A Consistent Microscopic Description of Rotational Motion in Even-Even Deformed Nuclei

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The idea of using sum rules expressing different single-particle transition operators in terms of higher powers of other operators, recently devised by D. H. E. Gross and the present author, is applied to the description of nuclear rotational motion in a proton-neutron system. For simplicity, we restrict ourselves to the single $j_p$-and $j_n$-shell model with a quadrupole interaction. Moment of inertia, quadrupole moments, transition probabilities and collective $g$-factor show excellent agreement with those given by the cranking model.

§ 1. Introduction

Recently microscopic descriptions for nuclear collective motions have been proposed by some authors from new viewpoints. The essential difference of these viewpoints from the conventional ones is in investigating microscopic structures of nuclear collective motions without introducing any "free" ground state, which plays an essential role in conventional microscopic theories. These new theories seem to be very useful for the investigation of rotation-like excitation. If we start from the cranking picture in order to describe rotational motion, we are forced to introduce the deformed Hartree or Hartree-Bogoliubov ground state and to use, to one degree or another, artificial devices of forced rotation. However, in order to have a consistent microscopic description of rotational excitation, we cannot but accept the viewpoint that rotational states may arise as a consequence of the interaction of particles in a spherical potential well.

On the basis of this viewpoint, the present author investigated nuclear rotational motion of one kind of nucleons with a quadrupole interaction in the case of the single-$j$ shell model, together with D.H.E. Gross (hereafter referred to as [I]). The theory [I] starts from the basic picture that a quadrupole mode should be strongly enhanced in rotational states, as is suggested by the conventional rotational model. From the equation of motion with respect to this strongly enhanced mode, we can get a sum rule with respect to matrix elements of transition operators which characterize the rotational motion (we call it the sum rule by dynamical restriction). On the other hand, by introducing some relations which govern kinematically the transition operators characterizing the rotational motion, we can obtain sum rules with respect to the matrix elements of the ope-
rators, which come from kinematical restrictions. From these sum rules, we can conclude that the rotational spectrum appears in a case where transition probabilities and quadrupole moments satisfy the relations which are predicted by the conventional rotational model. These are the essential points of [1].

The main aim of the present work is to show that the basic idea of [1] may be extended naturally to the proton-neutron system. In this case also, we shall start from the basic picture of a conventional rotational model; in other words, the quadrupole mode should be strongly enhanced. However, there is an additional aspect to take into account, which does not exist in the case of [1], i.e., proton and neutron systems are under the restriction that the directions of principal axes of both systems be the same. On the basis of such pictures, we can get sum rules which come from the dynamical restrictions. On the other hand, the above-mentioned kinematical sum rules can be obtained in a way analogous to that in [1]. From these sum rules, the same conclusion as in [1] can be obtained.

For simplicity, we start from the $j-j$ coupling shell model with a quadrupole interaction in which $N_p$ protons and $N_n$ neutrons are in the single particle states $j_p$ and $j_n$, respectively. After giving our basic Hamiltonian, we will discuss basic assumptions (§ 2). In § 3, we shall give equations of motion for quadrupole modes and investigate the detail structure of these equations. It will be emphasized that the behavior of matrix elements of transition operators of rank 3 is essential for the appearance of the rotational spectrum. Section 4 will be devoted to the derivation of sum rules which are sufficient to determine all transition probabilities, quadrupole moments and excitation energies. In § 5, we will predict that for the mass number $N_p \approx 0.35 \cdot (2j_p+1)$, $N_p \approx 0.65 \cdot (2j_p+1)$, $N_n \approx 0.35 \cdot (2j_n+1)$ and $N_n \approx 0.65 \cdot (2j_n+1)$ the spectrum is purely rotational, and transition probabilities and quadrupole moments satisfy the relations given by the rotational model. In § 6, we shall discuss the validity of our treatment, and comparison with a cranking model with axial symmetric deformed Hartree field will be made. One of our conclusions is that the proton-neutron interaction should be strongly attractive or repulsive, if the proton and the neutron systems have the same shapes or opposite shapes, respectively. Our results show excellent agreement with the cranking model with respect to the moment of inertia, intrinsic quadrupole moments and collective $g$-factor.

§ 2. Basic picture

i) Hamiltonian

For simplicity, we start from the $j-j$ coupling shell model with a quadrupole interaction in which $N_p$ protons and $N_n$ neutrons are in the single particle orbits $j_p$ and $j_n$, respectively (generally $j_p \approx j_n$, and $N_p$ and $N_n$ are even):
\[ H = -\frac{1}{2} \chi_\rho \sum_M (\tilde{Q}_{2 M}(\rho) \tilde{Q}_{2 M}(\rho) + \tilde{Q}_{2 M}'(\rho) \tilde{Q}_{2 M}'(\rho)) \\
- \frac{1}{2} \chi_\sigma \sum_M (\tilde{Q}_{2 M}(\rho) \tilde{Q}_{2 M}(n) + \tilde{Q}_{2 M}'(\rho) \tilde{Q}_{2 M}'(n)) \\
+ \frac{1}{2} \sum_{k, i = p_n} \gamma_{k, i}^{(2)} \sum_M \tilde{Q}_{2 M}(k) \tilde{Q}_{2 M}(l) , \tag{2.1a} \]

where
\[ \left( \begin{array}{c} \gamma_{pp}^{(2)} \\ \gamma_{np}^{(2)} \\ \gamma_{np}^{(2)} \end{array} \right) = \left( \begin{array}{cc} \rho & \sigma \\ \sigma & \rho \end{array} \right) \quad (\rho > 0) \tag{2.1b} \]

Here \( \chi_\rho \) is the strength of the proton-proton interaction, which is equal to the neutron-neutron interaction, and \( \chi_\sigma \) is the strength of the proton-neutron interaction. For the convenience of our discussion, let us introduce the following operators:

\[ \tilde{N}_{00}(k) = \sqrt{2j_\rho + 1} \tilde{B}_{00}(k) \] (particle number),
\[ \tilde{Q}_{SM}(k) = q_k \tilde{B}_{SM}(k) \] (quadrupole moment),
\[ \tilde{J}_{M}(k) = \frac{j_\rho (j_\rho + 1) (2j_\rho + 1)}{3} \tilde{B}_{SM}(k) \] (angular momentum),
\[ \tilde{J}_{M}(k) = -\sqrt{\frac{7}{8}} \sqrt{\frac{(2j_\rho - 1)(2j_\rho + 3)}{(2j_\rho - 2)(2j_\rho + 4)}} \sqrt{\frac{j_\rho (j_\rho + 1)(2j_\rho + 1)}{3}} \tilde{B}_{SM}(k) \]

where \( k \) means the proton or neutron and \( \tilde{B}_{SM}(k) \) is defined by
\[ \tilde{B}_{SM}(k) = \sum_{m_k m_k'} \langle j_\rho m_k j_\rho m_k' \mid J M \rangle C_{m_k}^+ (-y_{k+m_k} y_{k-m_k}) . \tag{2.3} \]

**Basic assumptions**

Now let \( |\gamma_\rho; LL_z \rangle \) represent one of the members of the ground-state rotational band which we intend to describe. The quantities \( L \) and \( L_z \) are the quantum numbers of angular momentum (in our case \( L = 0, 2, 4, \ldots \)) and its projection, respectively, and \( \gamma_\rho \) denotes a set of additional quantum numbers specifying the band. In the same way, we introduce the state \( |\gamma_1; LL_z \rangle \), which belongs to the other bands. Here \( \gamma_1 \), \( I \) and \( I_z \) are similar to \( \gamma_\rho \), \( L \) and \( L_z \).

