Mass function and assembly of dark haloes: an approach to inventory isolated overdense regions in random fields

C. Firmani\textsuperscript{1,2} and V. Avila-Reese\textsuperscript{1}\textsuperscript{*}
\textsuperscript{1}Instituto de Astronomía, Universidad Nacional Autónoma de México, A.P. 70-264, 04510, México, D.F., Mexico
\textsuperscript{2}Osservatorio Astronomico di Brera, via E.Bianchi 46, I-23807 Merate, Italy

Accepted 2013 April 9. Received 2013 April 1; in original form 2012 August 16

ABSTRACT

In order to attain a statistical description of the evolution of cosmic density fluctuations in agreement with results from the numerical simulations, we introduce a probability conditional formalism (CF) based on a complete inventory of isolated overdense regions in a density random field. This formalism is a useful tool for describing at the same time the mass function (MF) of virialized dark haloes, their mass aggregation histories (MAHs) and merging rates (MRs). The CF focuses on virialized regions in a self-consistent way rather than in mass elements, and it offers an economical description for a variety of random fields. Within the framework of the CF, we confirm that, for a Gaussian field, it is not possible to reproduce at the same time the MF, MAH and MR of haloes, for both a constant and moving barrier. Then, we develop an inductive method for constraining the cumulative conditional probability from a given halo MF description, and thus, using the CF, we calculate the halo MAHs and MRs. By applying this method to the MF measured in numerical simulations by Tinker et al., we find that a reasonable solution, justified by a mass conservation argument, is obtained if a rescaling – increment by $\sim30\perc$ – of the virial mass defined in simulations is introduced, and a (slight) deviation from Gaussianity is taken into account. Thus, both the MAH and MR obtained by a Monte Carlo merger tree agree now with the predictions of numerical simulations. We discuss the necessity of rescaling the virial mass in simulations when comparing with analytical approaches on the ground of the matter not accounted as part of the haloes and the halo mass limit due to numerical resolutions in the simulations. Our analysis supports the presence of a diffuse dark matter component that is not taken into account in the measured halo MFs inasmuch as it is not part of the collapsed structures.

Key words: methods: analytical – methods: statistical – galaxies: formation – galaxies: haloes – dark matter – large-scale structure of Universe.

1 INTRODUCTION

According to the contemporary cosmological paradigm, cosmic structures emerge from the gravitational growth of primordial dark matter density fluctuations. A central problem in the last four decades has been the connection between this primordial fluctuation field and the abundance and assembly history of the virialized dark matter haloes; in more detail, their mass function (MF), mass aggregation histories (MAHs) and merging rates (MRs). Since the seminal work by Press & Schechter (1974, hereafter PS), many statistically based analytical formalisms were developed in order to perform such a connection (see for reviews e.g. Zentner 2007; Mo, van den Bosch & White 2010).

In order to achieve an association between the density fluctuation field and the virialized haloes, one needs: a statistical description of the density fluctuation field, an inventory of the overdense regions which will be associated with the virialized objects and a model for the dynamical evolution of the overdensities, including an operational criterion of collapse (virialization).

PS assumed a Gaussian density field and used the spherical collapse model (Gunn & Gott 1972). A region of the density field is assumed to end as a virialized halo when its linearly evolving overdensity exceeds a critical value $\delta_c$. The virialized mass at a fixed scale is assumed to be given by the contributions of the regions of this scale overdense by $\delta_c$ (the PS ansatz) plus the underdense regions of the same scale contained in larger regions overdense by $\delta_c$; the latter statement enunciates the so-called cloud-in-cloud problem. In order to account for all mass, PS assumed that the second contribution equals to the first one, which justifies the fudge factor of 2 introduced in their inferred halo MF.
Based on a more rigorous statistical description, the excursion set (ES), also known as extended PS (EPS) formalism (Peacock & Heavens 1990; Bond et al. 1991), allows us to overcome the cloud-in-cloud problem. The ES formalism applied to a Gaussian field provides a tool to compute, besides the unconditional MF, the conditional MF that can be used for generating halo merger trees (Bond et al. 1991; Lacey & Cole 1993). In this formalism, the result depends critically on the filter used to average the linear density fluctuation field at different mass scales. The mathematical solution is straightforward when a sharp \( k \)-space filter is adopted. For this filter, the PS result including the fudge factor of 2 is recovered. In the calculation of the merger trees, the sharp \( k \)-space filter implies independent steps (Markovian random walks) along the mass trajectory. Note that the ES formalism has a conceptual problem for predicting the halo mass in which a particular mass element ends up (Mo et al. 2010, section 7.2.2), problem that may lead to an inaccurate buildup of MAHs and MRs.

With the advent of large cosmological N-body simulations, the whole non-linear process of dark matter gravitational evolution and collapse into virialized haloes could be followed, though with strong limitations due to mass resolution. Do approaches based on the Gaussian ES formalism allow us to describe correctly the results from the simulations? In particular, do results from these approaches agree at the same time with the MF, MAHs and MRs of haloes as measured in simulations?

