Reduced Widths of Levels in Sharp Resonance Region
and Calculation on Si$^{28} + p$

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An attempt is made to interpret the smallness of the reduced widths of some isolated resonance levels observed in the reaction processes of the lighter nuclei. Resonance scattering of a proton by Si$^{28}$ is discussed as an example, and it is found out that the small relative reduced width of the order of magnitude $\theta^2 = 10^{-3} - 10^{-4}$ of the above process may be successfully interpreted by a relatively simple model. Namely, the incident proton drops into a bound orbit, inducing simultaneously an excitation of the target nucleus (which we describe by the vibrational model) and subsequent reemission of the proton, leaving the target nucleus again in the ground state.

§ 1. Introduction

As is well known, nucleon-nucleus scattering in the low energy region, where distinct resonances are observed, is successfully described by the compound nucleus theories. The degree of complexity of a compound level then manifests itself in the magnitude of the reduced width.

Let us describe the physical implications in terms of perturbation theory. If the incident nucleon does not perturb the target nucleus the scattering is of a single particle nature. But in the case where the angular momentum of the incident nucleon coincides with that of the unfilled shell of the target nucleus (assuming the target nucleus to be well described by the shell model), recouplings of angular momenta take place and the situation deviates from that of the single particle scattering. The relative reduced widths in this case were calculated intensively by Lane$^1$ with the intermediate coupling theory. However, the computed relative reduced widths are found to be rather large, as it should be, because, in this case, the degree of complexity of the compound state is taken to be low, in the sense that the possible induced mixing of configurations is assumed to be negligible.

In this situation, it seems to be of considerable interest to investigate the reduction of the magnitude of reduced widths expected from the induced configuration perturbations. Again, in the perturbation theoretical language, the incident nucleon may, when its interaction with the target nucleus is not too small, drop

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virtually into some bound orbit, exciting simultaneously the target nucleus, and after repetitions of such virtual jumps the nucleon will be scattered leaving the target nucleus in its ground state or in some low excited state. These repetitions of virtual jumps in the compound state will be represented, in the time-independent formalism of collision theory, as a complicated nature of the wave function of the compound state, which becomes a superposition of each eigenstate of the target nucleus multiplied by the respective bound orbital of the virtually captured incident nucleon. Coefficients of this superposition may in principle be determined by the solution of a secular equation, but in practice, it would be very hard to work through the computations in the shell model.

However, in some particular nuclei, low excitations will be represented as the excitation of vibrational levels. Then the mathematics will be a bit simpler and actual constructions of the wave functions of the compound state and, therefore, the computation of the reduced widths for the resonance scattering will be more practicable. Calculation of the reduced width in the collective model was already worked out by Yoshida. However, as his intention was to analyze the data obtained from the stripping reactions and not from the resonance reactions, the above-mentioned virtual captures of the nucleon in various bound orbitals were not taken into account.

In this paper, we attempt, through an example of the resonance scattering of a proton with Si\textsuperscript{28} nucleus, a calculation of the reduced width of a compound level, taking into account only the simplest mode of the above-mentioned virtual capture of the incident proton accompanied by the core excitation. The vibrational model was adopted to describe such excitations of Si\textsuperscript{28}-core in the compound nucleus P\textsuperscript{29}. It is found out that the relative reduced width can become so small as 10^{-2}---10^{-3} in agreement with observations. In §2, an expression of the reduced width is derived, the method of which is nothing but a simplified version of the one given by Bohr and Mottelson, which is particularly suited for our purpose. In §3, the method is applied to the resonance scattering of proton by Si\textsuperscript{28} nucleus and numerical computations are carried out. The results are given and discussed in §4.

§2. Derivation of formula for reduced width

In the following we derive a formula for reduced width in the elastic scattering of a nucleon by an atomic nucleus. For simplicity, the treatment is restricted to the case where the incident energy is too low to cause any nuclear reactions other than elastic scattering and radiative capture.

