Singular Vertices in the Strong Coupling Phase of Four-Dimensional Simplicial Gravity

Tomohiro Hotta, Taku Izubuchi and Jun Nishimura

Department of Physics, University of Tokyo, Tokyo 113

National Laboratory for High Energy Physics (KEK), Tsukuba 305

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We study four-dimensional simplicial gravity through numerical simulation with special attention given to the existence of singular vertices, in the strong coupling phase, that are shared by abnormally large numbers of four-simplices. We attempt to solve this problem by adding a term to the action which suppresses such singular vertices. For a sufficiently large coefficient of the additional term, however, the phase transition disappears and the system is observed to be always in the branched polymer phase for any gravitational constant.

Due to precise measurements of gauge coupling constants at LEP, it is now customary to consider the three interactions which are described quite well by the Standard Model to be unified into a Grand Unified Theory at about an energy scale of $10^{16}$ GeV. On the other hand, it is well known that the gravitational interaction between particles becomes nonnegligible at the Planck scale, which is around $10^{19}$ GeV. These two energy scales are remarkably close to each other considering the ambiguity involved in the derivation of their values. The most natural interpretation of this fact is that, around such a high energy scale, all the four interactions, including gravity, are unified. There are two possibilities for such unified theories at present. One is string theory and the other is a unification within ordinary field theory including gravity.

The problem we encounter when we try to formulate quantum gravity within ordinary field theory in four dimensions is that we cannot renormalize it perturbatively. In the path integral formalism, quantization of the geometry is performed by integrating over the metric field. If we use lattice regularization, which enables a nonperturbative study, general coordinate invariance is not manifest and, whether or not it is restored in the continuum limit is a crucial problem. There are two kinds of lattice regularization of quantum gravity, dynamical triangulation and Regge calculus. The former is believed to restore general coordinate invariance in the continuum limit. It has been solved exactly in two dimensions, and its continuum limit is shown to reproduce Liouville theory, in which general coordinate invariance has been treated carefully. There is also a handwaving argument for the restoration of general coordinate invariance in the continuum limit of dynamical triangulation.

Although four-dimensional dynamical triangulation seems to be difficult to carry out analytically, it can be readily studied numerically. Employing the Einstein-Hilbert action as the lattice action and sweeping the gravitational constant, it has been discovered that the system undergoes a second order phase transition, suggesting the possibility of taking a continuum limit. The geometrical features of each
phase have the following characteristics. In the weak coupling phase, the configurations are made of relatively small baby universes connected through pinches. This phase is called the branched polymer phase. In the strong coupling phase, on the other hand, the configuration has a mother universe with very high connectivity, which seems to be due to the existence of vertices shared by a very large number of four-simplices. This phase is called the crumpled phase.

In the continuum theory, it is well-known that Euclidean gravity may have a problem related to the unboundedness of the Einstein term. Decomposing the metric into a conformal mode and a transversal mode, one finds that the unboundedness results from the fact that we can make the Einstein term arbitrarily large a negative value by choosing a rapidly varying conformal mode. This problem of the unboundedness of the Einstein term is, therefore, called 'the conformal mode instability'. Since the coefficient of the Einstein term is proportional to the inverse of the gravitational constant, it is natural for us to relate this problem to the branched polymer structure observed in the weak coupling phase.

In this respect, the strong coupling phase seems to be more promising than the weak coupling phase. At least there is a mother universe, although it might be plagued with singular vertices shared by abnormally large numbers of four-simplices. Let us call the number of four-simplices sharing a vertex the vertex order. In this paper, we measure the vertex order distribution with high statistics. We observe two singular vertices which show themselves as a sharp peak. Moreover, we find that the peak moves towards a larger value of vertex order linearly as we increase the system size. Considering this fact to be a potential obstacle to taking a continuum limit, we add a term in the action which suppresses such singular vertices. For a sufficiently large coefficient of the additional term, however, we find that the phase transition disappears and that the system is always in the branched polymer phase.