A non-negligible discrepancy between the MFs obtained in the ES formalism and the N-body simulations has been reported early. The introduction in the ES formalism of the elliptical gravitational collapse instead of the spherical one (mass-dependent instead of constant \( \delta_c \), respectively) helped us to overcome this problem (Sheth, Mo & Tormen 2001). However, in this case the ES formalism does not provide an analytic formula for the conditional MF, and the merger trees based on the Gaussian ES formalism show deviations with respect to simulations in the progenitor mass distributions (Sheth & Tormen 2002), the mass contained within all the progenitors (Neistein, van den Bosch & Dekel 2006), and the average main progenitor, MAHs and MRs (van den Bosch 2002; Wechsler et al. 2002). On the other hand, direct measures of the conditional MFs in N-body simulations show that they depart from the corresponding functions calculated with the ES formalism for a Gaussian field (Coles et al. 2008; Neistein, Macciò & Dekel 2010). Thus, a possible source of the discrepancies lies in the strict assumption of Gaussianity in the ES formalism. Note that the question is not about primordial non-Gaussianity; the N-body simulations use indeed a Gaussian density field that is evolved analytically (e.g. by means of the Zel’dovich approximation) until the quasi-linear regime is reached. The point is that non-negligible deviations from Gaussianity at the scales of interest have likely happen already during this regime (Coles & Jones 1991). Here, we set aside Gaussianity and the ES formalism and handle the problem of the inventory of overdense regions in arbitrary random fields.

We present an approach aimed to state the problem of the inventory of overdense regions in random (Gaussian or non-Gaussian) fields from a very general point of view. We assume that the density fluctuation field is fully characterized by its field conditional probability function. In order to face the problem of associating the halo mass to the overdensity in an alternative way with the ES formalism (see Mo et al. 2010) we proceed as follows. In our conditional formalism (CF), we make use of the concept of isolated regions introduced by Jedamzik (1995; see also Yano, Nagashima & Gouda 1996; Nagashima 2001) as a natural way to connect overdensities to virialized regions. We enunciate the isolated overdense regions inventory theorem, through which the given field conditional probability function of the random field is linked to the inventory of isolated overdense regions inside larger isolated regions of lower overdensity, the former being those that eventually collapse into virialized haloes if their overdensities are equal to \( \delta_c \).

The CF offers an alternative EPS formulation. It gives an easy way to develop a Monte Carlo algorithm for calculating the MAH and MR of the growing virialized haloes, because it supplies naturally the progenitor conditional probability function\(^1\) of finding a specific isolated region (progenitor) inside a given isolated region (descendent). When the problem of finding isolated overdense regions inside larger ones is extended to the overall Universe, such conditional probability reduces to an unconditional probability, which gives the halo MF. Nevertheless, if one starts from an unconditional probability function, for example on the basis of a halo MF obtained from N-body simulations, and tries to go back to the conditional probability function, in order to build merger trees compatible with the given MF, then multiple possibilities appear. The CF offers useful tools to make a choice. As we will discuss in Section 4.2, we decide to leave the Gaussian hypothesis even if, for simplicity, we retain the (Gaussian) PS ansatz in a generalized version.

The CF might be a powerful tool for economically describing in a consistent way the simulation results related to the MF, MAHs and MRs of the virialized haloes. These descriptions can then be easily implemented in semi-analytical and semi-empirical models of galaxy evolution, and can be extended to masses and epochs, where the resolution is a limit for the simulations. On the other hand, the CF may allow us to explore economically cosmic structure formation in alternative cosmologies or in cases where the density fluctuation field is intrinsically non-Gaussian. Our approach could be applied also in other astrophysical problems, where random density fields are introduced, e.g. the formation and evolution of dense gas structures in the interstellar medium as giant molecular clouds, massive clumps and cores (Hopkins 2012).

This paper is organized as follows. Section 2 is devoted to the CF; in Section 2.1, an equation for the isolated overdense region’s inventory is derived, and in Section 2.2 the algorithms to build up the merger trees are introduced. Section 3 deals with the particular case of Gaussian random fields. The halo MF is derived in Section 3.1, while the MAH and MR are calculated in Section 3.2. A discussion on the moving barrier effects is presented in Section 3.3. In view of the discrepancies obtained when using as input the Gaussian statistics, in Section 4 a heuristic approach, based on the use of the halo MF from simulations as input, is presented. In Section 4.1 we approach the problem of the difference on the mass estimates in the analytic method and in the simulations, and remark the necessity to introduce a mass rescaling, representative of a diffuse matter component, to connect the two standpoints. The results concerning MAHs and MRs are presented in Section 4.2. Our conclusions are given in Section 5. We adopt a flat \( \Lambda \) cold dark matter (LCDM) cosmology with \( \Omega_M = 0.27, \Omega_{\Lambda} = 0.73, h = 0.71, \sigma_8 = 0.9, n = 1 \).

