter pairs of $R_0$ and $R_{\text{ref}}$, $\alpha$ and $R_{\text{ref}}$, $\alpha$ and $c_{\text{sp}}$, $\Omega_1 p$ and $c_{\text{sp}}$, and $V_s$ and $\epsilon$.

reasonable Galactic model is given (this will be discussed more later). This ensures that the future VLBI observations with VERA, VLBA/BeSSeL and other VLBI arrays will contribute significantly to the accurate determination of the Galactic structure. We note that Sumi et al. (2009) obtained a similar result and that the Galaxy parameters (including spiral perturbation) can be determined well based on astrometry of $\sim 850$ sources if observed with the SIM.

5 Results II: uncertainty dependence on $N_{\text{obj}}$

In this section, we investigate how the Galactic parameter uncertainties vary with the number of objects for which astrometry results are obtained. We created various samples of simulated sources by alternating the number of objects $N_{\text{obj}}$ from 100 to 1000 in steps of 50, resulting in 19 sets of sample sources. The sample creation is done in the same manner as in sections 2 and 3, and then we run an MCMC method to determine the best parameters and their uncertainties, as described in section 4. In the spiral case, below we again use a pattern speed $\Omega_1 p$ of 20 km s$^{-1}$ kpc$^{-1}$, and in the bar case we use a pattern speed $\Omega_2 p$ of 40 km s$^{-1}$ kpc$^{-1}$. These pattern speeds were the same as those mainly used in the previous sections. However, as seen in the previous sections, we note that the parameter determination accuracy is independent of the pattern speed and consequently the location of the resonance.
with increasing $N_{\text{obj}}$ due to statistical effects. Also, the best estimates converge to the initial values, though in some cases there is a deviation slightly larger than the statistical error (this will be discussed later). These results indicate that the initial model parameters can be extracted based on the astrometric observations made with VLBI, provided that a reasonable model is given (see the next section).

Here we comment on the determination of each parameter. First, it is noteworthy that the Galactic constants $R_0$ and $\Theta_0$ can be accurately determined. In figure 10, the upper and lower horizontal-dashed lines in the plots for $R_0$ and $\Theta_0$ show ±5% deviation from the initial value. With $N_{\text{obj}}$ larger than 100, the resultant values of $R_0$ and $\Theta_0$ are well within the ±5% line, and for larger $N_{\text{obj}}$ even the 1%–2% level determination is possible, as already seen in the previous section for $N_{\text{obj}} = 500$.

As for the mean deviation from the Galactic rotation, $U_s$, $V_s$, and $W_s$, the upper and lower horizontal-dashed lines in the plot correspond to ±2 km s$^{-1}$, and their uncertainty is expected to be less than 1 km s$^{-1}$ for $N_{\text{obj}}$ larger than ~300. However, $V_s$ is tightly correlated with $V_\odot$, which is the $V$ component of the solar peculiar motion, as discussed in Honma et al. (2012). Throughout this paper $V_\odot$ is fixed and is not determined in the MCMC, but to obtain this important parameter we need different observations, and presumably the astrometry of relatively nearby stars with Gaia would be the most effective.

As for spiral parameters, accuracy could be slightly less than the parameters related to the circular rotation, but they are still better than 10% in most cases. In figure 10 for $\Omega_p$, $R_{\text{ref}}$, $i$, and $c_p$, the upper and lower horizontal-dashed lines are ±10% deviations. $\Omega_p$, $R_{\text{ref}}$, and $i$ can be obtained at a few percent level, provided that we have $N_{\text{obj}}$ larger than a few hundred. The spiral potential strength, $c_p$, is slightly less accurate but is determined with an accuracy of a few percent beyond $N_{\text{obj}} = 600$. On the other hand, it turns out that $\epsilon$ and $\lambda$ could have relatively larger uncertainty. From the results described above, one can expect that the astrometry of Galactic maser sources numbering ~300 or more, which is a reasonable number of targets for VERA and VLBA/BeSSeL, can in fact strongly constrain the Galactic parameters, including both the circular parameters and the spiral perturbations.