Let us first give a general outlook on this kind of approach. When we regularize four-dimensional quantum gravity with dynamical triangulation, the integration over the metric is replaced with a random summation over all the four-dimensional simplicial manifolds. It is this point of the formalism that is essential in recovering general coordinate invariance in the continuum limit, as can be seen from the hand-waving argument in Ref. 3). Note that the lattice action can therefore be chosen arbitrarily, as long as it is local without being restricted by general coordinate invariance. We must search for a sound second order phase transition, where we can take a sensible continuum limit, by changing the lattice action in various ways. Considering universality in quantum gravity, as well as in ordinary field theories, we expect that the continuum theory, if it exists at all, does not depend on the way we construct it.

It is natural for us to start with the Einstein-Hilbert action

$$ S = \int d^4 x \sqrt{g} \left( \Lambda - \frac{1}{G} R \right) $$

as a first trial, where $\Lambda$ is the cosmological constant, and $G$ is the gravitational constant. Let us denote the number of $i$-simplices in a simplicial manifold by $N_i$. One can easily find that, for a simplicial manifold,
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\[ \int d^4x \sqrt{g} = cN_4, \]  
\[ \int d^4x \sqrt{g} R = 2\pi N_2 - 10\alpha N_4, \]

where \( c \) is the volume of each four-simplex, and \( \alpha \) is the angle between two faces of a four-simplex, and is equal to \( \arccos(1/4) \). Therefore the Einstein-Hilbert action (1) can be expressed in terms of lattice variables as

\[ S_{\text{lat}} = \kappa_4 N_4 - \kappa_2 N_2, \]

where \( \kappa_4 \) and \( \kappa_2 \) are related to \( \Lambda \) and \( G \) through

\[ \kappa_4 = c\Lambda + \frac{10\alpha}{G}, \]
\[ \kappa_2 = \frac{2\pi}{G}. \]

The \( N_i \)'s \((i=0, 1, \cdots, 4)\) satisfy the following three relations:

\[ N_0 - N_1 + N_2 - N_3 + N_4 = x, \]
\[ 2N_1 - 3N_2 + 4N_3 - 5N_4 = 0, \]
\[ 5N_4 - 2N_3 = 0, \]

where \( x \) is the Euler number, which is two for the spherical topology. Therefore only two of the \( N_i \)'s are independent, which means that, actually, the lattice action (4) is the most general one that can be written as a linear combination of the \( N_i \)'s.

We consider an ensemble with fixed \( N_4 \) and spherical topology. There are well established methods for generating such an ensemble through numerical simulations, and the technical details of our simulation shall be given elsewhere,\(^6\) except for a few comments. The initial configuration is taken to be the surface of a five-simplex, and we use the so-called \((p, q)\)-moves\(^7\) to update a configuration. Since the topology is not changed by the moves, the spherical topology of the initial configuration is maintained throughout the run. Our code is written for arbitrary dimension following Ref. 8).

Let us turn to the results of our simulation. We first look at the second order phase transition, which can be seen through thermodynamic quantities such as the average curvature per unit volume:

\[ R_{\text{av}} = \frac{R_{\text{tot}}}{N_4}, \]
\[ (R_{\text{tot}} = \int d^4x \sqrt{g} R) \]
\[ = 2\pi \frac{N_2}{N_4} - 10\alpha. \]

Figure 1 shows our results for \( \langle R_{\text{av}} \rangle \) at various \( \kappa_2 \)'s. In contrast to the three-dimensional case,\(^9\) no hysteresis has been observed. Also, one sees that the size dependence of the data changes abruptly at \( \kappa_2 = 1.2 \sim 1.3 \). On the right there is little size dependence, whereas on the left, the curve goes lower and lower as we increase
the system size. The derivative of the average curvature gives the susceptibility

$$\chi_k = 2\pi \frac{\partial \langle R_{\text{rv}} \rangle}{\partial k_2} = \frac{\langle R_{\text{tot}}^2 \rangle - \langle R_{\text{tot}} \rangle^2}{N_4},$$

(12)

which represents the fluctuation of the total curvature. Figure 2 shows the result. As is expected from Fig. 1, the susceptibility has a peak around \(k_2 = 1.2 \sim 1.3\), which grows higher as the system size is increased. This implies that the correlation length of the local curvature diverges at the critical point, where we may hope to take a continuum limit. These results are in reasonable agreement with the data given in Refs. 4 and 12. Since \(k_2\) corresponds to the inverse of the gravitational constant, as is seen from (6), we call the large \(k_2\) phase the weak coupling phase and the small \(k_2\) phase the strong coupling phase.