First, we make the following assumptions in the same way as in \( \Pi \): (I) every state \( |\gamma_1; LL_z \rangle \), which belongs to the other bands, is much higher than the ground-state rotational band; and (II) we can neglect the couplings between the ground-state rotational band and the other bands completely. Under such assumptions, the following matrix elements are important for our description of rotational motion, as will be shown in \( \S \ 3 \):

\[ \langle \gamma_1; LL'_z \mid \tilde{N}_{00}(k) \mid \gamma_\rho; LL_z \rangle = N_k(L'), \quad (N_k(L') = N_k) \]
\[ \langle \gamma_1; LL'_z \mid \tilde{Q}_{SM}(k) \mid \gamma_\rho; LL_z \rangle = -Q_k(L') \sqrt{\frac{L'(L'+1)}{(2L'-1)(2L'+3)}}, \]
\[ \langle \gamma_1; LL'_z \mid \tilde{Q}_{SM}(k) \mid \gamma_\rho; LL_z \rangle = P_k(L') \sqrt{\frac{3}{2} \frac{(L'+1)(L'+2)}{(2L'+1)(2L'+3)}}, \quad (L = L' + 2) \]
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\[ \langle \gamma_0; L' | \hat{J}_0(k) | \gamma_0; L' \rangle = J_0^{(s)}(L') \sqrt{L'(L' + 1)} , \quad (\sum_k J_0^{(s)}(L') = 1) \]

\[ \langle \gamma_0; L' | \hat{J}_s(k) | \gamma_0; L' \rangle = J_s^{(s)}(L') \sqrt{\frac{3}{2} \sqrt{\frac{(L' - 1)(L' + 2)}{(2L' - 1)(2L' + 3)}} \sqrt{L'(L' + 1)}}, \]

\[ \langle \gamma_0; L' | \hat{J}_e(k) | \gamma_0; L' \rangle = -I_e^{(s)}(L') \sqrt{\frac{5}{4} \sqrt{\frac{(L' + 2)(L' + 3)}{(2L' + 1)(2L' + 3)}} \sqrt{L'(L' + 1)}}, \]

\[ (L = L' + 2) \] (2.4)

where we used the Wigner-Eckart theorem

\[ \langle \gamma_0; L' L_e' | \hat{T}_{JM} | \gamma_0; LL_e \rangle = \langle \gamma_0; L' | \hat{T}_{JM} | \gamma_0; LL_e \rangle \langle LL_e JM | L' L_e' \rangle, \] (2.5)

and \( k \) in Eq. (2.4) means proton or neutron.

According to the conventional rotational model, the following picture is essential: (III) the system is strongly deformed with a quadrupole shape and (IV) the proton and the neutron systems rotate under the restriction that the directions of the principal axes of both deformed systems are the same. This will be the starting point of our treatment. Condition (IV) tells us that the deformation parameters of proton system, \( \alpha_2M(p) \), should be proportional to those of neutron system, \( \alpha_2M(n) \), with \( M \)-independent proportional constant. If \( \alpha_2M(p) \) and \( \alpha_2M(n) \) do not satisfy this relation, it is impossible to transform the proton system from the laboratory to the body-fixed frame with the \( D \)-function by which the neutron system is transformed. Quantum-mechanically, this means that there must be a certain linear combination of \( \tilde{Q}_{2M}(p) \) and \( \tilde{Q}_{2M}(n) \):

\[ \tilde{Q}_{2M}(v) = -\beta_n \tilde{Q}_{2M}(p) + \beta_p \tilde{Q}_{2M}(n), \] (2.6)

\[ \beta_p + \beta_n = 1, \] (2.7)

which has negligible matrix elements between states of the ground-state rotational band, because, classically, the quadrupole moment is proportional to the deformation parameter. Therefore \( P_v(L') \) and \( Q_v(L') \) should be negligible:

\[ \begin{align*}
    P_v(L') &= -\beta_n P_p(L') + \beta_p P_n(L') = 0, \\
    Q_v(L') &= -\beta_n Q_p(L') + \beta_p Q_n(L') = 0.
\end{align*} \] (2.8)

Condition (III) means that the matrix elements of the following operator, which is linear independent to Eq. (2.6), are strongly enhanced compared to the other ones:

\[ \tilde{Q}_{2M}(n) = \alpha_p \tilde{Q}_{2M}(p) + \alpha_n \tilde{Q}_{2M}(n), \] (2.9)

\[ \alpha_p \beta_p + \alpha_n \beta_n = 1. \] (2.10)

Because from the relation \( \tilde{Q}_{2M}(p) + \tilde{Q}_{2M}(n) = \tilde{Q}_{2M}(u) + (\alpha_p - \alpha_n) \tilde{Q}_{2M}(v) \), we see that under condition (IV) the matrix elements of \( \tilde{Q}_{2M}(u) \) are equal to those of the total quadrupole moment. Then \( P_u(L') = (\alpha_p P_p(L') + \alpha_n P_n(L')) \) and \( Q_u(L') = (\alpha_p Q_p(L') + \alpha_n Q_n(L')) \) are much larger than the other quantities. Hereafter
we use the following compact representations for $\bar{Q}_{2M}(u)$ and $\bar{Q}_{2M}(v)$:

$$\bar{Q}_{2M}(r) = \sum_{k=p, n} \alpha_k \bar{Q}_{2M}(k), \quad (r = u \text{ or } v) \quad (2.11a)$$

where

$$\begin{pmatrix} \alpha_{up} & \alpha_{un} \\ \alpha_{vp} & \alpha_{vn} \end{pmatrix} = \begin{pmatrix} \alpha_p & \alpha_n \\ -\beta_n & \beta_p \end{pmatrix}. \quad (2.11b)$$

Inverse relations to Eq. (2.11a) are given by

$$\bar{Q}_{2M}(k) = \sum_{r=u, v} \beta_k \bar{Q}_{2M}(r), \quad (k = p \text{ or } n) \quad (2.12a)$$

where

$$\begin{pmatrix} \beta_{up} & \beta_{vp} \\ \beta_{un} & \beta_{vn} \end{pmatrix} = \begin{pmatrix} \beta_p & -\alpha_n \\ \beta_n & \alpha_p \end{pmatrix}. \quad (2.12b)$$

Now, under the above-mentioned conditions, we can get the following relations:

$$J_k(1) (L') = \alpha_k \beta_k \Rightarrow J_k(1). \quad (2.13)$$

With the aid of the following relation and Eq. (2.8):

$$\left[ J_k(1) (L) \right. (L' + 2) (L' + 3) - J_k(1) (L) L' (L' + 1)] \langle \gamma_0; L'L_s | \bar{Q}_{2M}(u) | \gamma_0; LL_s \rangle$$

$$= \langle \gamma_0; L'L_s' | [\bar{Q}_{2M}(u), \sum_k \left[ \hat{J}_{1k}(k), (\hat{J}_{1k}(p) + \hat{J}_{1k}(n)) \right]] \rangle \langle \gamma_0; LL_s \rangle, \quad (L = L' + 2) \quad (2.14)$$

we can obtain the following relation:

$$J_k(1) (L) (L' + 1) (L' + 3) - J_k(1) (L) L' (L' + 2) = \alpha_k \beta_k (2L' + 3). \quad (2.15)$$

Solutions of Eq. (2.15) are given by Eq. (2.13). From Eq. (2.13) with Eq. (2.10), it is easily shown that projection of $\hat{J}_{1k}(k)$ in the direction of total angular momentum is $J_k(1) \sqrt{L'(L' + 1)}$ and the orthogonal component vanishes. This means that the angular momenta of the proton and the neutron systems have the same directions as that of total angular momentum, and the total angular momentum is shared between the motion of the proton and the neutron systems in a very simple way, which can be predicted from the conventional rotational model. It is interesting to see that collective $g$-factor, $g_R$, is given by $J_p(1)$ ($= \alpha_{up} \beta_p$).