In figure 10, for some parameters such as $R_0$, there occasionally exists discrepancy between the initial and output values that is slightly larger than the formal statistical error. This systematic error is caused by the fluctuations of the simulated sample, including both outliers and nonoutliers; since we generate sample sources based on Monte Carlo simulations, the distribution of the source positions and motions randomly vary with different choice of $N_{\text{obj}}$ and this difference introduces systematic error in addition to

5.1 Spiral case

Figure 10 presents the best estimates of the Galactic parameter vs. $N_{\text{obj}}$ in the spiral case. The initial model parameters are the same as those we used in subsection 4.1. Filled circles in each panel in figure 10 correspond to the best estimate of the parameter with its uncertainty. As seen in the figure, the basic trend is that the parameter uncertainties decrease

\[ V_s \approx 250 \pm 20 \text{ km s}^{-1} \]

\[ U_s \approx 120 \pm 20 \text{ km s}^{-1} \]

\[ W_s \approx 100 \pm 20 \text{ km s}^{-1} \]

\[ \Omega_p \approx 20 \pm 2 \text{ km s}^{-1} \text{ kpc}^{-1} \]

\[ R_0 \approx 8.0 \pm 0.2 \text{ kpc} \]

\[ \Theta_0 \approx 100 \pm 10 \text{ km s}^{-1} \]

\[ c_p \approx 0.1 \pm 0.01 \]

\[ \epsilon \approx 1 \pm 0.2 \]

\[ \lambda \approx 2 \pm 0.5 \]

\[ U_s \approx 120 \pm 20 \text{ km s}^{-1} \]

\[ V_s \approx 250 \pm 20 \text{ km s}^{-1} \]

\[ W_s \approx 100 \pm 20 \text{ km s}^{-1} \]

\[ \Omega_p \approx 20 \pm 2 \text{ km s}^{-1} \text{ kpc}^{-1} \]

\[ R_0 \approx 8.0 \pm 0.2 \text{ kpc} \]

\[ \Theta_0 \approx 100 \pm 10 \text{ km s}^{-1} \]

\[ c_p \approx 0.1 \pm 0.01 \]

\[ \epsilon \approx 1 \pm 0.2 \]

\[ \lambda \approx 2 \pm 0.5 \]
Fig. 10. Parameter determinations with different sample of observed sources (from $N_{\text{obj}} = 100$ to 1000, in steps of 50) in the spiral case. Dashed lines in $R_0$ and $\Omega_0$ plots are for $\pm 5\%$ deviation from the initial value, those in $U_s$, $V_s$, and $W_s$ are for $\pm 2 \, \text{km} \, \text{s}^{-1}$ deviation, and those in $\Omega_1^p$, $R_{\text{ref}}$, $i$, and $c_{sp}$ are for $\pm 10\%$ deviation.

statistical error in the parameter estimates. Hence, one can see how the outliers, which are assumed to be 10% of the sources in the present paper, could affect the best estimates of the parameters. However, as already discussed above, the systematic effect does not alter drastically the parameter estimates, and the Galactic parameters should be determined at a certain level as long as the velocity distribution of the real outliers is not quite different from what we assumed in this paper.

5.2 Bar case

Figure 11 is a set of the plots of the best parameter estimates with various $N_{\text{obj}}$ in the bar case. The initial model parameters are the same as in subsection 4.2. As in figure 10 in the spiral case, filled circles with error bars in each panel in figure 11 correspond to the best estimates of the parameters. The same as in the spiral model, the estimated parameters gradually converge on the initial value, with some systematic deviations on some occasions. Also, the uncertainties of the parameters decrease with increasing $N_{\text{obj}}$. The results confirm again that the parameters can be extracted based on the astrometric observations with VLBI, if a good model is given.