Total Hamiltonian can be written as follows,

\[ H = T + H_\text{r} + V(r, \xi) \]

where \( T \) is the energy of the relative motion and \( H_\text{r} \) is the Hamiltonian of the
target nucleus. \( V(r, \xi) \) is interaction between the nucleon and the target nucleus, where \( \xi \) stands for all variables except relative distance \( r \). The wave function \( \Psi \) of the whole system with incident energy \( E \) can be expanded as follows,

\[
\Psi = \frac{1}{r} u_{ij} (r) \Phi (lj, \alpha I; JM) + \sum_{\sigma_{ij}''} \frac{1}{\sigma_{ij}'' r} v_{ij''} (r) \Phi (lj', \alpha' I' ; JM),
\]

where \( \Psi \) is eigenfunction of the total angular momentum \( Jh \) and its \( z \) component \( Mh \) of the whole system, \( \Phi \)'s are composed of the ground state eigenfunction of the target nucleus \((\alpha, I, \text{energy zero}) \) or of the excited state eigenfunction \((\alpha', I', \text{excitation energy } W_r) \) and the spin-angle eigenfunctions for the impinging nucleon having orbital angular momentum \( lh \) or \( l'h \) and total angular momentum \( jh \) or \( j'h \) respectively. \( I, I' \) are the spin quantum numbers and \( \alpha, \alpha' \) are all other quantum numbers specifying the states of the target nucleus. \( u(r) \) and \( v(r) \) are the unknown radial functions for the incident channel and closed channels respectively. Hereafter we drop the symbols \( \alpha, \alpha' \) and superscripts \( ij \), \( ij' \) for the brevity of notations.

Equations which determine \( u(r) \) and \( v(r) \)'s read as follows,

\[
(T_i + U_{ij} (r) - E) u_{ij} (r) = - \sum_{ij'} \int \Phi^* (lj, I; JM) V \Phi (lj', I'; JM) d \xi v_{ij'},
\]

\[
(T_{ij'} + U_{ij'} - E + W_{ij'}) v_{ij'} = - \int \phi^* (lj', I'; JM) V \phi (lj, I; JM) d \xi u_{ij} - \sum_{ij''} \int \phi^* (lj', I'; JM) V \phi (lj'', I''; JM) d \xi v_{ij''},
\]

with

\[
T_i = - \frac{\hbar^2}{2 \mu} \left\{ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right\},
\]

where \( \mu \) is the reduced mass of the system and \( U(r) \)'s are matrix elements of \( V \) diagonal with respect to \( \Phi \)'s. Hitherto our \( \Phi \)'s are free from any nuclear model but we assume hereafter that we could choose some nuclear model so as to make nondiagonal elements of \( V \) small and neglect the second term on the right-hand side of Eq. (4). Diagonal elements \( U \)'s may, of course, have large values.

Then from (3) and (4) we can derive an equation including only \( u_{ij}(r) \), which comes out to be

\[
(T_i + U_{ij}(r) - E) u_{ij}(r) = \sum_{ij'} \frac{1}{T_{ij'} + U_{ij'} - E + W_{ij'}} \langle lj'l' | V | lj' I' \rangle u_{ij'},
\]
where

\[ \langle ij|V|l'j'\rangle = \int \Phi^*(ij, I; JM) V(r, \xi) \Phi(l'j', I'; JM) d\xi. \]

If the right-hand side term of (5) for the formation of a compound state was absent, we would obtain a simple potential scattering from

\[ (T_I + U_{ij}(r) - E) u_{ij}^0(r) = 0. \]  (6)

The logarithmic derivative

\[ f_{ij}(E, a) = a \frac{u_{ij}^{(0)}(a)}{u_{ij}(a)} \]  (7)

differs from that for the potential scattering

\[ f_{ij}^{(0)}(E, a) = a \frac{u_{ij}^{(0)}(a)}{u_{ij}^{(0)}(a)} \]  (8)

through the formation of compound state.