2 THE CONDITIONAL FORMALISM

The ES formalism provides a means to derive the halo MF, accounting for the cloud-in-cloud problem, and to build up the halo MAHs. However, as mentioned above, the Gaussian ES formalism seems to have difficulties in predicting at the same time the MF, MAH and MRs in agreement with numerical simulations. Also,\(^1\) For brevity, hereafter we will omit the specification ‘progenitor’ when referring to the conditional probability.
with the ES formalisms it is not easy to deal with non-Gaussian random fields (Inoue & Nagashima 2002), though some progress has been made recently by introducing a stochastic barrier and non-Markovian corrections (Maggiore & Riotto 2010), by manipulating the step-size distribution of the random walks (Lam & Sheth 2009) or by accounting for correlations between steps (Musso & Paranjape 2012). On the other hand, being the ES formalism focused on a generic mass element, the virialized halo mass is associated with a smoothing scale, which may lack a real physical meaning. In order to overcome these shortcomings, we introduce below the CF inspired by the Jedamzik (1995) approach. This formalism is based on a complete inventory of the isolated overdense regions of a random density field described by a cumulative field conditional probability function.

### 2.1 A complete inventory of isolated overdense regions

Let us start with some key definitions.

**Isolated regions:** an isolated region with overdensity $\delta$ is a connected region not included in a larger connected region with overdensity $\geq \delta$, while at the same time, outside of the totality of isolated regions with overdensity $\delta$ there does not exist any region with overdensity $\geq \delta$. Hereafter, we will refer to an isolated region with overdensity $\delta$ by means of the symbol $I(\delta)$.

**Virialized regions:** a virialized region is an isolated region whose overdensity $\delta$ is equal to a critical value $\delta_c$.

$\phi(M, \delta|M, \delta) d\delta$ is the field conditional probability to find a region of mass $M$ with an overdensity between $\delta$ and $\delta + d\delta$, contained inside a larger region of mass $M$ and overdensity $\delta$.

$N(M, \delta|M, \delta) dM$: is the number of isolated regions, i.e. the conditional MF, with overdensity $\delta$ and mass between $M$ and $M + dM$, contained inside a larger region $S$ with overdensity $\delta$ and mass $M$.

The cumulative field conditional probability to find a region of mass $M$ with overdensity $\geq \delta$ inside a larger region of mass $M$ and overdensity $\delta$ is

$$F(M, \delta|M, \delta) = \int_{\delta}^{\infty} \phi(M, \delta'|M, \delta) d\delta'.$$

Our CF concerns random fields that can be defined by such conditional probability.

In the following, a theorem focused on an isolated overdense region’s inventory and with the aim of establishing a connection between the cumulative field conditional probability, $F(M, \delta'|M, \delta)$, and the conditional MF, $N(M, \delta'|M, \delta)$ with $\delta' \geq \delta$, is presented.

Consider a region $S$ with overdensity $\delta$ and mass $M$. Define an arbitrary mass $M' \leq M$. A decomposition of the mass range $(M', M)$ into $n$ ordered steps, $dM_i$, identifies a sequence of $dN_i = N(M_i', \delta|M, \delta) dM_i$ isolated regions $I(\delta)$ with overdensity $\delta'$ and masses between $M_i$ and $M_i + dM_i$ inside $S$.

For any given $\delta'' \geq \delta'$, the amount of mass $S$ assembled by regions with mass $M_i$ and overdensity between $\delta'$ and $\delta'' + d\delta''$ is

$$M(M_i', \delta'|M, \delta) d\delta''.$$  

This mass is also given by the sum of the regions of mass $M_i$ with overdensity between $\delta'$ and $\delta'' + d\delta''$ located in each one of the isolated regions $I(\delta)$ with masses between $M_i$ and $M_i + dM_i$, and overdensity $\delta'$ of the above defined sequence:

$$\sum_{i=1}^{n} M_i(\delta', \delta'|M_i', \delta) d\delta'' dN_i.$$  

No other contribution exists because outside the totality of the isolated regions $I(\delta)$ in $S$ there are no regions with overdensity $\geq \delta'$. By equating the last two contributions, we can write

$$\phi(M_i', \delta'|M, \delta) d\delta'' = \sum_{i=1}^{n} M_i(\delta', \delta'|M_i', \delta) d\delta''$$

from which we derive the following integral equation:

$$\phi(M', \delta'|M, \delta) d\delta'' = \int_{M'}^{M} dM'' \frac{M''}{M} N(M'', \delta'|M, \delta) \times \phi(M', \delta''|M', \delta) d\delta''.$$  

By integration on $\delta''$ between $\delta'$ and $\infty$, the previous equation writes in terms of the cumulative field conditional probability:

$$F(M', \delta'|M, \delta) = \int_{M'}^{M} dM'' \frac{M''}{M} F(M', \delta'|M'', \delta) N(M'', \delta'|M, \delta).$$  

(1)

An alternative formulation of equation (1) is obtained introducing the conditional probability $p(M', \delta'|M, \delta) dM''$ of finding an isolated region of mass between $M'$ and $M'' + dM''$ and overdensity $\delta'$ inside a region $S$. Given a region $S$ with overdensity $\delta$ and mass $M$, our interest is now to identify inside it an isolated region with overdensity $\delta'$. The quantity

$$\Delta M'' = M'' N(M'', \delta'|M, \delta) dM''$$

gives the amount of mass of $S$ assembled by isolated regions with overdensity $\delta'$ and mass between $M''$ and $M'' + dM''$. Therefore, the probability that an arbitrary mass element of $S$ is part of an isolated region with overdensity $\delta'$ and mass between $M''$ and $M'' + dM''$ is

$$p(M', \delta'|M, \delta) dM'' = \Delta M''$$

that is

$$p(M', \delta'|M, \delta) = \frac{M''}{M} N(M'', \delta'|M, \delta).$$  

(2)

