Asymptotic Behavior of the Eigenvalues in the Wick-Cutkosky Model for Large Principal Quantum Number

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The Wick-Cutkosky eigenvalue problem for the bound state in the Bethe-Salpeter equation is transformed into an operator form. With the aid of the representation thus obtained, the asymptotic behavior of the eigenvalues is found for large principal quantum number.

§ 1. Introduction

The Wick-Cutkosky model of the Bethe-Salpeter equation describes the bound states formed out of two massive scalar particles by the exchange of a massless scalar particle in the ladder approximation. For a given bound state energy, the eigenvalue problem for the coupling constant is described by a second order ordinary differential equation. The property of the eigenvalues has been investigated extensively. The behavior near the threshold was studied by Wick and by Cutkosky. The asymptotic form of the eigenvalues near the zero bound state energy was obtained to the first order by Nakanishi, to second order by Kyriakopoulos and to sixth order by Tanaka and Nakanishi. Numerical analysis of the eigenvalues as a function of the bound state energy was carried out by Cutkosky and more thoroughly by zur Linden. The Regge trajectories in the Wick-Cutkosky model were also studied perturbationally by Nakanishi, Seto and Müller, theoretically by Gatto and Menotti and numerically by zur Linden and by Murai, Nakamura and Ezawa.

There exist, also, systematic approaches to the eigenvalue problem. Hadji-oi and Kyriakopoulos have transformed the Wick-Cutkosky equation into a one-dimensional Schrödinger equation with a suitable potential function. They have obtained approximate eigenvalues which exhibit the threshold singularity. In Ref. 7), Tanaka and Nakanishi have derived the asymptotic form of the ground state eigenvalue for large principal quantum number. The present paper is, in some sense, a continuation of the work in Refs. 7) and 13). The method of analysis employed here is, however, quite different from theirs.

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In the next section, a general formalism for the eigenvalue problem in the Wick-Cutkosky model is presented. The eigenvalue problem in the form of a differential equation is transformed into a matrix form on a suitable Hilbert space. In §3, the matrix, or the operator, is expressed in terms of the creation and annihilation operators in a one-dimensional quantum harmonic oscillator. With the aid of the representation thus obtained, the perturbation calculation for large principal quantum number is performed and the asymptotic form of the eigenvalues is determined to fourth order. The final section is devoted to discussion.

§2. General formalism

The eigenvalue problem in the Wick-Cutkosky model\(^{(5, 20)}\) for the equal-mass case (common masses are set unity) with the bound state energy \(s(s < 4)\) is expressed in the form of an ordinary differential equation as

\[
\left\{ (1-z^2) \frac{d}{dz} \left( \frac{d}{dz} \right) + 2(n-1)z \frac{d}{dz} - n(n-1) + \frac{\lambda_n(s)}{1-(1-z^2)s/4} \right\} g_n(z, s) = 0,
\]

\((-1 \leq z \leq 1)\) \hspace{1cm} (2·1)

with the boundary condition \(g_n(z = \pm 1, s) = 0\). In the above equation, \(n\) denotes the principal quantum number \((n = 1, 2, \cdots)\), \(\lambda_n(s)\) is the eigenvalue to be found and \(g_n(\cdot, s)\) is the Wick-Cutkosky weight function involved in the Bethe-Salpeter amplitude in a well-known way. (We follow the notations in Ref. 4.) For the case of the zero bound state energy, \(s = 0\), Eq. (2·1) can be solved explicitly. The results are

\[
\lambda_n(0) = (n + \kappa)(n + \kappa + 1), \quad (\kappa = 0, 1, \cdots)
\]

\[
g_n(z, 0) = 2^n T \left( n + \frac{1}{2} \right) \left( \frac{(n + \kappa + \frac{1}{2}) \kappa!}{\pi (2n + \kappa)!} \right)^{1/2} (1 - z^2)^n C_{\kappa + 1/2}^n(z),
\]

where \(C_{\kappa + 1/2}^n\) is the Gegenbauer polynomial. The set \(\{g_{kn}(\cdot, 0), \kappa = 0, 1, \cdots\}\) forms a complete orthonormal system in the Hilbert space \(\mathcal{H}_n := L^2([-1, 1]; dz/(1-z^2)^{s})\). (The left-hand side of the symbol :=is defined by the right-hand side of it.)