The right-hand side of (7) can be calculated from (5) and (6) by the standard procedure, as follows,

\[ f_{ij}(E, a) = f_{ij}^{(0)}(E, a) - \frac{2mu}{h^2} \frac{1}{u_{ij}(a)} \sum_{m'j'} \left( \int \frac{dr}{\rho_{ij'}} \langle ij'|V|lj\rangle u_{ij'} \right) \left( \int \frac{dr}{\rho_{ij'}} \langle ij|V|l'j'\rangle u_{ij'} \right) \frac{E_{nlj} + W_{ij'} - E}{E_{nlj} + W_{ij'} - E}. \]  (9)

where \( w_{n lj'} \)'s are the normalized radial eigenfunction of the wave equation

\[ (T_{ij} + U_{ij} - E_{n ij}) w_{n ij} (r) = 0. \]  (10)

Isolated resonance energy \( E_{ij} \) can be calculated by equating \( f_{ij}(E, a) \) to the well-known level shift parameter \( A_i \) as

\[ f_{ij}(E, a) = A_i \]  (11)

and the reduced width of this level is obtained by

\[ \tau_{ij} = - \left[ \frac{df_{ij}(E, a)}{dE} \right]_{E=E_{ij}}. \]  (12)

If the collective vibrational model is adopted for the core excitation and the incident nucleon is assumed to interact with the target nucleus only on the surface \( r=a \), then, \( \langle ij|V|l'j'\rangle \) will contain a delta function \( \delta(r-a) \) and (9) will simply be reduced to

\[ f_{ij}(E, a) = f_{ij}^{(0)}(E, a) \]

\[ - \frac{2mu}{h^2} \sum_{m'j'} \int_0^a dr \frac{w_{n lj}^m \langle lj'|V|lj\rangle w_{n lj}}{E_{n lj} + W_{ij'} - E}. \]  (13)
Then we obtain an isolated resonance energy $E_{r}^{j}$, extracting only one term from the right-hand side of (13), that is,

$$E_{r}^{j} = W_{ll}^{j} + E_{nl_{l}l_{l'}}^{j}$$

$$\frac{2\mu a}{\hbar^{2}} \left| \int dr \, w_{nl_{l}l_{l'}}^{*}(l'j' I'| V| lj I)_{A} \right|^2$$

The reduced width $\gamma_{r}^{j}$ is given, from (12), (13) and (14), by

$$\gamma_{r}^{j} = -\left( \frac{d f_{ll'}^{j}}{dE} \right)_{E_{r}^{j}}$$

$$+ \frac{\hbar^{2}}{2\mu a^{2}} \left| \int_{0}^{\infty} dr \, w_{nl_{l}l_{l'}}^{*}(l'j' I'| V| lj I)_{A} \right|^2$$

This shows explicitly the dependence of $\gamma_{r}^{j}$ on the matrix element of the interaction. This formula is already given in Bohr-Mottelson's paper in a more general form, namely, our matrix element in (15)

$$\int w_{nl_{l}l_{l'}}^{*}(l'j' I'| V| lj I)_{A} \Phi^{*}(l'j', I'; JM) \Phi(lj, I; JM) d\xi$$

is replaced there by

$$\int \mathcal{F}^{*}((\lambda, JM) V \Phi(lj, I; JM) d\xi,$$

where $\mathcal{F}(\lambda, JM)$'s in (17) are assumed to constitute a complete orthonormal set orthogonal to $\Phi(lj, I; JM)$ of the ground state. If the interaction $V(r, \xi)$ is not so weak as we have assumed above, we should not disregard the couplings between excited states $\Phi(l'j', I' ; JM)$ in the right-hand side of Eq. (4). Then, $\mathcal{F}(\lambda, JM)$ would become a linear combination

$$\mathcal{F}(\lambda, JM) = \sum_{nl'j'I'} b(nl'j'I') w_{nl_{l}l_{l'}}^{*} \Phi(l'j', I' ; JM),$$

the coefficients $b(nl'j'I')$ in which would be determined from solutions of a secular equation. Generally speaking, such a complexity of $\mathcal{F}(\lambda, JM)$ will make the matrix element (17) small even if the interaction $V(r, \xi)$ may be rather strong, resulting in the small value of $\gamma_{r}^{j}$. However, in such a strong interaction case, quantitative evaluation of $\gamma_{r}^{j}$ is very hard and we have, up to now, only the well-known statistical estimation given by Feshbach, Peaslee and Weisskopf. Our simplified model which leads to (15) describes the scattering through the compound nucleus formation simply as a two step process, namely, absorption of the incident nucleon in a bound level $(nl'j')$ with simultaneous excitation of the target nucleus to $(l'j')$ level and reemission, the target nucleus going back to the ground state.
again to the ground level. Though our model is very simple, as can be seen from (16), \(\tilde{r}_{ij}\) will still come out to be small if the interaction \(V(r, \tilde{r})\) is sufficiently weak, and it will be of considerable interest to examine the magnitude of \(\tilde{r}_{ij}\) calculated by (15) for the collective model. This we shall do through an example of resonance scattering of a proton by \(\text{Si}^{28}\).