The previous statement is equivalent to say that $p(M', \delta'|M, \delta) dM''$ is the probability to find an isolated region with overdensity $\delta'$ and mass between $M''$ and $M'' + dM''$ inside $S$. Introducing equation (2) into equation (1), we obtain

$$F(M', \delta'|M, \delta) = \int_{M'}^{M} dM'' F(M', \delta'|M'', \delta') p(M'', \delta'|M, \delta)$$  

(3)

which is basically a Volterra integral equation.
The fundamental aspect of equation (1) is that for any given cumulative field conditional probability \( F(M', \delta | M, \delta) \), it gives the conditional MF, \( N(M', \delta | M, \delta) \). Its main difference with the Jedamzik (1995) result is that equation (1) is applied to any region \( S \) with overdensity \( \delta \) and mass \( M \) and not only to the overall universe. For this reason the cumulative field conditional probability \( F(M', \delta | M, \delta) \) appears now on both sides of the equation. Such property highlights the intrinsic meaning of the CF. It is interesting to remark that the previous isolated overdense region’s inventory theorem solves automatically the cloud-in-cloud problem. Equation (1) represents the generalization of the PS formalism. Such formalism may be recovered for the special random fields wherein

\[
F(M', \delta | M, \delta) = k
\]

with \( k \) constant. In such case equation (1) becomes

\[
p(M', \delta | M, \delta) = \frac{1}{k} \frac{dF(M', \delta | M, \delta)}{dM'}.
\]

(5)

The PS ansatz in terms of the conditional probability and including already the fudge factor of 2 is obtained in the particular case of \( k = 1/2 \) (corresponding namely to Gaussian fields; see below).

A random field for which equation (4) is satisfied is very simple to handle in terms of equation (5). Note that Gaussian fields obey not only equation (4) but also a variety of non-Gaussian fields. Therefore, in order to find an analytical formalism in agreement with simulations and to avoid the complex numerical integration of equation (1), it is natural for us to look at first among random fields which satisfy equation (4). This is the strategy followed in Section 4.

2.2 Merger trees build up

A merger tree realization needs to identify progenitors of a given descendant. This is done by means of a Monte Carlo algorithm that uses the conditional probability and MF. The choice of a Monte Carlo algorithm is not unique and may hide subtle physical and technical questions (for a review see Zhang, Fakhouri & Ma 2008).

The simplest criterion establishes that the Monte Carlo algorithm has to reproduce the conditional probability and MF predicted by the theory. However, this criterion is not sufficient, because the overall meaning of the involved probability, i.e. the stochastic nature of the problem, has to be taken into account. We will introduce a Monte Carlo algorithm optimized to build up a merger tree. Afterwards we will check our algorithm with tests that involve the conditional probability and MF, as well as the MAH and MR in order to verify also the agreement with the stochastic nature of the problem.

Only in this section, to ease the reading of the formulae, we will indicate the variables of the extracted regions by a lower case letter.

Integrating \( p(M', \delta | M, \delta) dM' \) between 0 and \( m \) and identifying the obtained cumulative probability with a random number \( \alpha \) uniformly distributed in the interval [0–1], we write

\[
\alpha = \int_0^m p(M', \delta | M, \delta) \, dM',
\]

(6)

which represents the basis for a Monte Carlo approach to build up merger trees.

Given an isolated region \( I_0(\delta_0) \) with overdensity \( \delta_0 \) and mass \( M_0 \), the extraction of the random number \( \alpha \) by equation (6) allows us to identify inside it a first isolated region \( i_1(\delta') \) of mass \( m_1 \) with a given overdensity \( \delta' \). Taking into account that the isolated overdense region’s inventory theorem has been proved for general regions (see footnote2), after the identification of the isolated region \( i_1(\delta') \), the extraction may continue on the complementary region \( S_1(\delta_1) \) with mass \( M_1 = M_0 - m_1 \) and volume \( V_1 = V_0 - v_1 \) obtained from \( I_0(\delta_0) \) removing \( i_1(\delta') \) from its inside. In terms of the density, from the volume relation we write

\[
\frac{M_0}{\rho_0} = \frac{m_1}{\rho'} + \frac{M_1}{\rho_1}.
\]

(7)

Being the region density \( \rho = \rho(\delta + 1) \), where \( \rho \) is the background density and \( \delta \) the overdensity, we have

\[
\frac{M_0}{\delta_0 + 1} = \frac{m_1}{\delta' + 1} + \frac{M_1}{\delta_1 + 1},
\]

and finally

\[
\delta_1 = \frac{M_0 - m_1}{M_0/m_1 - \delta_1 + 1} - 1.
\]

At this point the extraction of the next isolated region can be carried out on the region \( S_1(\delta_1) \) with overdensity \( \delta_1 \) and mass \( M_1 \). To perform such extraction, for simplicity, we will use equation (6) with a new random number \( \alpha_1 \), applied on the region \( S_1(\delta_1) \). By proceeding in this way we implicitly assume that the same statistics of \( I_0(\delta_0) \) holds for \( S_1(\delta_1) \), excluding any sort of correlation. For a discussion of possible correlations, see Sheth & Lemson (1999). According to what has been said above, afterwards we will proceed to test the robustness of such hypothesis.