Here we give short comments on each parameter. The Galactic constants $R_0$ and $\Theta_0$ can be accurately determined, the same as in the case for the spiral model. In fact, for $N_{\text{obj}} = 500$ or larger, $\Theta_0$ can be determined at a level of 1%, while $R_0$ shows a slight deviation ($\sim 2\%$) from the initial value. The mean deviations from the Galactic rotation,
Fig. 11. Parameter determinations with a different sample of observed sources (from $N_{\text{obj}} = 100$ to 1000, in a steps of 50) in the bar case. Dashed lines in $R_0$ and $\Omega_0$ plots are for $\pm 5\%$ deviation from the initial value, those in $U_s$, $V_s$, and $W_s$ are for $\pm 2 \text{ km s}^{-1}$ deviation, those in $\Omega_p$, $R_{\text{bar}}$, $\phi_{\text{bar}}$, and $c_{\text{bar}}$ are for $\pm 20\%$ deviation.

$U_s$, $V_s$, and $W_s$, are determined better than 1 km s$^{-1}$ for $N_{\text{obj}}$ larger than $\sim 300$. Therefore, Galactic constants and circular rotation law can be accurately determined in both spiral and bar cases as long as reasonable models are given as input. In the case where priori knowledge is unavailable for a reasonable model, we can select a better model from among different types of model based on the Bayesian information criterion, as will be discussed in the next section. As for the bar parameters, the horizontal-dashed lines in figure 11 show the parameter space within $\pm 20\%$ of the initial value. The figure shows that the bar parameters such as the pattern speed $\Omega_p$ and the bar strength $c_{\text{bar}}$ are determined with an accuracy of 10% for $N_{\text{obj}}$ larger than $\sim 500$.

As already seen in the spiral case, there occasionally exists slight deviation of the estimated parameter from the initial value. As we have already discussed in the previous subsection, the reason for this is the statistical fluctuations in the simulated sample of both outliers and nonoutliers, i.e., variation in sample source distributions in the positions and velocities with varying $N_{\text{obj}}$. Another point which has to be taken into account in the bar case is the effect of the source distribution in the inner regions. In order to better constrain the bar parameters, of course having more sources in the central regions would be more effective in terms of statistics. However, on the current sample creation criteria, the number of sources near the Galactocentric region is quite limited, and hence the fluctuation in the number of sources near the bar region could also affect the parameter determination.
6 Discussion

6.1 Model selection

We have seen that if the underlying dynamical model is known, the dynamical parameters of the Galaxy can be precisely determined based on VLBI astrometry of hundreds of maser sources. However, in practice, the dynamical model of the Galaxy is still under investigation, and thus we cannot exactly know the true model and the set of the parameters which describe the source motions best. To overcome this issue, presumably the most practical way is to prepare different models to be tested and to run the analysis as described in this paper with these models, and to select the best model based on the goodness of the model.

In such a procedure, the key is how to select a better model from among several possible ones. Basically such a model selection can be done based on statistical techniques such as \( \chi^2 \) and/or the posterior probability obtained by an MCMC method. In fact, if the number of parameters is the same within a different class of models, simply comparing \( \chi^2 \) or the posterior probability works well to judge which model represents the observations better. However, if the models considered have different numbers of parameters, direct comparison based on such as \( \chi^2 \) is not easy, as generally a model with more parameters provides better results, and would lead to an overfitted model with number of parameters more than necessary.

For a comparison between the models with different numbers of parameters, here we use the Bayesian information criterion (BIC; see, e.g., Feigelson & Babu 2012), which is often used when comparing models with different numbers of parameters. The value, \( BIC \), is expressed as a sum of the term related to posterior probability and the penalty term as follows,

\[
BIC = -2 \ln(L) + k \ln(n). \tag{16}
\]

Here \( L \) is the likelihood of observations when a model is given, \( k \) is the number of parameters, and \( n \) is the number of observational data that is equal to \( 3N_{\text{obj}} \), as the observables to be compared with the model are three-dimensional motions. The first term describes the goodness of model, and the second term is the penalty for introducing more parameters. By definition, a model with a smaller \( BIC \) is preferred.

We calculated the BIC for the simulation data with different models, and discuss if model selection can be properly done based on the BIC. We created simulation data as described in the previous sections using the spiral or the bar model as the initial model. Here the number of objects is fixed at \( N_{\text{obj}} = 500 \), and we run an MCMC analysis using different fitting models: (1) an axisymmetric model (without spiral or bar), (2) the spiral model, and (3) the bar model. The axisymmetric model has only six parameters \( (k = 6) \), while the spiral and bar models have 12 parameters \( (k = 12) \).