§ 3. Calculation example of \(\text{Si}^{28} + p\)

We deal with an example of \(\text{Si}^{28} + p\) by adopting vibrational model for the target nucleus \(\text{Si}^{28}\). Pandya\(^5\) used the vibrational model and Bromley et al.\(^6\) and Oostrum et al.\(^7\) applied the rotational model to discuss the properties of \(\text{Si}^{29}\), the mirror nucleus to \(\text{P}^{29}\) which is the compound nucleus in our case, with some successes.

We restrict the relevant states to those ones composed of zero or one phonon state (Pandya included up to two phonon state) and \(2s1/2\) or \(1d3/2\) single particle orbit which comes into play in the mass region concerned, and assume a simple form of average potential as

\[
U_{ij}(r) = \frac{Ze^2}{r} \quad \text{for } r > a
\]

\[
= -v - d(\text{ls}) \quad \text{for } r < a.
\]

\hspace{1cm} (19)

The nondiagonal elements of interaction are constructed from

\[
-\frac{h\omega}{2C} \delta(r-a) \sum_{\mu} (b_{\mu} + (-)^{s}b_{\mu}^{*}) Y_{2\mu}(\Omega),
\]

\hspace{1cm} (20)

where \(k\) is a coupling constant, \(C\) is the nuclear rigidity and \(b, b^{*}\) are annihilation, creation operator for surphon. \(H_{i}\) is assumed to be an oscillator Hamiltonian for surphons with the energy \(h\omega\) which is equated to the excitation energy 1.78 MeV of the first excited state \((2^+)\) of \(\text{Si}^{28}\). The matrix elements of the interaction (20) to be concerned are

\[
\langle lj, N=0 \mid R=0; J=j \mid M=M|V|l'j', 12 \mid JM \rangle
\]

\[
= -k \sqrt{\frac{h\omega}{2C}} \langle j|l|j'\rangle \delta(r-a),
\]

\hspace{1cm} (21)

where \(N\) is the number of phonon, \(R\) its angular momentum and \(\langle j|h|j'\rangle\) is a geometrical coefficient given in B–M.\(^9\) This expression will be used in (14) and (15). \(w_{ij}(r)\)'s in the region \(r > a\) are evaluated by the WKB approximation.

There are four unknown quantities \(a, k\sqrt{h\omega/2C}\) and the binding energies of orbitals \(E(2s1/2), E(1d3/2)\) in our theory. The potential parameter \(v\) and \(d\) in the formula (19) are derived from the assumed values of these four quantities. We take \(a\) as a free parameter and, to decide the value of \(k\sqrt{h\omega/2C}\) and \(E's\), use is made of the experimental data of the lowest three levels of \(\text{P}^{29}\) (Fig. 1).
That is, the total Hamiltonian is diagonalized within the restricted states mentioned above and the three parameters are adjusted to give the experimental energy values. Spin, parity assignments of these levels were recently completed by Okano et al. and proved to be the same as those of the mirror nucleus Si$^{29}$.

The results are shown in Table I and Fig. 1. The values in Table I are almost constant with the free parameter $a$ from 3.6 to 4.6 $\times 10^{-13}$ cm, though that of $v$ varies from 46 to 30 MeV and $d$ from 5.7 to 4.2 MeV. In the middle of Fig. 1, calculated bound levels are shown and in the left levels in the zero-th order approximation and in the right experimental values.