§ 3. Equations of motion for quadrupole modes

The starting point of our approach for the description of rotational motion is the following equations of motion with respect to the quadrupole modes:

$$[\bar{Q}_{2M}(r), \hat{H}] = -\sqrt{\frac{3}{2}} \sum_{\ell=1,3} \sum_{s=\pm 1} \sum_{k=p, n} \sum_{\gamma_{1k}} \sum_{2AIK} \langle 2AIK \mid 2M \rangle [\hat{J}_{1k}(k), \bar{Q}_{1s}(s)], \quad (3.1a)$$
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\[ \gamma_{\nu}^{(\ell)} = c_{\nu}^{(\ell)} \alpha_{\nu} \sum_{k=p,n} \beta_{\nu} \delta_{\nu}^{(\ell)}, \]  

(3.1b)

where \( Q_{2M}(r) \) means \( Q_{2M}(u) \) or \( Q_{2M}(v) \), and \( c_{\nu}, \gamma_{\nu}^{(\ell)} \) and \( \delta^{(\ell)} \) are given by

\[
\begin{align*}
\gamma_{\nu}^{(\ell)} &= 1, \\
\gamma_{\nu}^{(3)} &= \frac{8}{7} \frac{(2j_{\nu} - 2)(2j_{\nu} + 4)}{(2j_{\nu} - 1)(2j_{\nu} + 3)}, \\
\delta_{\nu}^{(1)} &= 1, \\
\delta_{\nu}^{(3)} &= \sqrt{\frac{7}{2}}.
\end{align*}
\]

(3.2)

Then the energy spectrum is given by the following equation:

\[
\omega_{LL'} \langle \gamma_{0}; L'L' | Q_{2M}(r) | \gamma_{0}; LL' \rangle = \langle \gamma_{0}; L'L' | [Q_{2M}(r), H_{-}] | \gamma_{0}; LL' \rangle, \quad (L = L' + 2)
\]

(3.3)

where \( \omega_{LL'} \) is the energy difference between the states \( | \gamma_{0}; LL' \rangle \) and \( | \gamma_{0}; L'L' \rangle \). With the aid of the quantities defined in Eqs. (2.4) together with conditions (2.8) and Eq. (2.13), Eq. (3.3) can be written down as

\[
\omega_{LL'} \cdot P_{\nu}(L') = \frac{1}{2} \sum_{k=p,n} \left[ \gamma_{\nu}^{(\ell)} J_{\nu}^{(\ell)} P_{\nu}(L') \right.
\]

\[
+ \frac{\gamma_{\nu}^{(3)}}{\gamma_{\nu}^{(\ell)}} \left\{ \frac{3}{4} \frac{(L' + 3)(L' + 4)}{(2L' + 3)^2} J_{\nu}^{(3)}(L) + \frac{3}{4} \frac{(L' - 1)L'}{(2L' + 3)^2} J_{\nu}^{(3)}(L') \right\} P_{\nu}(L')
\]

\[
+ \frac{5}{2} \frac{L'(L' + 3)}{(2L' + 3)^2} I_{\nu}^{(3)}(L') \cdot \frac{Q_{\nu}(L) + Q_{\nu}(L')}{2} \}
\]

\[
R_{LL'},
\]

(\( R_{LL'} \equiv L(L + 1) - L'(L' + 1) = 4L' + 6 \)).

(3.4)

By dividing both sides of Eq. (3.4) for \( r = u \) by \( P_{\nu}(L') \), we can get

\[
\omega_{LL'} = \frac{1}{2R_{LL'}} \cdot R_{LL'},
\]

\[
\frac{1}{R_{LL'}} = \sum_{k=p,n} \left[ \gamma_{\nu}^{(\ell)} J_{\nu}^{(\ell)}
\right.
\]

\[
+ \frac{\gamma_{\nu}^{(3)}}{\gamma_{\nu}^{(\ell)}} \left\{ \frac{3}{4} \frac{(L' + 3)(L' + 4)}{(2L' + 3)^2} J_{\nu}^{(3)}(L) + \frac{3}{4} \frac{(L' - 1)L'}{(2L' + 3)^2} J_{\nu}^{(3)}(L')
\]

\[
+ \frac{5}{2} \frac{L'(L' + 3)}{(2L' + 3)^2} I_{\nu}^{(3)}(L') \cdot \frac{Q_{\nu}(L) + Q_{\nu}(L')}{2P_{\nu}(L')} \}
\]

\[
\left. \right\}.
\]

(3.5)

On the other hand, Eq. (3.4) for \( r = v \) should vanish, because \( P_{\nu}(L') \) is assumed to be negligibly small. Equation (3.5) has the following structure: if \( P_{\nu}(L') \) is equal to \( Q_{\nu}(L') \) and does not depend on \( L' (= Q_{\nu}) \) and also \( I_{\nu}^{(3)}(L') \) is equal to \( J_{\nu}^{(3)}(L') \) and independent of \( L' (= J_{\nu}^{(3)}) \), \( R_{LL'} \) becomes independent of \( L' \) and \( L\):
This is just a rotational spectrum. It should be noticed that the above-mentioned condition for \( P_u(L') \) and \( Q_u(L') \) means that the transition probabilities and quadrupole moments satisfy the relations which are predicted by the rotational model (intrinsic quadrupole moment \( Q_0 \) is given by \( \sqrt{16\pi/5} Q_u \)), and under this condition the rotational spectrum appears. Furthermore, it may be important for the understanding of the rotational motion to investigate the behavior of \( J_{ik}(k) \).

In order to understand the effect of \( J_{ik}(k) \) much more transparently, let us introduce an effective Hamiltonian on analogy with \( \Pi \), which approximates the original equations of motion for the quadrupole mode \( \hat{Q}_{1M}(u) \):

\[
\hat{H}_{\text{eff}} = \frac{1}{2} \sum_{j=1,3} \left[ \rho^{(t)} \sum_{k} (\hat{J}_{ik}(p) \hat{J}_{ik}(p) + \hat{J}_{ik}(n) \hat{J}_{ik}(n)) + \sigma^{(t)} \sum_{k} (\hat{J}_{ik}(p) \hat{J}_{ik}(n) + \hat{J}_{ik}(n) \hat{J}_{ik}(p)) \right]
\]

\[
= \frac{1}{2} \sum_{j=1,3} \sum_{l=L_{\text{min}}}^{L_{\text{max}}} \gamma_{hk}^{(t)} \sum_{k} \hat{J}_{ik}(k) \hat{J}_{ik}(l),
\]

where

\[
\begin{pmatrix}
\tilde{\chi}^{(t)} \\
\tilde{\chi}^{(t)}
\end{pmatrix} = \begin{pmatrix}
\rho^{(t)} & \sigma^{(t)} \\
\sigma^{(t)} & \rho^{(t)}
\end{pmatrix}.
\]