However, as mentioned earlier, the properties in both phases seem to be rather strange. In particular, considering the possibility of taking the continuum limit from the strong coupling phase, the singular vertices that are shared by abnormally large numbers of four-simplices might be an obstacle. We would like to study this phenomenon in detail.

Let us look at the vertex order distribution, which can be defined as

$$\rho(n) = \frac{1}{N_0} \langle \sum_v \delta_{\varnothing(v), n} \rangle,$$

(13)

where \(\varnothing(v)\) is the order of the vertex \(v\). This quantity is measured every 100 sweeps and averaged over 100 configurations. In order to reduce the fluctuations of the distribution, we smear the data over bins of size 10.

We first note that the average vertex order \(\varnothing(v)\) can be given as

$$\overline{\varnothing(v)} = \frac{1}{N_0} \sum_v \varnothing(v) = \frac{5N_4}{N_0}.$$
In the last equality, we used the relation
\[ \sum_v o(v) = 5N_4, \] (15)
which comes from the fact that each four-simplex has five vertices. Thus one finds that the average curvature per unit volume \( R_{av} \) can be written in terms of \( o(v) \) as
\[ R_{av} = 2\pi \frac{N_2^2}{N_4^2} - 10a, \] (16)
\[ = 20\pi \overline{o(v)^{-1}} - (10a - 4\pi) - \frac{8\pi}{N_4}, \] (17)
\[ \sim 20\pi \overline{o(v)^{-1}} - 0.614, \quad (N_4 \gg 1) \] (18)
where in the second equality we used the relation
\[ N_2 = 2(N_0 + N_4 - 2). \] (19)

Therefore, we can deduce from Fig. 1 how the average vertex order \( \overline{o(v)} \) should behave. First of all, the average vertex order should be small in the weak coupling phase and large in the strong coupling phase. As for its behavior for increasing system size, in the weak coupling phase it should not change much, whereas in the strong coupling phase it should grow larger and larger.

In Fig. 3 we show the vertex order distribution \( \rho(n) \) for \( \kappa_2 = 0.0, 1.267 \) (near the critical point) and 2.0 with \( N_4 = 32000 \). For \( \kappa_2 = 2.0 \), the distribution damps quite rapidly for large vertex order, and we have confirmed that it remains almost unchanged when we increase the system size. For \( \kappa_2 = 0.0 \), on the other hand, we find that there is a peak of very large vertex order, as large as one third of the total number of four-simplices. We have also confirmed that the peak consists of two

![Fig. 3. The vertex order distribution for \( \kappa_2 = 0.0 \) (solid curve), 1.267 (dotted curve) and 2.0 (dashed curve) with \( N_4 = 32000 \).](https://academic.oup.com/ptp/article-abstract/94/2/263/1851249/Singular-Vertices-in-the-Strong-Coupling-Phase-of/608736)

![Fig. 4. The vertex order distribution for \( \kappa_2 = 0.0 \) with \( N_4 = 8000 \) (dash-dotted curve), \( N_4 = 16000 \) (long dashed curve), \( N_4 = 32000 \) (dashed curve), \( N_4 = 64000 \) (dotted curve) and \( N_4 = 128000 \) (solid curve).](https://academic.oup.com/ptp/article-abstract/94/2/263/1851249/Singular-Vertices-in-the-Strong-Coupling-Phase-of/608737)
vertices. In Fig. 4 we display the size dependence of the vertex order distribution for $\kappa_2=0.0$. We find that the very large vertex order grows linearly as $N_4$ is increased.