A recurrent algorithm may be particularly useful now. From the \( n \)th complementary region \( S_n(\delta_n) \) of mass \( M_n \) with overdensity \( \delta_n \) calculated by the previous method, using equation (6) with a random number \( \alpha_{n+1} \) we can identify inside it an isolated region \( i_{n+1}(\delta') \) with mass \( m_{n+1} \) and overdensity \( \delta' \). At this point the next complementary region \( S_{n+1} \) with mass

\[
M_{n+1} = M_n - m_{n+1}
\]

will have the overdensity

\[
\delta_{n+1} = \frac{M_n - m_{n+1}}{M_n/m_1 - \delta_{n+1}} - 1.
\]

(8)

Equations (7) and (8) guarantee a mass conservation and make the procedure particularly handy because they do not make explicit reference to the previous \( i \)-steps.

Equations (6)–(8) are particularly useful for constructing the halo merger trees. Suppose a simple case when the isolated virialized regions are defined by a critical overdensity \( \delta_c \) function of \( z \) only. Given a descendant at \( z \) with overdensity \( \delta_c(z) \), the progenitors at \( z + \Delta z \) have overdensity \( \delta_c(z + \Delta z) \). The finding of such progenitors (branches) is performed by applying the Monte Carlo extraction of equation (6) to the (complementary) region described by equations (7) and (8). This procedure identifies a sequence of progenitors at \( z + \Delta z \) that can be sorted in order of decreasing mass. The largest (primary) progenitor identifies the main progenitor by which the MAH is obtained. The overall population of less massive progenitors fulfills the merger tree and allows us to find the merger rate, MR, along the MAH.

Note that the condition \( \rho = 0 \) implies \( \delta = -1 \). Therefore, for the complementary regions, the progenitor extractions can be performed meanwhile \( \delta_c \geq -1 \). This means that the progenitors are extracted until all matter in the volume is exhausted, which implies that the entire mass contained in the descendant should come from virialized structures. However, the physics of halo mass assembly is more complex. A fraction of the mass can be present as diffuse matter. The strict limit \( \delta \geq -1 \) implies lack of diffuse matter. In this
paper we explore solutions with $\delta \geq -1$; however we will have in mind the physical consequences of such a hypothesis.

The above described method has been extensively used in the case of Gaussian CDM density fields for generating the initial conditions (the MAHs) for the virialization process of dark haloes (Avila-Reese, Firmani & Hernandez 1998), and the subsequent formation and evolution of disc galaxies inside them (Firmani & Avila-Reese 2000, 2009; Avila-Reese & Firmani 2000). The obtained halo properties agree with those found in cosmological numerical simulations (Avila-Reese et al. 1999).

It is interesting to remark that the mass distribution of the first progenitor obtained with the Monte Carlo method can be compared with the conditional probability function, while the mass distribution of all the progenitors can be compared with the conditional MF. This offers a first test of our Monte Carlo algorithm by comparing its one-step results with the straight analytic formulae from the statistics (in Section 3.2 we will show this point in some detail).

It is important to highlight that the mean halo MAH has to be a convergence solution of the Monte Carlo algorithm for the integration redshift step $\Delta z \to 0$, otherwise it is physically meaningless. Because of the stochastic nature of the problem, this property is dependent on the behaviour of the given cumulative field conditional probability. We will focus on well-behaved cumulative field conditional probabilities, such that guaranty $\Delta z \to 0$ convergence in the MAH buildup.

3 GAUSSIAN RANDOM FIELDS

In the case of a Gaussian random field, the cumulative field conditional probability to find a region with mass $M'$ and overdensity $\delta'$ contained inside the region of mass $M$ with overdensity $\delta$ is given (see Bower 1991) by

$$F(M', \delta'|M, \delta) = \frac{1}{\sqrt{\pi}} \int_{\gamma}^{\infty} e^{-y^2} dy,$$

where

$$\gamma = \frac{\sqrt{2(\sigma^2 - \sigma_M^2)}}{\sigma_M},$$

and $\sigma_M$ is the mass variance calculated from the density fluctuation power spectrum of the random field. Making the limit $\delta' \to \delta$, equation (9) gives

$$F(M', \delta|M, \delta) = \frac{1}{2} (M' < M),$$

which is equation (4) with $k = 1/2$. Then equation (5) reduces to

$$p(M', \delta'|M, \delta) = -2 \frac{dF(M', \delta|M, \delta)}{dM'}.$$

This is just the PS ansatz in terms of the conditional probability with the fudge factor of 2 included. Introducing equation (9) into equation (12) we obtain

$$p(M', \delta'|M, \delta) = \frac{2}{\sqrt{\pi}} e^{-\gamma^2} \frac{d\gamma}{dM'}.$$

(13)

It is easy to verify that such conditional probability is normalized, i.e. its integral in $M'$ from 0 to $M (\gamma$ from 0 to $\infty$) is 1.