From the recurrence formula for the Gegenbauer polynomials\(^{(15)}\)

\[
(1-z^2)C_{\kappa + 1/2}(z) = \frac{\kappa + 1}{2}\frac{(n + \kappa + 1) + n^2 - 1}{(2n + 2\kappa - 1)(2n + 2\kappa + 3)} C_{\kappa + 1/2}(z) + \frac{2(n + \kappa)(n + \kappa + 1)}{(2n + 2\kappa - 1)(2n + 2\kappa + 3)} C_{\kappa + 1/2}(z)
\]

it immediately follows that
where the functions $g$ and $h$ are defined by

$$g(\kappa, n) = \frac{2[(n+\kappa)(n+\kappa+1)+n^2-1]}{(2n+2\kappa-1)(2n+2\kappa+1)}$$

$$h(\kappa, n) = \left(\frac{\kappa (2n+\kappa)}{(2n+2\kappa-1)(2n+2\kappa+1)}\right)^{1/2}.$$ (2·3)

As is easily verified, the differential operator

$$(\lambda_n(s) - d_n(z, s)) g_{\kappa n}(z, 0)$$

$$= \left[\lambda_n(s) - \lambda_n(0) \left(1 - (1-z^2) \frac{s}{4}\right)\right] g_{\kappa n}(z, 0).$$ (2·4)

The matrix elements of the differential operator in $\mathcal{H}_n$ can be calculated from Eqs. (2·2) and (2·4) to be

$$\langle n'\kappa' | \lambda_n(s) - d_n(\cdot, s) | n\kappa \rangle = \left\{\left(\lambda_n(s) - \lambda_n(0) \left(1 - (1-z^2) \frac{s}{4}\right)\right)\delta_{\kappa\kappa'} - \frac{s}{4} h(\kappa+1, n) h(\kappa+2, n) \delta_{\kappa\kappa'-2} \right\} \lambda_n(0),$$ (2·5)

where we have used Dirac’s ket notation $|n\kappa\rangle$ for the vector $g_{\kappa n}(\cdot, 0)$ in $\mathcal{H}_n$.

Hereafter we shall put $\sigma := s/4$ for notational convenience. By introducing an Hermitian operator $L_n(\sigma)$ defined by

$$\langle n\kappa' | L_n(\sigma) | n\kappa \rangle = \sqrt{\lambda_n(0)} \left\{(1-\sigma g(\kappa, n)) \delta_{\kappa\kappa'} + \sigma h(\kappa+1, n) \times h(\kappa+2, n) \delta_{\kappa\kappa'-2} + \sigma h(\kappa'+1, n) h(\kappa'+2, n) \delta_{\kappa\kappa'-2} \right\} \sqrt{\lambda_n(0)},$$ (2·6)

we can readily recognize that the differential operator can be rewritten as

$$\lambda_n(s) - d_n(\cdot, s) = L_n(0)^{1/2} (\lambda_n(s) - L_n(\sigma)) L_n(0)^{1/2}.$$ (2·1)

The eigenvalue equation $d_n(\cdot, s) |\kappa\rangle = \lambda_n(s) |\kappa\rangle$ is, therefore, equivalent to the equation $L_n(0)^{-1/2} (\lambda_n(s) - L_n(\sigma)) L_n(0)^{1/2} |\kappa\rangle = 0$. The positive definitness of $L_n(0)$ implies $(\lambda_n(s) - L_n(\sigma)) L_n(0)^{1/2} |\kappa\rangle = 0$ and also $L_n(0)^{1/2} |\kappa\rangle \neq 0$ provided that $|\kappa\rangle \neq 0$. Thus the eigenvalue problem (2·1) is reduced to the eigenvalue problem
for the operator $L_n(\sigma)$.

§ 3. Asymptotic behavior of eigenvalues for large $n$

In the preceding section, we have reduced the eigenvalue problem (2·1) to that of the operator $L_n(\sigma)$ in (2·6). Let us introduce in $S_n$ the creation and annihilation operators and the number operator, $a'|n\kappa\rangle = \sqrt{\kappa+1}|n\kappa+1\rangle$, $a|n\kappa\rangle = \sqrt{\kappa}|n\kappa-1\rangle$ and $N = a' a$. Then $L_n(\sigma)$ is expressed as

$$L_n(\sigma) = \sqrt{(n+N)(n+N+1)} \left[ 1 - \sigma g(N,n) + \sigma \left( \frac{\hbar(N,n)}{\sqrt{N}} - a^2 \right)^2 + \sigma \left( \frac{\hbar(N,n)}{\sqrt{N}} \right)^2 \right] \sqrt{(n+N)(n+N+1)}. \quad (3·1)$$