The commutation relation of \( \hat{H}_{\text{eff}} \) with respect to \( \hat{Q}_{1M}(u) \), \([\hat{Q}_{1M}(u), \hat{H}_{\text{eff}}]_r \), is reduced to Eq. (3.1a) for \( r=u \), with the omission of terms related to \( \hat{Q}_{1M}(v) \) and \( \hat{B}_{1M}(k) \) which are assumed to be negligible, if \( \rho^{(t)} \) and \( \sigma^{(t)} \) satisfy the following relations:

\[
\rho^{(t)} = \frac{1}{\alpha_{\beta_p} - \alpha_{\beta_n}} [c_{\beta_p} \tilde{\chi}^{(t)} \alpha_{\beta_p}^2 \beta_p (\rho_\beta_n + \sigma_\beta_n) - c_{\beta_n} \tilde{\chi}^{(t)} \alpha_{\beta_n}^2 \beta_n (\rho_\beta_n + \sigma_\beta_n)],
\]

\[
\sigma^{(t)} = \frac{1}{\alpha_{\beta_p} - \alpha_{\beta_n}} [c_{\beta_n} \tilde{\chi}^{(t)} \beta_n (\rho_\beta_n + \sigma_\beta_n) - c_{\beta_p} \tilde{\chi}^{(t)} \beta_p (\rho_\beta_n + \sigma_\beta_n)].
\]

Then, as was shown in \( \Pi \), we can get

\[
\omega_{LL'} = \langle \gamma_0; L\vert \hat{H}_{\text{eff}} \vert \gamma_0; L \rangle - \langle \gamma_0; L'\vert \hat{H}_{\text{eff}} \vert \gamma_0; L \rangle. \quad (L=L'+2)
\]

Here \( \langle \gamma_0; L\vert \hat{H}_{\text{eff}} \vert \gamma_0; L \rangle \) is given by

\[
\langle \gamma_0; L'\vert \hat{H}_{\text{eff}} \vert \gamma_0; L' \rangle = \frac{1}{2} \sum_{j=1,3} \sum_{l=L_{\text{min}}}^{L_{\text{max}}} \gamma_{hk}^{(t)} \left[ J_{ik}^{(t)} J_{ik}^{(t)} + \frac{3}{2} \frac{(L'-1)(L'+2)}{(2L'-1)(2L'+3)} J_{ik}^{(t)} (L') J_{ik}^{(t)} (L') + \frac{5}{4} \frac{(L' + 2)}{(2L'+1)(2L'+3)} I_k^{(t)} (L') I_k^{(t)} (L') + \frac{5}{4} \frac{(L' - 2)}{(2L'-1)(2L'+1)} I_k^{(t)} (L') I_k^{(t)} (L') \right] L' (L' + 1). \quad (L'' = L' - 2)
\]
If the second terms on the right-hand side of Eq. (3·11) are negligible compared to the first ones, it is clear that our spectrum is rotational. However, even if the second terms are not negligible, we can expect a rotational spectrum in the case where the second terms do not depend on $L'$. This corresponds to the condition that $I_k^{(3)}(L')$ is equal to $J_k^{(3)}(L')$ and independent of $L'$.

Finally, we must point out that our effective Hamiltonian should satisfy the following relation in the same way as $[\mathcal{Q}_2M(v), \hat{H}]$: 

$$0 = \langle \gamma_0; L'L'' | [\mathcal{Q}_2M(v), \hat{H}_{\text{eff}}] | \gamma_0; LL'' \rangle \quad \text{for } r = v. \quad (3·12)$$

Equation (3·12) can be given explicitly by

$$\left[ \langle \gamma_0; L | \hat{H}_{\text{eff}} | \gamma_0; L' \rangle - \langle \gamma_0; L' | \hat{H}_{\text{eff}} | \gamma_0; L'' \rangle \right] P_r(L')$$

$$= \frac{1}{2} \sum_{k,m,n} \left( \sum_{r} q^{(1)}_{rka} J_k^{(1)} P_u(L') \right)$$

$$+ \sum_{r} q^{(3)}_{rka} \left( \frac{3}{4}, \frac{(L' + 3)(L' + 4)}{(2L' + 3)^2} J_k^{(3)}(L) + \frac{3}{4}, \frac{(L' - 1)L'}{(2L' + 3)^2} J_k^{(3)}(L') \right) P_u(L')$$

$$+ \frac{5}{2} \frac{L'(L' + 3)}{(2L' + 3)^2} I_k^{(3)}(L') \right] P_r(L')$$

$$= \frac{5}{2} \frac{L'(L' + 3)}{(2L' + 3)^2} I_k^{(3)}(L') \right] P_r(L')$$

$$\text{for } r = v,$$

$$\text{for } r = u.$$ (3·13)

where

$$q^{(3)}_{rka} = \sum_{l=1}^{r} \alpha_{rl} \beta_{s} \beta_{kl}.$$

(3·14)

Of course, Eq. (3·13) for $r = u$ is the same as Eq. (3·4) for $r = u$.

§ 4. Generalized sum rules

As was already discussed in § 3, the problems for our description of rotational motion are to find the condition under which $P_u(L')$ is equal to $Q_u(L')$ and does not depend on $L'$, and to show that $I_k^{(3)}(L')$ is equal to $J_k^{(3)}(L')$ and independent of $L'$ under the above condition for $P_u(L')$ and $Q_u(L')$. In order to do this, the use of sum rules is very powerful as was already demonstrated in [1].

First, we should notice that we can calculate $\mathfrak{T}_{LL'}$ from Eq. (3·5), if we know six quantities, $Q_u(L)$, $P_u(L')$, $J_p^{(3)}(L)$, $I_p^{(3)}(L')$, $J_n^{(3)}(L)$ and $I_n^{(3)}(L')$, together with $\alpha_p$, $\beta_n$, $\beta_p$, and $\beta_n$. Of course, for each $\mathfrak{T}_{LL'}$, there are altogether nine quantities which must be given; however, $Q_u(L)$, $J_p^{(3)}(L')$ and $J_n^{(3)}(L')$ are already known from the calculation of the lower energy step $\mathfrak{T}_{L''}(L'' = L' - 2)$.

Let us start from the derivation of the sum rules which come from dynamical restrictions. We look after a system for which the matrix elements of $\mathcal{Q}_2M(u)$
are dominant and those of $\tilde{Q}_M(v)$ and $\tilde{B}_{\text{int}}(k)$ are negligible. Then combining Eq. (3·5) with Eq. (3·10) and using Eq. (3·11), we can get the following equation:

$$
\sum_{k,l} \gamma^{(3)}_{kl} \left[ \frac{3}{2} \frac{(L' + 1) (L' + 4)}{(2L' + 3) (2L' + 7)} J_k^{(3)}(L) J_l^{(3)}(L) + \frac{5}{4} \frac{(L' + 4) (L' + 5)}{(2L' + 5) (2L' + 7)} I_k^{(3)}(L) I_l^{(3)}(L) + \frac{5}{4} \frac{L'(L' + 1)}{(2L' + 3) (2L' + 5)} I_k^{(3)}(L') I_l^{(3)}(L') \right] \frac{(L' + 2) (L' + 3)}{4L' + 6} 
$$

$$
= \sum_{k} \zeta^{(3)}_{\text{int}} J_k^{(3)}(L) - \sum_{k,l} \gamma^{(3)}_{kl} J_k^{(1)} J_l^{(1)}, \quad (L = L' + 2, L'' = L' - 2) \tag{4·1}
$$