3.1 Unconditional mass function for haloes

Equation (13) is applied to a region of mass $M$ and overdensity $\delta$. We can extend such region to the entire universe making $M \to \infty$.

$$\sigma_M \to 0$$

and $\delta \to 0$. Hereafter, in such limit case of the entire universe, the superscript $'$ will be omitted. A recipe for the collapse criterion may be obtained from the top-hat spherical gravitational collapse, which establishes that an isolated region at redshift $z$ is virialized when its overdensity is equal to a critical value $\delta_c$ (Navarro, Frenk & White 1997). For a flat cosmology with cosmological constant ($\Omega_M + \Omega_{\Lambda} = 1$):

$$\delta_c = \frac{1.686 \Omega^{-0.055}_M D^+(z)}{D^+(z)}$$

(14)

where $D^+(z)$ is the linear growth factor normalized to 1 at $z = 0$.

By defining the number density per mass unit

$$\frac{dN}{dM} \equiv \rho \frac{dN}{dM} p(M, \delta|\infty, 0),$$

(15)

definition that is not limited to Gaussian fields, and introducing

$$\nu = \frac{\delta_c}{\sigma_M},$$

we obtain the halo MF

$$\frac{M}{\nu} \frac{dN}{dM} = \sqrt{\frac{2}{\pi}} e^{-\nu^2} \frac{d\nu}{dM}$$

(17)

which coincides with the PS MF.

In Fig. 1, the solid red curves show the Gaussian-case (PS) halo MF obtained from equation (17) at $z = 0, 1, 2, 3, 4$, while black solid curves are the accurate fitting formulae to cosmological N-body simulations provided by Tinker et al. (2008, hereafter Tinker08); the parameters for the virial mass at the overdensity $\Delta(z)$ (Bryan & Norman 1998) corresponding to our cosmology were used. The comparison reveals the well-known excess (deficit) of intermediate (high) mass haloes predicted by the PS formalism. The Tinker08 fitting curves are valid for halo masses above $\sim 10^{11} h^{-1} M_\odot$, which corresponds to the minimal mass at which haloes are reasonably resolved in the simulations studied by these authors; this limitation makes uncertain the normalization of the overall MF. Just in order to explore possibilities to approximate the PS MF to the simulation

Figure 1. Halo MFs plotted as ($M^2/\rho$) $dN/dM$ from Tinker08 (solid black lines) compared to the PS MFs obtained with the CF (solid red lines) for $z = 0, 1, 2, 3, 4$. The dotted red lines show the PS MFs with rescaled critical overdensity and lowered amplitude (see text).
MF, δ, has been rescaled by a factor of 0.86 to fit the high-mass cut-off of the N-body (Tinker08) curves (for similar earlier attempts, see e.g. Carlberg & Couchman 1989; Klypin & Rhee 1994), and the normalization has been lowered by a factor of 0.61 to fit the maxima (red dotted curves). Note that the PS MF is normalized independently of δ, and that the dotted curves imply now a not normalized MF. It is evident that in spite of these mass-independent transformations of the PS MF, the comparison with simulation results continues failing. The exercise presented above is congruent with the one carried out in Sheth & Tormen (1999), where they showed that besides these operations, the PS MF should be multiplied by a υ (mass)-dependent factor in order to fit the numerical simulations analysed by them.

3.2 Mass aggregation histories and merger rates

For a Gaussian field and the critical overdensity given by equation (14), if M → 0, then \( \sigma_{M^i} \rightarrow \infty, \gamma \rightarrow 0 \) and \( F(M^i, \delta | M, \delta) \rightarrow 1/2 \). Using equations (12) and (9), equation (6) becomes

\[
\alpha = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2} \, d\xi.
\]

(18)

This operation is equivalent to making a random choice of a number \( \gamma \) with a normal deviate, zero mean and variance 0.5. Hence, using equation (10) and making reference to Section 2.2, we obtain:

\[
\sigma_{\delta}^2 = \left( \delta_0 - \delta_i \right)^2 \frac{2 \gamma^2}{\pi} + \sigma_M^2.
\]

(19)

where the symbols are the same as in Section 2.2, and \( \delta_0 = \delta(z) \) is the (rescaled critical) overdensity of the descendant at redshift \( z \), while \( \delta_i = \delta(z + \Delta z) \) is the (rescaled critical) overdensity of a progenitor at redshift \( z + \Delta z \). The \( i \) sequence defines the complementary regions of mass \( M_i \) with overdensity \( \delta_i \), as well as the progenitors of mass \( M_i^\prime \) with overdensity \( \delta_i^\prime \). Equations (7), (8) and (19), together, define the Monte Carlo algorithm to build the merger tree of a virialized halo with a given mass at a given redshift.

In Fig. 2, we plot the average of 2 × 10^4 different MAHs corresponding to a present-day halo of mass \( M_h = 10^{13} \Msol \) (red solid line; in this case, the critical overdensity is not rescaled). As in Fig. 1, the red dotted line corresponds to the rescaling in the critical overdensity and the amplitude-reduction correction of the MF mentioned above. The dashed line represents the fit to the corresponding average MAH measured in the Millennium Simulations (Fakhouri et al. 2010).