As mentioned earlier, in the strong coupling phase, one already knows from the behavior of $\langle R_{av} \rangle$ that the average vertex order should grow as the system size is increased. However, what we have found by looking at the vertex order distribution is that the distribution does not shift to the right uniformly but that the two singular vertices move to the right linearly with increasing system size, leaving the continuum part which, as we also observe, moves to the right but much more slowly than the singular vertices. This seems to be a potential obstacle to taking a sensible continuum limit from the strong coupling phase.

As we emphasized earlier, we still have the freedom to modify the action. We, therefore, attempt to solve the above problem by adding the following term to the action,\(^*\)

$$S_u = \frac{u}{2} \sum_v (\sigma(v) - C)^2.$$  \hspace{1cm} (20)

We note that, since a change of the constant $C$ can be absorbed into a redefinition of $\kappa_2$ and $\kappa_3$, we can fix it to an arbitrary value without loss of generality. We choose $C=5$, considering $\sigma(v) \geq 5$ to be guaranteed from the requirement that the configuration should be a simplicial manifold in a strictly mathematical sense. We have performed simulations with $u=5 \times 10^{-3}, 5 \times 10^{-4}$ and $5 \times 10^{-6}$. The vertex order distribution is measured every 100 sweeps and averaged over 100 configurations as in the $u=0$ case. In order to reduce the fluctuations of the distribution, we smear the data over bins of size 5.

In Fig. 5 we show the vertex order distribution for $u=5 \times 10^{-3}$ with $N_4=32000$. One sees that there is no singular vertex with very large order for all $\kappa_2$. On the other

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\(^*\) For some studies on other types of modified actions with different motivations, see Refs. 11) and 12).
hand, we find that for \( u = 5 \cdot 10^{-6} \), the results resemble the ones for \( u = 0 \).

Let us next turn to the curvature and the susceptibility. Figure 6 shows \( \langle R_{av} \rangle \) as a function of \( \kappa_2 \) for \( u = 5 \cdot 10^{-3} \), \( 5 \cdot 10^{-4} \) and \( 5 \cdot 10^{-6} \). We have replotted the data for \( u = 0 \). For \( u = 5 \cdot 10^{-6} \), the curves resemble the \( u = 0 \) case. On the other hand, for \( u = 5 \cdot 10^{-3} \) and \( 5 \cdot 10^{-4} \), the curves for \( N_4 = 8000 \) and 32000 coincide throughout the \( \kappa_2 \) region examined. Figure 7 shows the susceptibility \( \chi' \) as a function of \( \kappa_2 \) for \( u = 5 \cdot 10^{-3} \), \( 5 \cdot 10^{-4} \) and \( 5 \cdot 10^{-6} \). Although we see a broad peak even for \( u \geq 5 \cdot 10^{-4} \), it remains unchanged when we increase the system size, in contrast to the \( u = 0 \) case.

These results imply that the second order phase transition which is observed in the \( u = 0 \) case (Fig. 2) disappears for sufficiently large \( u \). We also observe that the configurations for \( u \geq 5 \cdot 10^{-4} \) have much the same properties as the ones in the weak coupling phase of the \( u = 0 \) case, i.e., no mother universe, low Hausdorff dimension, little size dependence in the vertex order distribution, etc. Therefore we claim that in the presence of the additional term, with a sufficiently large coefficient in the action, the system is in the branched polymer phase for any \( \kappa_2 \).

To summarize, we found that, in the strong coupling phase, there are two singular vertices with large vertex order which grows linearly as the system size is increased. We consider this to be a potential obstacle to taking a sensible continuum limit from the strong coupling phase. Adding a term like (20) turns out to have as strong an influence on the system as to erase the phase transition for a sufficiently large coefficient. A local term having such a large influence on the system is in itself an interesting fact. Also, that we find a branched polymer structure with a large average vertex order is worth mentioning. The fact that the size dependence completely disappears for a sufficiently large coefficient of the additional term provides an intuitive picture of the branched polymer structure: the typical configuration is composed of baby universes which are fluctuating independently, and increasing the system size merely contributes to increasing the number of baby universes without changing the local properties of the configuration. In order to take a sensible continuum limit, it seems that we need to fine-tune \( u \) or try other types of modifications of the action.

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