Next, from Eqs. (3·4) and (3·13) for $r = v$ under the condition that $P_v(L')$ is negligible, we can obtain the following two kinds of sum rules:

$$
\sum_{k} \zeta^{(5)}_{\text{int}} \left[ \frac{3}{4} \frac{(L' + 3) (L' + 4)}{(2L' + 3)^2} J_k^{(5)}(L) + \frac{5}{4} \frac{(L' + 1) L'}{(2L' + 3)^3} J_k^{(5)}(L') \right] - \sum_{k} \zeta^{(3)}_{\text{int}} J_k^{(1)} = - \sum_{k} \zeta^{(1)}_{\text{int}} J_k^{(1)}, \tag{4·2}
$$

$$
\sum_{k} \zeta^{(5)}_{\text{int}} \left[ \frac{3}{4} \frac{(L' + 3) (L' + 4)}{(2L' + 3)^2} J_k^{(5)}(L) + \frac{5}{4} \frac{L' (L' + 3)}{(2L' + 3)^3} J_k^{(5)}(L') \right] - \sum_{k} \zeta^{(3)}_{\text{int}} J_k^{(1)} = - \sum_{k} \zeta^{(1)}_{\text{int}} J_k^{(1)}. \tag{4·3}
$$

These three relations are the sum rules which come from dynamical restrictions. Clearly it is impossible to understand our system completely in the framework of the above three relations. Therefore we must introduce additional sum rules. They come from kinematical restrictions which govern the operators introduced in Eq. (2·3). By direct extension of the relation in Eqs. (4·2) of $\Pi$, we get the following relation:
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\[ \mathcal{B}_{m_{2}, m_{1}}(k) = \sum_{J_{3} J_{2} J_{1}} Z_{k}^{(1)}(J_{3} J_{2} J_{1}) \sum_{M_{3} M_{2} M_{1}} \langle J_{3} M_{3} J_{2} M_{2} J_{1} M_{1} | J_{3} M_{3} J_{2} M_{2} J_{1} M_{1} \rangle \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \]

\[ \mathcal{B}_{m_{2}, m_{1}}(k) = \sum_{J_{3} J_{2} J_{1}} Z_{k}^{(2)}(J_{3} J_{2} J_{1}) \sum_{M_{3} M_{2} M_{1}} \langle J_{3} M_{3} J_{2} M_{2} J_{1} M_{1} | J_{3} M_{3} J_{2} M_{2} J_{1} M_{1} \rangle \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \]

\[ + \sum_{J_{3} J_{2} J_{1}} Z_{k}^{(3)}(J_{3} J_{2} J_{1}) \sum_{M_{3} M_{2} M_{1}} \langle J_{3} M_{3} J_{2} M_{2} | J_{3} M_{3} J_{2} M_{2} \rangle \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \]

\[ \times \left[ \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \mathcal{B}_{m_{2}, m_{1}}(k) \right], \quad (4.4a) \]

where

\[ Z_{k}^{(1)}(J_{3} J_{2} J_{1}) = \frac{1}{2} \left( 1 + (-)^{J_{3}^{2} + J_{2}^{2} + J_{1}^{2}} \right) \sqrt{(2J_{3} + 1)(2J_{2} + 1)} W(j_{k} j_{k} j_{k} J_{3} J_{2} J_{1}), \]

\[ Z_{k}^{(2)}(J_{3} J_{2} J_{1}) = \frac{1}{4} \frac{1}{j_{k} + 1} \left( 1 - (-)^{J_{3}^{2} + J_{2}^{2} + J_{1}^{2}} \right) \sqrt{(2J_{3} + 1)(2J_{2} + 1)} W(j_{k} j_{k} j_{k} J_{3} J_{2} J_{1}). \]

(4.4b)

With the omission of coupling to the other bands, \( \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; L' \rangle \) is given by

\[ \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; L' \rangle \]

\[ = \sum_{J_{3} J_{2} J_{1}} Y_{k}^{(1)}(J_{3} J_{2} J_{1}; L' L' I) \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle \]

\[ + \sum_{J_{3} J_{2} J_{1}} Y_{k}^{(2)}(J_{3} J_{2} J_{1}; L' L' I) \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle \]

\[ - \sum_{J_{3} J_{2} J_{1}} Y_{k}^{(3)}(J_{3} J_{2} J_{1}; L' L' I) Y_{k}^{(3)}(J_{3} J_{2} J_{1}; L' L' I) \]

\[ \times \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; I \rangle, \quad (4.5a) \]

where

\[ Y_{k}^{(a)}(J_{3} J_{2} J_{1}; L_{1} L_{2} L_{3}) = Z_{k}^{(a)}(J_{3} J_{2} J_{1}) \sqrt{(2J_{1} + 1)(I_{5} + 1)} W(I_{1} I_{2} I_{3} J_{1} J_{2} J_{3}), \]

\[ (\alpha = 1 \text{ and } 2) \]

(4.5b)

Applying this relation to the case of \( \langle \gamma_{0}; L' | \mathcal{B}_{m_{2}, m_{1}}(k) | \gamma_{0}; L' \rangle \) with the aid of Eqs. (2.2), (2.4), (2.8) and (2.13), we can get the following relation:

\[ J_{k}^{(1)} \left[ 1 + \frac{7}{5} \frac{(2J_{k} - 1)(2J_{k} + 3)}{(2J_{k} - 3)(2J_{k} + 5)} \frac{Q_{k}(L')}{{\sum} Q_{k}} \right] \]

\[ \times \left[ \frac{3}{2} \frac{(L' - 1)(L' + 2)}{(2L' - 1)(2L' + 3)} Q_{k}(L') J_{k}^{(3)}(L') \right] \]

\[ + \frac{5}{4} \frac{(L' + 2)(L' + 3)}{(2L' + 1)(2L' + 3)} P_{k}(L') I_{k}^{(3)}(L') \]

\[ + \frac{5}{4} \frac{(L' - 2)(L' - 1)}{(2L' - 1)(2L' + 1)} P_{k}(L') I_{k}^{(3)}(L') \]

\[ = \frac{3}{4} \frac{1}{j_{k} + 1} \left[ \frac{(j_{k})}{(j_{k})^{3}} \left[ 1 + \frac{7}{5} \frac{(2J_{k} - 1)(2J_{k} + 3)}{(2J_{k} - 3)(2J_{k} + 5)} \frac{Q_{k}(L')}{{\sum} Q_{k}} \right] \right] \]

\[ \times \left[ \frac{3}{2} \frac{(L' - 1)(L' + 2)}{(2L' - 1)(2L' + 3)} Q_{k}(L') J_{k}^{(3)}(L') \right] \]

\[ + \frac{5}{4} \frac{(L' + 2)(L' + 3)}{(2L' + 1)(2L' + 3)} P_{k}(L') I_{k}^{(3)}(L') \]

\[ + \frac{5}{4} \frac{(L' - 2)(L' - 1)}{(2L' - 1)(2L' + 1)} P_{k}(L') I_{k}^{(3)}(L') \]

\[ = \frac{3}{4} \frac{1}{j_{k} + 1} \left[ \frac{(j_{k})}{(j_{k})^{3}} \left[ 1 + \frac{7}{5} \frac{(2J_{k} - 1)(2J_{k} + 3)}{(2J_{k} - 3)(2J_{k} + 5)} \frac{Q_{k}(L')}{{\sum} Q_{k}} \right] \right] \]
\[
\times \left[ \frac{1}{(2L' - 1)(2L' + 3)} Q_u(L')^3 - \frac{L' + 2}{(2L' + 1)(2L' + 3)} P_u(L')^3 + \frac{L' - 1}{(2L' - 1)(2L' + 1)} P_u(L'')^3 \right]
\]
\[
+ \frac{9}{4} \left( \frac{2L - 2}{2L + 5} \right) \frac{1}{(2L - 3)(2L + 5)} \beta_k^2 \frac{1}{q_k} \sum Q^0_i
\]
\[
\times \left[ \frac{(L' + 2)(L' + 3)}{(2L' + 1)(2L' + 3)} (Q_u(L) - Q_u(L')) P_u(L')^3
\]
\[
+ \frac{(L' - 2)(L' - 1)}{(2L' - 1)^2(2L' + 1)} (Q_u(L') - Q_u(L'')) P_u(L'')^3 \right]
\]
\[
+ \frac{27}{4} \left( \frac{2L - 2}{2L + 5} \right) \frac{1}{(2L - 3)(2L + 5)} \beta_k^2 \frac{1}{q_k} \sum Q^0_i
\]
\[
\times \left[ \frac{8(L' - 1)(L' + 2)}{(2L' - 1)^2(2L' + 3)^2} Q_u(L')^3 + \frac{(L' - 1)(L' + 2)}{(2L' + 1)(2L' + 3)^2} P_u(L')^3
\]
\[
- \frac{(L' - 1)(L' + 2)}{(2L' - 1)^2(2L' + 1)} P_u(L'')^3 \right] Q_u(L'),
\]
\[
(L = L' + 2, \quad L'' = L' - 2, \quad k = p \quad \text{and} \quad n)
\]