We cannot, however, solve the eigenvalue problem exactly for general value of $\sigma(\sigma<1)$. We shall expand the right-hand side of (3·1) in the power series of $1/n$, and obtain the asymptotic form of the eigenvalues for large $n$. The operator $L_n(\sigma)$ has the asymptotic expansion

$$L_n(\sigma) = n^2 (1-\sigma) + nH_0 + H_1 + \frac{1}{n} H_2 + \cdots, \quad (3·2)$$

where the operators $H_0$, $H_1$, and $H_2$ are given by

$$H_0 := \frac{2-\sigma}{2} (2N+1) + \sigma \left( a^{z^2} + a^2 \right),$$

$$H_1 := \frac{2-\sigma}{2} (N^2 + N) + \frac{\sigma}{2} \left( \frac{2N-1}{4} a^{z^2} + a^2 \frac{2N-1}{4} \right) - \frac{\sigma}{4},$$

$$H_2 := \frac{3\sigma}{8} (2N+1) - \frac{25\sigma}{64} (a^{z^2} + a^2).$$

The unperturbed Hamiltonian $H_0$ is of quadratic in $a'$ and $a$, so that it can be brought into a diagonal form by a suitable linear transformation. In terms of the new creation and annihilation operators

$$b' := \alpha a' + \beta a,$$

$$b := \beta a' + \alpha a, \quad (3·3)$$

with

$$\alpha := \frac{1}{2} \left( (1-\sigma)^{-1/4} + (1-\sigma)^{1/4} \right),$$

$$\beta := \frac{1}{2} \left( (1-\sigma)^{-1/4} - (1-\sigma)^{1/4} \right),$$

$H_0$ takes the following form:

$$H_0 := 2\sqrt{1-\sigma} \left( b'b + \frac{1}{2} \right). \quad (3·4)$$
Note that $\alpha$ and $\beta$ satisfy $\alpha^2 - \beta^2 = 1$, so that the usual commutation relation $[b, b'] = 1$ is valid. The transformation (3.3) is generated by the unitary operator $U: = \exp(\text{ln}(1 - \sigma)(a'^2 - a^2)/8)$; $b' = UaU^{-1}$ and $b = UaU^{-1}$. Using the equality

$$\frac{2 - \sigma}{2} (2N + 1) + \frac{\sigma}{2} (a'^2 + a^2) = 2\sqrt{1 - \sigma} (b'b + \frac{1}{2}), \quad (3.5)$$

we can express $H_1$ and $H_2$ in terms of $N$ and $b'b$ as

$$H_1 = \frac{\sqrt{1 - \sigma}}{2} \left( (N + \frac{1}{2}) (b'b + \frac{1}{2}) + (b'b + \frac{1}{2}) (N + \frac{1}{2}) \right) - \frac{2 + \sigma}{8},$$
$$H_2 = \frac{50 - \sigma}{32} \left( (N + \frac{1}{2}) \right)^2 - \frac{25}{16} \sqrt{1 - \sigma} (b'b + \frac{1}{2}). \quad (3.6)$$

The old number operator $N$ is related to $b'$ and $b$ through the relation

$$N + \frac{1}{2} = \frac{2 - \sigma}{2\sqrt{1 - \sigma}} (b'b + \frac{1}{2}) - \frac{\sigma}{2} \frac{1}{2\sqrt{1 - \sigma}} (b'^2 + b^2). \quad (3.7)$$

In order to diagonalize $H_0$, we choose new basis vectors $|\kappa\rangle := U|n\kappa\rangle$ ($\kappa = 0, 1, \cdots$) in $S$. In the new base, the new creation and annihilation operators satisfy $b'|\kappa\rangle = \sqrt{\kappa + 1}|\kappa + 1\rangle$ and $b|\kappa\rangle = \sqrt{\kappa} |\kappa - 1\rangle$, and $H_0$ is of a diagonal form

$$H_0|\kappa\rangle = 2\sqrt{1 - \sigma} (\kappa + \frac{1}{2}) |\kappa\rangle. \quad (3.8)$$

The matrix elements of $N + \frac{1}{2}$ in (3.7) are given by

$$\langle \kappa'|N + \frac{1}{2}|\kappa\rangle = \frac{2 - \sigma}{2\sqrt{1 - \sigma}} (\kappa + \frac{1}{2}) \delta_{\kappa\kappa'} - \frac{\sigma}{2} \frac{1}{2\sqrt{1 - \sigma}} (\sqrt{\kappa' - 1}) \delta_{\kappa\kappa' - 2} + \sqrt{(\kappa - 1) \delta_{\kappa\kappa' - 2}}. \quad (3.9)$$