For present-day haloes of \( M_h = 10^{13} \Msol \), Fig. 3 shows the mean MR histories per unit of redshift with merger ratios greater than \( \xi > 0.3 \) and \( >0.03 \), blue lines for \( \xi > 0.1, 0.01 \) and 0.001 (blue lines). The coding of solid and dotted lines is the same as in Fig. 2. The average halo MRs as a function of the \( \xi \) threshold at \( z = 0 \) compares well with the results reported in fig. 3(a) of Fakhouri et al. (2010). The small change of these mean MRs with \( z \) is also in general agreement with the numerical simulation results reported in Fakhouri & Ma (2008) and Fakhouri et al. (2010).

The main conclusion from the exercise presented here is that for a Gaussian density field (PS), the predicted average halo MAH and MRs agree reasonably well with the results from numerical simulations. However, the predicted halo MF, as it is well known, has an excess at intermediate masses and a deficit at high masses as compared to simulations at the mass range they are able to resolve (e.g. Tinker08).

3 For the amplitude-reduced case with rescaled critical overdensity, the MAH and MRs do not change significantly (Figs 2 and 3). In what follows, unless otherwise stated, we will use the rescaled critical overdensity case.

Figure 2. Average MAH (solid red line) for a \( M_h = 10^{13} \Msol \) present-day halo for the PS case, and the same for the amplitude-reduced case with rescaled critical overdensity (dotted red line, see text). The dashed black line shows the corresponding mean MAH obtained from the Millennium Simulations (Fakhouri, Ma & Boylan-Kolchin 2010).

Figure 3. Average MRs for the PS case (from bottom to up, green lines for \( \xi > 0.3 \) and \( >0.03 \), blue lines for \( \xi > 0.1, 0.01 \) and 0.001) along the average MAHs of Fig. 2 (solid lines for the main case and dotted lines for the amplitude-reduced case with rescaled critical overdensity, see text). Taking into account the mass variation, this result compares well with the Millennium Simulations at \( z \approx 0 \) (the thick in the vertical axis indicates the corresponding \( z = 0 \) MRs reported in Fakhouri et al. 2010). The evolutionary behaviour also agrees with these simulations.
Particularly interesting are the one-step results shown in Fig. 4 for a descendant mass of $10^{13} M_{\odot}$ at $z = 0$ and redshift intervals of $\Delta z = 0.05$ (upper panel) and $\Delta z = 0.005$ (lower panel), respectively. The red curve (left abscissa) shows the analytic conditional probability $p(M', \delta' | M, \delta)$ per unitary progenitor mass natural logarithm, while the green curve (right abscissa) shows the conditional MF $N(M', \delta' | M, \delta)$ per unitary progenitor mass natural logarithm as a function of the progenitor mass. Here $\delta$ corresponds to $z = 0$ while $\delta'$ to $z = \Delta z$. The dots show an average distribution obtained with our Monte Carlo algorithm for the first progenitor (bottom region) and for the entire progenitor collection (top region). The total number of extractions, not reported here because of being unnecessary, is regulated in each case to reduce the scatter at reasonable levels. The marginal differences appearing in the graphics can arise from numerical effects, as well as from some departure from the hypothesis made in Section 2.2 about the complementary regions. Fig. 5 shows a similar test carried out for the masses $10^{14} M_{\odot}$ and $10^{15} M_{\odot}$ at $z = 0$ (upper panel) and for $10^{15} M_{\odot}$ but at $z = 10$ (lower panel), being in both cases $\Delta z = 0.05$. We conclude that the agreement between the analytic conditional probability and MF and the Monte Carlo results as well as the comparison of the MAHs and MRs with simulations (see above) are fully satisfactory, and it indicates that our merger tree algorithm works properly. For our main calculations we will assume a redshift step close to 0.05, while for step invariance tests we will assume 0.005.

### 3.3 The effect of mass dependence in $\delta_c$

So far, we have used the spherical collapse critical overdensity (equation 14), which is the function of $z$ but not of mass; this corresponds to a fixed barrier in the ES formalism. The adoption of a mass-dependent critical overdensity, i.e. a moving barrier, compatible with an ellipsoidal collapse, improves the agreement between the analytic and the numerical results regarding the halo (unconditional) MF (Sheth et al. 2001; Sheth & Tormen 2002). We extend now the test to the halo MAH and MR. We assume the same Gaussian random field as in Section 3 described by equations (9)–(13) and equations (15)–(17), and adopt the moving barrier

$$\delta_c = a \delta_c (1 + f(\chi)),$$

where $\chi = \sigma_M / \delta_c$, $f$ is a function that in Sheth et al. (2001) is $f = b \chi^c$, and $a$, $b$, and $c$ are free parameters. The CF can be used now to explore the effects of a mass-dependent critical overdensity on the MF, MAH and MR, under the ansatz that the cumulative field conditional probability is the same as given in equation (9). Using equation (17) with the parameter values given in Sheth et al. (2001), $a = 0.87$, $b = 0.47$ and $c = 1.23$, we recover almost the same halo MF proposed by Sheth & Tormen (1999) as a fit to numerical simulations, and justified in Sheth et al. (2001) as an indication of elliptical collapse (Fig. 6; note that both the obtained MF and the Sheth & Tormen one are close to the MF given in Tinker08). The excellent agreement shows that our ansatz is quite reasonable. Within the framework of the ES approach, recent work on random walks with correlated steps suggests that substituting $\delta_c$ into equation (10) could indeed be a good approximation (cf. Musso & Sheth 2012; Paranjape, Lam & Sheth 2012).