where
\[
Q^0_k = \frac{7}{\sqrt{5}} \frac{j_k + 1}{(2L - 3)(2L + 5)} \frac{j_k - 1}{(2L - 3)(2L + 5)} \frac{q_k}{2L + 1} (2L + 1 - 2N_k).
\]

In the same way, we can obtain the following relations from \( \langle \gamma_6; L''| \bar{B}_s^+ (k)| \gamma_6; L' \rangle \) and \( \langle \gamma_6; L'| \bar{B}_s^+ (k)| \gamma_6; L' \rangle \):
\[
- \frac{1}{2} \left( \frac{2L' - 3}{2L' - 1} \right) \frac{2L' + 5}{2L' + 3} Q_u(L')^3 + \frac{3}{4} \left( \frac{2L' - 1}{2L' + 1} \right) \frac{2L' + 4}{2L' + 3} P_u(L')^3
\]
\[
+ \frac{3}{2} \left( \frac{2L' - 2}{2L' - 1} \right) \frac{2L' + 3}{2L' + 1} P_u(L')^3 = Q_u(L') \cdot \frac{Q^0_k}{\beta_k}, \quad (L'' = L' - 2)
\]
\[
\frac{L' + 1}{2L' - 1} \frac{2L' + 3}{2L' + 1} Q_u(L')^3 + \frac{3}{2} \left( \frac{L' + 1}{2L' + 1} \right) \frac{L' + 2}{2L' + 3} P_u(L')^3
\]
\[
+ \frac{3}{2} \left( \frac{L' - 1}{2L' - 1} \right) \frac{L'}{2L' + 1} P_u(L'')^3 = \frac{1}{2L + 1} \left( \frac{q_k}{\beta_k} \right)^2 N_k (2L + 1 - N_k). \quad (L'' = L' - 2)
\]

Here we should notice that the left-hand sides of Eqs. (4·8) and (4·9) are independent of proton and neutron. Therefore, we have the following relations:
\[
\frac{Q^0_p}{\beta_p} = \frac{Q^0_n}{\beta_n},
\]
\[
\frac{1}{2j_p + 1} \left( \frac{q_p}{\beta_p} \right)^2 N_p (2j_p + 1 - N_p) = \frac{1}{2j_n + 1} \left( \frac{q_n}{\beta_n} \right)^2 N_n (2j_n + 1 - N_n). \tag{4·11}
\]

From Eqs. (4·10) and (2·7), \(\beta_p\) and \(\beta_n\) are given by

\[
\beta_k = \frac{Q_k^{(0)}}{\sum_i Q_i^{(0)}}. \tag{4·12}
\]

Putting Eqs. (4·12) and (4·7) into Eq. (4·11), we get the following equation:

\[
\frac{(2j_p - 3)(2j_p + 5)(2j_p + 1 - N_p)}{j_p(j_p + 1)(2j_p - 1)(2j_p + 3)} \cdot \frac{N_p(2j_p + 1 - N_p)}{(2j_p + 1 - 2N_p)^3} = \frac{(2j_n - 3)(2j_n + 5)(2j_n + 1 - N_n)}{j_n(j_n + 1)(2j_n - 1)(2j_n + 3)} \cdot \frac{N_n(2j_n + 1 - N_n)}{(2j_n + 1 - 2N_n)^3} \equiv b. \tag{4·13}
\]

The above relation (4·13) means that the restriction which we give the quadrupole modes in the proton and the neutron systems can be realized only if \(N_p\) and \(N_n\) satisfy Eq. (4·13). Putting Eqs. (4·12) and (4·13) into Eqs. (4·8) and (4·9), we can obtain the following two relations:

\[
-\frac{1}{2} \frac{(2L' - 3)(2L' + 5)(2L' + 1 - L')}{(2L' - 1)(2L' + 3)} Q_u(L')^2 + \frac{3}{4} \left( \frac{2L' - 1}{2L' + 1} \right) P_u(L')^2 = \frac{3}{4} \left( \frac{2L' - 2}{2L' + 1} \right) P_u(L')^2 = \frac{3}{2} \left( \frac{2L' - 1}{2L' + 3} \right) P_u(L')^2, \tag{4·14}
\]

\[
\frac{L'(L' + 1)}{(2L' - 1)(2L' + 3)} Q_u(L')^2 + \frac{3}{2} \left( \frac{L' + 1}{2L' + 1} \right) P_u(L')^2 = \frac{5}{49} b (\sum_k Q_k^{(0)})^2. \tag{4·15}
\]

Finally we have the following equation from Eqs. (2·10) and (4·12):

\[
\sum_k \alpha_k Q_k^{(0)} = \sum_k Q_k^{(0)} \tag{4·16}
\]

With the help of Eqs. (4·1), (4·2), (4·3), (4·6), (4·14), (4·15) and (4·16), together with Eqs. (4·7) and (4·12), we are now in a position to calculate the unknown quantities. First, from Eqs. (4·14) and (4·15), we can obtain \(P_u(L')\) and \(Q_u(L')\) successively from lower \(L'\). Next, substituting these solutions and Eq. (4·12) into Eqs. (4·1), (4·2), (4·3), (4·7) and (4·16), we can get six relations with respect to \(I_p^{(3)}(L'), J_p^{(3)}(L'), I_n^{(3)}(L'), J_n^{(3)}(L'), \alpha_p\) and \(\alpha_n\), and solve these equations successively in the same way as in the case of \(P_u(L')\) and \(Q_u(L')\). Here we should notice the following two points: i) our equations are applicable only in a system which contains \(N_p\) and \(N_n\) given by Eq. (4·13), and \(\alpha_p\) and \(\alpha_n\) should be independent of \(L'\) in spite of solving \(\alpha_p\) and \(\alpha_n\) in each step from lower \(L'\). Finally we should stress again that our basic
equations in this section were derived following the basic assumption which was discussed in § 2.