The relevant virtual state corresponding to the lowest resonance scattering is that in which

\[ \langle n'l' j' \rangle = (1d 3/2) \) and \( I' = 2 \) (one phonon) so that

\[ E_v = W_{\nu} + E_{n't'} = h\omega + E(1 d 3/2) \]

\[ = 1.07 \text{ MeV}. \] (22)

Then from angular momentum coupling and parity conservation, the partial waves in the incident wave that could resonate with the virtual state are $s 1/2$, $d 5/2$ and $g 7/2$. Putting (21) into (13) we get

\[ f_{ij}(E, a) = f_{ij}^0(E, a) \]

\[ - 2\mu a^2 \frac{h\omega}{k^2} \frac{a (\omega_{_W}(a))^2}{2Ca^2} \frac{j}{h |3/2|}^3, \] (23)
where $\omega(a)$ is the value at $r=a$ of the normalized radial wave function of orbital $1d\,3/2$ (see the Appendix) and $f_0^{3d}(E, a)$ is expressed in terms of spherical Bessel function.

The numerical evaluation shows that, the values of

$$\frac{2\mu a^2}{\hbar^2} \frac{\hbar^2 \omega}{2C d^2 a} (\omega(a))^2 \langle j|\hbar|3/2\rangle^2$$

are 0.28—0.46 for $s\,1/2$ wave, 0.14—0.23 for $d\,3/2$, 0.04—0.06 for $d\,5/2$ and 0.18—0.30 for $g\,7/2$ (all in MeV) for the free parameter $a=3.6-4.6 \times 10^{-13}$ cm (Table II). It is also found out that potential resonances do not appear below $E=2.5$ MeV. Thus a sharp resonance can be expected in the neighbourhood of $E=1.07$ MeV. Resonance energy $E_r^f$ for the incident $(i)$ wave is obtained by (14) where the level shift parameters $\Delta_i$'s are evaluated in the WKB approximation. We determine $E_r^f$ form (14) graphically and calculate $r$'s by (15). As expected, $E_r^f$'s are found to lie very close to $E_o=1.07$ MeV.

### § 4. Results and discussions

Results obtained are shown in Table II.

<table>
<thead>
<tr>
<th>$a$ (10$^{-13}$ cm)</th>
<th>$E_r$ (MeV)</th>
<th>$\gamma$ (keV)</th>
<th>$\theta^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>1.19</td>
<td>44</td>
<td>0.89</td>
</tr>
<tr>
<td>4.0</td>
<td>1.18</td>
<td>35</td>
<td>0.86</td>
</tr>
<tr>
<td>4.6</td>
<td>1.18</td>
<td>23</td>
<td>0.75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$ (10$^{-13}$ cm)</th>
<th>$E_r$ (MeV)</th>
<th>$\gamma$ (keV)</th>
<th>$\theta^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>1.08</td>
<td>530</td>
<td>0.016</td>
</tr>
<tr>
<td>4.0</td>
<td>1.07</td>
<td>3</td>
<td>10$^{-4}$</td>
</tr>
<tr>
<td>4.6</td>
<td>1.07</td>
<td>810</td>
<td>0.027</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$ (10$^{-13}$ cm)</th>
<th>$E_r$ (MeV)</th>
<th>$\gamma$ (keV)</th>
<th>$\theta^2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>1.16</td>
<td>50</td>
<td>1.0</td>
</tr>
<tr>
<td>4.0</td>
<td>1.17</td>
<td>48</td>
<td>1.2</td>
</tr>
<tr>
<td>4.6</td>
<td>1.18</td>
<td>45</td>
<td>1.5</td>
</tr>
</tbody>
</table>

The lowest resonance level already known lies at the proton energy $E=367$ keV$^9$ equivalent to 3.12 MeV excitation from $\text{P}^{29}$ ground state. There is some discrepancy between this value and our corresponding theoretical value $1.07 + 2.76 = 3.83$ MeV (2.76 being the binding energy of the proton). But this difference might be due to our simplification of the problem (neglect of configurations with more than one phonon).