Table 3 summarizes the likelihood, \( \chi^2 \) (only for reference), and \( BIC \) for the data generated by the spiral and bar models. The first set of three cases is that for the data generated by the spiral model and fitted with the (original) spiral model, the bar model, and the axisymmetric model without perturbation components. As seen in the table, the spiral sample sources are best represented by the spiral model in terms of \( BIC \). If one only considers axisymmetric potential, the fitting result is not as good as that based on the spiral case. Also, if one uses a wrong model, such as the bar model, the fitting residual such as \( \chi^2 \) as well as \( BIC \) is worse than when the correct model is used.

The same applies to the bar case, where the data are best described by the original bar model. These results indicate that one can safely discriminate different models using \( BIC \), which leads us consequently to the best model of the Galaxy.

6.2 Future prospect for the Gaia era

The Gaia mission (e.g., Lindegren et al. 2008) was launched in late 2013, and it will conduct astrometric observations of numerous stars at optical bands. While the number of the sources for which astrometry data will be available is totally different between VLBI astrometry \( [O(10^3)] \) and the Gaia mission \( [O(10^9)] \), these astrometric observations are complementary to each other. Gaia observes stars, which are collisionless systems, and VLBI arrays observe star-forming regions, which are gaseous and collisional systems. Therefore, the responses of these two classes of objects to the Galactic potential are totally different.

With this in mind, one can expect to constrain the Galaxy model more strongly calculating stellar orbits and
comparing them with the observations as well. In fact, stellar motions and velocity dispersions will be the quantities to be compared with Gaia’s observational data, and will provide an opportunity to test if the Galaxy model obtained with VLBI astrometry indeed represents the true dynamical properties of the Galaxy including stellar components. Such a test will be carried out in the early 2020s when Gaia’s results are available, and hopefully this will improve our understanding of the structure of the Galaxy significantly.

One of the major issues in galactic dynamics is whether the spiral arm is a persistent density wave just as treated in the present paper or whether it is a transient material wave. Recent numerical simulations suggested that the spiral arms are more or less transient and recurrent rather than persistent waves (e.g., Wada et al. 2011; Fuji et al. 2011; Grand et al. 2012; Baba et al. 2013). Also there are some observational studies showing that the pattern speed measurements based on the Tremaine–Weinberg method (Tremaine & Weinberg 1984) are in favor of a material wave with a radially varying pattern speed rather than a density wave with a constant pattern speed (e.g., Westpfahl 1998; Merrifield et al. 2006; Meidt et al. 2009; Speights & Westpfahl 2012).

However, the current situation is not at all simple, and indeed there could be large variety among galaxies; for instance, pattern speed measurement for galaxies claims that there are also some galaxies with a nearly constant pattern speed (e.g., Meidt et al. 2009). Also, simulation work by Roca-Fàbrega et al. (2013) suggested that a density wave with a constant pattern speed could occur in their strong bar model. Therefore, the nature of the spiral and/or bar may differ from galaxy to galaxy.

In order to determine the nature and characteristics of our own Galaxy’s spiral structure, we definitely need more observations of both gaseous (i.e., VLBI astrometry of maser sources) and stellar (i.e., optical astrometry with Gaia and others) components, which should be compared with theoretical models. Recently, Kawata et al. (2014) tried to compare their simulation results with ~100 sources observed with VLBI. Their approach to directly compare the observations and simulation results is a promising way, but the issue is that the comparison should be made with numerous “snap-shots” of the simulated galaxy. Alternatively, if the parametrized approximation used in the current paper is also available for transient/recurrent arms the same as the models used in this paper for persistent density wave, then the approach introduced here would also be applicable and the BIC technique discussed above would be useful for discriminating the different types of model. In any case, comparisons based on both persistent and recurrent spiral models are definitely needed and will hopefully be crucial to an understanding of the spiral structure of the Galaxy.

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**References**


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