§ 5. Determination of the energy spectrum, transition probabilities and quadrupole moments

We are now able to obtain a complete description of our system. This section will be devoted to finding the solutions which express nuclear rotational motion, following the discussion in § 3. Using the procedure of § 4, we can find from Eqs. (4·13), (4·14) and (4·15) the solutions with \( P_u(L') = Q_u(L') \) (\( = Q_u \); independent of \( L' \)) in the following mass regions:

\[
\frac{N_k}{2j_k + 1} = \frac{1}{2} \left[ 1 + \frac{\sqrt{5} (2j_k - 3) (2j_k + 5)}{\sqrt{196j_k (j_k + 1) (2j_k - 1) (2j_k + 3) + 5 (2j_k - 3)^2 (2j_k + 5)^2}} \right],
\]

(5·1a)

and in these cases \( Q_k(0) \) is given by

\[
Q_k(0) = \pm q_k \sqrt{2j_k + 1} \cdot \frac{7j_k (j_k + 1) (2j_k - 1) (2j_k + 3)}{\sqrt{196j_k (j_k + 1) (2j_k - 1) (2j_k + 3) + 5 (2j_k - 3)^2 (2j_k + 5)^2}}.
\]

(5·1b)

Therefore, \( Q_u \) and \( \beta_k \) is given by

\[
Q_u = \sum_k Q_k(0),
\]

(5·2)

\[
\beta_k = \frac{Q_k(0)}{\sum Q_k(0)}.
\]

(5·3)

Next, substituting the above results into Eqs. (4·1), (4·2), (4·3), (4·6) and (4·16), the following equations are obtained:

\[
\sum_{k\ell} \gamma_{k\ell} \left[ \left\{ \frac{3}{2} \left( \frac{L' + 1}{(2L' + 3)(2L' + 7)} \right) J^2_{k(0)}(L) J^2_{k(0)}(L') \right. \\
+ \frac{5}{4} \left( \frac{L' + 4}{(2L' + 5)(2L' + 7)} \right) I_k^{(3)}(L) I_{k}^{(3)}(L) \\
+ \frac{5}{4} \left( \frac{L' + 1}{(2L' + 3)(2L' + 5)} \right) I_k^{(3)}(L') I_k^{(3)}(L') \left\} \frac{(L' + 2)(L' + 3)}{4L' + 6} \right] \\
- \left\{ \frac{3}{2} \left( \frac{L' - 1}{(2L' - 1)(2L' + 3)} \right) J^2_{k(0)}(L') J^2_{k(0)}(L) \\
+ \frac{5}{4} \left( \frac{L' + 2}{(2L' + 1)(2L' + 3)} \right) I_k^{(3)}(L') I_k^{(3)}(L') \\
+ \frac{5}{4} \left( \frac{L' - 2}{(2L' - 1)(2L' + 1)} \right) I_k^{(3)}(L') I_k^{(3)}(L') \left\} \frac{L'(L' + 1)}{4L' + 6} \right].
\]
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By solving these equations successively from the lowest \((L'')=0\) state, we can obtain the following results:

\[ I_k^{(3)} (L') = J_k^{(3)} (L') = J_k^{(3)} = \alpha_k \beta_k, \]  

and

\[ \alpha_p = \frac{d_n (\rho \beta_n + \sigma \beta_p)}{d_n \beta_p (\rho \beta_n + \sigma \beta_p) + d_n \beta_n (\rho \beta_p + \sigma \beta_n)}, \]  
\[ \alpha_n = \frac{d_p (\sigma \beta_p + \rho \beta_n)}{d_n \beta_p (\rho \beta_n + \sigma \beta_p) + d_p \beta_n (\rho \beta_p + \sigma \beta_n)}, \]

where

\[ d_k = c_k (\gamma_k^{(1)} + \gamma_k^{(3)}). \]

Putting these results into Eq. (3.5), the following moment of inertia can be obtained:

\[ \mathcal{I}_L = \mathcal{I} = \mathcal{I}_p + \mathcal{I}_n, \]

where

\[ \mathcal{I}_p = \frac{Q_p^{(0)}}{\rho Q_p^{(0)} + \sigma Q_n^{(0)}} \cdot \frac{1}{\chi \alpha_p}. \]
\[ I_n = \frac{Q_n^{(0)}}{\rho Q_n^{(0)} + \sigma Q_p^{(0)}} \cdot \frac{1}{2d_n}, \tag{5.14} \]

and \( J_k^{(1)} \) and \( J_k^{(5)} \) are given by

\[ J_k^{(1)} = J_k^{(5)} = \frac{I_k}{I}. \tag{5.15} \]

Equation (5·12) tells us that the spectrum is rotational, and it appears in the case that transition probabilities and quadrupole moments satisfy the relations which are given by a rotational model of the \( K = 0 \) band. The intrinsic mass, proton and neutron quadrupole moments are given by \( \sqrt{16\pi/5} (Q_p^{(0)} + Q_n^{(0)}) \), \( \sqrt{16\pi/5} Q_p^{(0)} \) and \( \sqrt{16\pi/5} Q_n^{(0)} \), respectively. The oblate and prolate shapes of the proton system are favourable in the regions \( 0 < N_p/(2j_p + 1) < 0.5 \) and \( 0.5 < N_n/(2j_n + 1) < 1 \), respectively. The situation in the case of the neutron system is also the same as the proton system. Collective \( g \)-factor, \( g_n \), is given by \( I_n/I \).

\section{6. Discussion}

i) Consistency of the results with the assumptions

In the previous section, we arrived at some relations which characterize the ground-state rotational band of even-even deformed nuclei. However, it is not self-evident that the results are consistent with the assumptions. In this subsection we will give the following physically acceptable conditions under which there is no inconsistency:

(A) Our results are consistent under the conditions that \( j_p \) and \( j_n \) are large and \( L \) is small (low excited states).

(B) Our moment of inertia has its meaning in the case that \( I_p \) and \( I_n \) are positive quantities.

(C) Our results are reliable under the condition that the proton-neutron interaction is strongly attractive or repulsive in the case that proton and neutron systems have the same shapes or opposite shapes, respectively.

In order to recall our starting assumptions, we will summarize them as follows:

(I) \( |\gamma_1; H_\ell \rangle \) which belongs to the other bands is much higher than the ground-state rotational band.

(II) We can ignore the coupling between the ground-state rotational band and any other band.

(III) Within the ground-state rotational band, the mode \( \tilde{Q}_{1M}(u) \) is strongly enhanced, compared to the other ones.

(IV) The matrix elements of \( \tilde{Q}_{1M}(v) \) within the ground-state rotational band are negligible.

In the case of (A), we have already given such conditions in [II]. The situ-
ation is unchanged in our case. If the proton and the neutron systems have the same shapes, our results confirm assumption (III) automatically under condition (A). However, in the case that the shapes of both systems are opposite, there is no certainty that the mode \( \tilde{Q}_{2M}(u) \) is strongly enhanced, in spite of the strong enhancement of \( \tilde{Q}_{2M}(m) \) and \( \tilde{Q}_{2M}(n) \) under condition (A). It depends on \( j_m, j_n, q_p \) and \( q_n \).