The $\kappa$-th eigenvalue of $L_n(\sigma)$, $\lambda_{\kappa n}(\sigma)$, is expressed, by the usual perturbation method in quantum mechanics, to be

$$\lambda_{\kappa n}(\sigma) = n^2 (1 - \sigma) + 2n\sqrt{1 - \sigma} \left( \kappa + \frac{1}{2} \right) + \langle \kappa|H_1|\kappa\rangle$$
$$+ \frac{1}{n} \left[ \langle \kappa|H_2|\kappa\rangle + \sum_{\kappa' = \kappa \pm 2} |\kappa'| \langle H_1|\kappa'\rangle^2 \right] + \cdots.$$ 

The matrix elements of the Hamiltonian $H_1$ and $H_2$ can be readily calculated with the help of (3.6) and (3.9) and the final result is

$$\lambda_{\kappa n}(\sigma) = n^2 (1 - \sigma) + 2n\sqrt{1 - \sigma} \left( \kappa + \frac{1}{2} \right) + \left( \frac{2 - \sigma}{2} \left( \kappa + \frac{1}{2} \right)^2 \right) - \frac{2 + \sigma}{8}$$
$$+ \frac{1}{n} \left[ - \frac{\sigma^2}{8\sqrt{1 - \sigma}} \left( \kappa + \frac{1}{2} \right)^2 + \frac{3\sigma}{4\sqrt{1 - \sigma}} \left( 1 - \frac{\sigma}{8} \right) (\kappa + \frac{1}{2}) \right] + \cdots. \quad (3.10)$$

This is a generalization of Eq. (3.2.2) in Ref. 14) and Eq. (4.2) in Ref. 7).
§ 4. Discussion

We shall discuss some points of the results obtained in the present paper and add a few remarks.

1) We have studied only the equal-mass case. Since the eigenvalue problem in the unequal-mass case can be reduced to that in the equal-mass case by the Wick-Cutkosky transformation,\(^1\) the analogous formula (3·10) corresponding to the unequal-mass case is easily recovered.

2) As was stated in §2, the set \(\{n^{-1/2}(1-x^2/n)^{-n/2}g_{\kappa,n}(x,0), \kappa=0,1,\ldots\}\) forms a complete orthonormal system in \(L^2([-1,1];\sqrt{n} \, dz)\), and hence the set \(\{n^{-1/2} \times (1-x^2/n)^{-n/2}g_{\kappa,n}(x/\sqrt{n},0), \kappa=0,1,\ldots\}\) forms a complete orthonormal system in \(L^2([-\sqrt{n},\sqrt{n}];dx)\). If the limit \(n\to\infty\) is taken, each function approaches to \((\pi^{1/2} \times 2^{*12}e^{-x^2/2}H_n(x)^{16})\) and the Hilbert space approaches (formally) to \(L^2((-\infty, \infty); dx)\). These are nothing but the wave functions for a one-dimensional harmonic oscillator. We expect, therefore, that in the limit \(n\to\infty\) the Wick-Cutkosky model may be described, even in the case \(s\neq 0\), by a suitable version of a harmonic oscillator.

3) A glance at (3·10) reveals that the effective parameter of the expansion is \(1/n\sqrt{1-\sigma}\) rather than \(1/n\). It is, therefore, hopeless to recover Cutkosky's result\(^2\) concerning the behavior of \(\lambda_{\kappa,n}(s)\) near the threshold.

4) Equation (3·5) fails at the threshold \(\sigma=1\); the left-hand side is equal to \(p^\dagger\) in the representation \(a^\dagger=(p+ix)/\sqrt{2}\) and \(a=(p-ix)/\sqrt{2}\), while the right-hand side is formally equal to zero. In this case we have to modify the perturbation scheme by taking, say, \(\hat{H}_0+\hat{H}_2/n^2\) as an unperturbed Hamiltonian. This deserves separate investigation.

5) Equation (3·2) is essentially an anharmonic oscillator problem. As Bender and Wu have shown,\(^17\) the perturbation expansion with respect to the anharmonicity parameter never converges. Consequently, the expansion (3·10) also will be an asymptotic expansion. It is hoped that the \(\text{Pade}\) approximation method will be more helpful.\(^18\)

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References

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