Recent experiments and analyses by Okano et al.$^9$ determined definitely $J^*=5/2^+$ character of the resonance level. Fujimoto et al.$^{10}$ and Macfarlane et
al.$^{11}$ tabulated nucleon reduced widths for various nuclei calculated directly from the data of $(d, p)$ or $(d, n)$ reactions. The relation between reduced widths and cross sections for $(d, p)$ or $(d, n)$ reactions is discussed in references 10) and 12). A part of their table is transcribed here after translating their reduced width in $\Theta$, the value measured in the unit of sum rule limit$^{18}$ (Table III). If, as Okano et al.$^8$ has stressed as to a level scheme (see Fig. 2), general correspondences between mirror nuclei $^{29}$P and $^{29}$Si do hold good, then, proton reduced widths of $^{29}$P will be equal to neutron reduced width of $^{29}$Si for the

![Diagram](https://example.com/diagram.png)

Fig. 2. Level scheme of the mirror pair $^{29}$P and $^{29}$Si.

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Final state excitation</th>
<th>$I$-value</th>
<th>$J^*$</th>
<th>$\Theta(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{28}$Si$(d, n)^{29}$</td>
<td>0</td>
<td>0</td>
<td>$1/2^+$</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>1.30</td>
<td>2</td>
<td>$3/2^{++}$</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>1.92</td>
<td>2</td>
<td>$5/2^{++}$</td>
<td>0.5</td>
</tr>
<tr>
<td>$^{28}$Si$(d, p)^{29}$</td>
<td>0</td>
<td>0</td>
<td>$1/2^+$</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>1.28</td>
<td>2</td>
<td>$3/2^+$</td>
<td>1.9</td>
</tr>
<tr>
<td></td>
<td>2.03</td>
<td>2</td>
<td>$5/2^+$</td>
<td>0.5</td>
</tr>
<tr>
<td></td>
<td>3.07</td>
<td>2</td>
<td>$5/2^{++}$</td>
<td>0.2$^{**}$</td>
</tr>
<tr>
<td></td>
<td>3.62</td>
<td>3</td>
<td>$(7/2^-)$</td>
<td>(1.3)</td>
</tr>
<tr>
<td></td>
<td>4.90</td>
<td>1</td>
<td>$(3/2^-)$</td>
<td>(2.9)</td>
</tr>
<tr>
<td></td>
<td>4.93</td>
<td>1</td>
<td>$(1/2^-)$</td>
<td>(2.3)</td>
</tr>
<tr>
<td></td>
<td>6.36</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^{*}$ This assignment was not definite at that time of reference 10).

$^{**}$ As $J$ of this level was not known then,$^{11}$ only the value $(2J+1)\Theta$ is given in the original table.
Reduced Widths of Levels in Sharp Resonance Region

corresponding levels. Such expectation is actually realized, as can be seen in Table III for the lowest three levels. Thus the proton reduced width of the resonance level in question may be presumed to have the same value, or at least the same order of magnitude, as the neutron reduced width of 3.07 MeV ($5/2^+$) excited state of Si$^{29}$, that is, $\theta' = 0.002$.

In our calculation for $d \ 5/2$ wave, $E_r$ happens to come in the very close vicinity of the point where $f_r^{(0)}$ tends to infinity (this means the average potential $U_{\text{at/2}}(r)$ operates just like a hard sphere), thus a small variation of free parameter $\alpha$ is found to have a decisive influence on the value of $f_{d \ 5/2}$ through the denominator in (15) changing even its order of magnitude though anyhow the value rests always very small. We have also evaluated $f_{d \ 5/2}^{(0)} - \Delta r$ not for the theoretical $E_r$, but for the empirical $E_r$ (which is 0.7 MeV less than the theoretical value), but the situation (Table II) remains the same, that $\theta'^{d \ 5/2} < 0.03$ for $d \ 5/2$ wave when $\alpha$ varies from 3.6 to $4.6 \times 10^{-13}$ cm. As the situation is very critical at the energy near hard sphere scattering, some refinements of the calculation are necessary for the quantitative conclusion, such as inclusion up to two phonon state and more precise evaluation of $\Delta r$ and so on. It is, however, interesting to remark that experimental $\theta'^{d \ 5/2}$ shows minimum at $5/2^+$ levels in the mirror nucleus (Table III) and that just this tendency can be seen in our results (Table II).

For partial waves other than $d \ 5/2$, calculated reduced width are found to be of the same order as or smaller than the values observed but the precise identification is impossible. As for resonance energies $E_r$'s we need more refined calculation including, for instance, the second intermediate state of the whole system. By such refinements it might be expected that closely degenerate resonance levels we got here for $s \ 1/2$, $d \ 3/2$, and $g \ 7/2$ partial waves would be found splitted.