Next, let us discuss condition (B). This can be obtained by combining our basic assumptions and results on the basis of double commutator \( \sum_M [[\tilde{Q}_M^+(v), \tilde{H}], \tilde{Q}_{2M}(v)] \) as follows:

\[
\frac{1}{2} \langle \gamma_0; 00 \vert \sum_M [[\tilde{Q}_M^+(v), \tilde{H}], \tilde{Q}_{2M}(v)] \vert \gamma_0; 00 \rangle
\]

\[=
(E_{r,s} - E_{r,\phi}) \langle \gamma_0; 0 \vert \tilde{Q}_s(v) \vert \gamma_0; 2 \rangle^2 + \sum_{r_1} (E_{r,\phi} - E_{r,\phi}) \langle \gamma_0; 0 \vert \tilde{Q}_s(v) \vert \gamma_1; 2 \rangle^2
\]

\[\geq (E_{r,s} - E_{r,\phi}) \langle \gamma_0; 0 \vert \tilde{Q}_s(v) \vert \gamma_0; 2 \rangle^2 + \sum_{r_1} \langle \gamma_0; 0 \vert \tilde{Q}_s(v) \vert \gamma_1; 2 \rangle^2, \quad (6.1)
\]

where \( E_{r,J} \) are the eigenvalues of the states \( |J, J_z \rangle \) and we used \( E_{r,\phi} - E_{r,\phi} \geq E_{r,\phi} - E_{r,\phi} \) which comes from assumption (I). By substituting our basic assumptions and results in relation (6.1), we can obtain the following inequality:

\[
\frac{7}{3} \beta_p \beta_n \left( \frac{\beta_p}{\beta_p + \sigma \beta_n} \cdot \frac{1}{d_p} + \frac{\beta_n}{\beta_p + \sigma \beta_p} \cdot \frac{1}{d_n} \right) \left( \rho \beta_p + \sigma \beta_n \cdot d' - \rho \beta_p + \sigma \beta_p \cdot d'' \right) \geq 0,
\]

where

\[
d'_k = \frac{3}{7} c_k (r_k^{(3)}) + \frac{7}{2} r_k^{(3)} \approx d_k. \quad (for \ large \ j_k) \tag{6.3}
\]

Therefore the relation (6.2) can be rewritten approximately by

\[
\frac{7}{3} \beta_p \beta_n \left( \frac{1}{d_p} + \frac{1}{d_n} \right) \geq 0. \quad (6.4)
\]

As \( \mathcal{D}_p + \mathcal{D}_n = \mathcal{D} \) should be positive, we can get

\[
\mathcal{D}_p > 0 \quad and \quad \mathcal{D}_n > 0. \quad (6.5)
\]

This is exactly condition (B). We can see from \( J_k^{(3)} = \mathcal{D}_k / \mathcal{D} > 0 \) that proton and the neutron systems rotate in the same directions, and this fact supports our basic picture.

Condition (C) can be derived from the following argument. First, we set up the following double commutator:

\[
\frac{1}{2} \langle \gamma_0; 00 \vert \sum_M [[J_{1M}^+(v), \tilde{H}], J_{1M}(v)] \vert \gamma_0; 00 \rangle
\]

\[=
\sum_{r_1} (E_{r,\phi} - E_{r,\phi}) \langle \gamma_0; 0 \vert \tilde{J}_s(v) \vert \gamma_1; 1 \rangle^2
\]

\[\geq (E_{r,\phi} - E_{r,\phi}) \sum_{r_1} \langle \gamma_0; 0 \vert \tilde{J}_s(v) \vert \gamma_1; 1 \rangle^2, \quad (6.6)
\]
where
\[ J_{1M}(\nu) = -\alpha_\nu \beta_n J_{1,M}(\nu) + \alpha_p \beta_p J_{1,M}(\nu). \] (6.7)

In the same way as in the case of \( \tilde{Q}_{3M}(\nu) \), we can obtain the following relation from Eq. (6.6):
\[ \beta_p \beta_n \left( \frac{\beta_p x}{\beta_p + \beta_n x} \cdot \frac{1}{d_p} + \frac{\beta_n x}{\beta_n + \beta_p x} \cdot \frac{1}{d_n} \right) = f(x) \geq 0, \] (6.8)

where
\[ x = \frac{\sigma}{\rho}. \] (6.9)

The function \( f(x) \) is monotone increasing or decreasing if \( \beta_p \beta_n > 0 \) or \( \beta_p \beta_n < 0 \), respectively, and \( f(0) = 0 \). From this behavior of \( f(x) \), we can get condition (C). In both cases, \( \tau_p \) and \( \tau_n \) are positive; then condition (C) is consistent with (B).

ii) **Comparison of the results with the cranking model**

Finally, we compare our results with the cranking model with the axial symmetric deformed Hartree model. First we list the main features of the cranking model. The intrinsic quadrupole moments of proton or neutron systems \( Q_k^{(0)} \) which corresponds to \( Q_k \) in our description are given by
\[ Q_k^{(0)} = \begin{cases} 
-q_k \sqrt{2j_k + 1} \cdot \sqrt{\frac{5}{4}} \cdot \frac{(2j_k + 1)^3}{\sqrt{j_k(j_k + 1)(2j_k - 1)(2j_k + 3)}} n_k(1 - n_k)(1 + n_k), & \text{(for prolate shape)} \\
q_k \sqrt{2j_k + 1} \cdot \sqrt{\frac{5}{4}} \cdot \frac{(2j_k + 1)^3}{\sqrt{j_k(j_k + 1)(2j_k - 1)(2j_k + 3)}} n_k(1 - n_k)(2 - n_k), & \text{(for oblate shape)} 
\end{cases} \] (6.10)

where
\[ n_k = \frac{N_k}{2j_k + 1}. \] (6.11)

The moment of inertia \( \tau \) can be written down as
\[ \tau = \tau_p + \tau_n, \] (6.12)
\[ \tau_p = \frac{Q_p^{(0)}}{\rho Q_p^{(0)} + \sigma Q_n^{(0)}} \cdot \frac{1}{\chi d_p}, \] (6.13)
\[ \tau_n = \frac{Q_n^{(0)}}{\rho Q_n^{(0)} + \sigma Q_p^{(0)}} \cdot \frac{1}{\chi d_n}, \] (6.14)
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where

\[
\bar{d}_k = \begin{cases} 
  c_k \cdot \frac{5(2j_k + 1)^2}{(2j_k - 1)(2j_k + 3)} n_k^2, & \text{for prolate shape} \\
  c_k \cdot \frac{5(2j_k + 1)^2}{(2j_k - 1)(2j_k + 3)} (1 - n_k)^2, & \text{for oblate shape}
\end{cases}
\]  

(6.15)

The collective \( g \)-factor can be given by

\[
\bar{g}_n = \frac{C_p}{C_t}.
\]  

(6.16)

The above solutions were obtained under the following conditions:

\[
\frac{Q_p^{(0)}}{\rho Q_p^{(0)} + \sigma Q_n^{(0)}} > 0 \quad \text{and} \quad \frac{Q_n^{(0)}}{\rho Q_n^{(0)} + \sigma Q_p^{(0)}} > 0.
\]  

(6.17)

As was already shown in \( \Pi \), the oblate and the prolate shapes are favourable in the regions \( 0 < n_k < 0.5 \) and \( 0.5 < n_k < 1 \), respectively. In these cases the results obtained by using the cranking model are reliable in the regions \( 0.125 \leq n_k \leq 0.425 \) and \( 0.575 \leq n_k \leq 0.895 \). In Table I, we will show some numerical results of our theory and cranking model in the mass regions \( n_k = 0.348 \) and \( 0.652 \) where rotational motion can be expected for large \( j_k \) from our theory. We can see that our results show excellent agreement with those by the cranking model by comparing Eqs. (5.12) \( \sim \) (5.15) with Eqs. (6.12) \( \sim \) (6.16) and Table I.

Table I. Comparison of the present method with the cranking model with respect to some important quantities in the case of large \( j_k \).

<table>
<thead>
<tr>
<th>( n_k )</th>
<th>our method</th>
<th>cranking model</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.348</td>
<td>( Q_k^{(0)} = 0.476 q_k \sqrt{2j_k} + 1 )</td>
<td>( \tilde{Q}_k^{(0)} = 0.419 q_k \sqrt{2j_k} + 1 )</td>
</tr>
<tr>
<td>0.652</td>
<td>( Q_k^{(0)} = -0.476 q_k \sqrt{2j_k} + 1 )</td>
<td>( \tilde{Q}_k^{(0)} = -0.419 q_k \sqrt{2j_k} + 1 )</td>
</tr>
</tbody>
</table>

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M. Yamamura

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