§ 5. Conclusion

It is shown through an example of Si$^{28} + p$, that small reduced widths $\theta^{2} = 10^{-2} \sim 10^{-3}$ can be explained by rather simple model for the formation of compound nucleus and its subsequent decay process. Namely, for the example cited here, resonance scattering of protons of energy 367 keV by Si$^{28}$ nucleus ($\theta^{2} \sim 10^{-3}$) might be caused by a drop-in of the incident proton with angular momentum $d \ 5/2$ into the bound orbit $d \ 3/2$ with simultaneous excitation of one surphon vibration of Si$^{28}$ core and the subsequent reemission. For the sake of mathematical convenience we have described the core excitation by the vibrational model, and it remains to be studied whether the values of $\theta^{2}$ of the same order as above can be predicted by other models, e.g. the shell model or the rotational model. It is also very desirable to investigate further whether or not any model (certainly a little more complex ones) might be capable to describe resonance reactions with even smaller relative reduced widths.
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Appendix

A real function \(w(r)\) which satisfies an equation
\[
\frac{d^2 w}{dr^2} + U(r)w(r) + \lambda w(r) = 0
\]  
(A·1)

satisfies also a relation as follows,
\[
\int_\alpha^\beta w(r)^2 dr = -\left[\frac{\alpha^\beta}{\lambda} \frac{d}{dr} \frac{w'}{w}\right]_\alpha^\beta.
\]  
(A·2)

If, for \(\lambda<0\), we define two particular solutions \(w_+(r)\) and \(w_-(r)\) by the boundary conditions as
\[
w_+(\infty) = w_-(0) = 0,
\]  
(A·3)

then the eigenvalue \(\lambda_0\) of (A·1) with the boundary condition \(w(0) = w(\infty) = 0\) can be determined from the equation,
\[
\frac{w_+'(a)}{w_+(a)} = \frac{w_-'(a)}{w_-(a)} \quad \text{for arbitrary } a.
\]  
(A·4)

And the eigenfunction is given by \(w = w_+ = w_-\). If we further normalize the eigenfunction as
\[
\int_0^\infty w(r)^2 dr = 1,
\]  
(A·5)

it follows, from (A·2),
\[
[aw(a)^2]^{-1} = \frac{d}{d\lambda} \left\{ \frac{w_+(a)}{w_+(a)} - \frac{w_-(a)}{w_-(a)} \right\} |_{\lambda=\lambda_0}
\]  
(A·6)

Now, in our case,
\[
w_- = N_- r j_1(Kr),
\]
\[
w_+ = N_+ \exp \left\{ -\int_0^r \sqrt{\kappa^2 + \frac{2\mu}{\hbar^2} \frac{Ze^2}{r} - \frac{l(l+1)}{r^2}} \right\} \text{(WKB)},
\]  
(A·7)

where \(K\) and \(\kappa\) is determined from
\[
\frac{\hbar^2}{2\mu} K^2 = E + v + d(l\hbar),
\]  
(A·8)
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\[ -\frac{\hbar^2}{2\mu} \kappa^2 = E \quad (E < 0), \quad \text{(A·9)} \]

and \( N_\pm \) is a normalization constant. The value of the wave function at \( r = a \) can then be calculated from (A·6) and (A·7) which now read,

\[ \left[ a\omega(a)^2 \right]^{-1} = \frac{\hbar^2}{2\mu a^2} \frac{d}{dE} \left[ \frac{a}{w_-(a)} \frac{w'_-(a)}{w'_+(a)} - 1 \right], \quad \text{(A·10)} \]

where

\[ a \frac{w'_-(a)}{w_-(a)} = K a \frac{j_1(Ka)}{j_1(ka)} - 1 \quad \text{(A·11)} \]

and

\[ a \frac{w'_+}{w_+} = -\int \frac{2\mu a^2}{\hbar^2} \left( |E| + \frac{Ze^2}{a} \right) + l(l+1). \quad \text{(A·12)} \]

References

13) T. Teichman and E. P. Wigner, Phys. Rev. 87 (1952), 123.