To find the MAH and MR, we use equations (9)–(12) with $\delta_c$ given by equation (20). For $M' \rightarrow 0$ ($\sigma_M \rightarrow \infty$), $\gamma \rightarrow 0$ only if $c < 1$, while $\gamma \rightarrow ab \sqrt{c}$ when $c = 1$ and $\gamma \rightarrow \infty$ for $c > 1$. In
these three cases, \( \phi \equiv F(0, \delta_c, M, \delta_c) \) is 1/2, a value between 0 and 1/2, and 0, respectively. Using equation (12), equation (6) now becomes

\[
\alpha - 2\varphi + 1 = \frac{2}{\sqrt{\pi}} \int_0^{\gamma} e^{-\xi^2} d\xi. \tag{21}
\]

This equation does not have a solution when \( c > 1 \), while it does have a solution when \( c < 1 \) for each value of 0 < \( \alpha < 1 \), and for \( c = 1 \). only if \( 2\varphi = 1 \alpha < 2\varphi \). Within the framework of our approach, we conclude that a mass-dependent critical overdensity, as the one inferred from the moving barrier analysis suggested in Sheth et al. (2001), does not offer a consistent description of the halo MAH and MR, though the limiting ‘square-root’ moving barrier (\( c = 1 \)) suggested by Moreno, Giocoli & Sheth (2008) in principle could offer a solution.

The results obtained by the Monte Carlo method using the square-root moving barrier show that both the average halo MAH and MR depend significantly from those found in the simulations. The MAH is now too extended towards the past, revealing a low contribution of major mergers in the halo assembly. The major MRs (\( \xi > 0.1 \)) are indeed rare with respect to the simulation results. We have explored the obtained effects varying the \( a, b, c \) parameters without any interesting result.

Nonetheless, the most serious difficulty in our formalism when using a mass-dependent critical overdensity is that the convergence of the Monte Carlo algorithm, when redshift step \( \Delta z \to 0 \), does not accomplish anymore. It is not surprising that some rough agreement with the results from simulations is obtained only when \( \Delta z \gtrsim 0.5 \), owing to the ‘correct’ behaviour of \( \gamma \) as a function of the progenitor mass in this case (see below). Indeed, by using the large \( \Delta z \gtrsim 0.5 \) time-step, Moreno et al. (2008) have shown that the moving barrier predictions agree with the simulation conditional probability.

Simple considerations help us to understand the failure of the mass-dependent critical overdensity in our scheme. Fig. 7 allows us to understand the reduction of the major merger rate. For a fixed barrier (\( \delta_c \) independent of mass; dashed lines), \( \gamma \) increases monotonically with mass and equation (13) leads to a conditional probability that makes possible the existence of low-mass progenitors. For the ‘square-root’ moving barrier (Moreno et al. 2008; solid lines), \( \gamma \) shows a decreasing shape on a very extended low-mass range of progenitors; here the conditional probability equation (13) is physically meaningless and no low-mass progenitors are possible. Only in a small range immediately below the descendant mass, \( \gamma \) increases with mass, allowing at this range for a physical conditional probability, such that the progenitors are more massive than in the fixed barrier case. When \( \Delta z > 0.5 \) the ‘correct’ monotonic increasing of \( \gamma \) with the progenitor mass is recovered. Furthermore, a necessary condition to obtain a redshift-step convergence is that \( \gamma \to 0 \) when \( \Delta z \to 0 \) with \( M' < M \). This tendency is not fulfilled for the moving barrier case.

We conclude that, when using a mass-dependent critical overdensity (moving barrier) in our approach, the halo MF is correctly described but the predicted halo MAH and MR fail in reproducing the simulation results. An exhaustive analysis of this shortcoming and the reason behind it is deferred to future work.

### 4 A SEMI-EMPIRICAL STRATEGY

In view of the difficulty of proposing a random fluctuation field described by an analytic cumulative field conditional probability function and being able to generate at the same time the halo MF, MAH and MR obtained in the numerical simulations, we turn on to an inductive (semi-empirical) approach: the probability function will be inferred numerically from the Tinker08 halo MF, which was obtained from simulations. We split the problem into two parts. First, in Section 4.1 we extend the unconditional probability corresponding to the Tinker08 MF (defined only above \( \sim 10^{11} \, M_\odot \)) down to small masses by an algebraic extrapolation, taking care that such an extrapolation does not imply a sharp change in the functionality; furthermore, we introduce a mass rescaling in order to take into account any halo mass defect due to some diffuse matter (see Section 2.2). Secondly, in Section 4.2, based on the properties of equation (1), we extend the definition of the unconditional probability to the conditional probability, and using it within the context of the CF, we calculate the halo MAH and MRs.

#### 4.1 Halo mass rescaling in the numerical simulations

Tinker08 provided an analytical fit to the halo MFs measured in numerical simulations at different redshifts and above a
minimum (resolution) mass. We can calculate from these MFs the cumulative unconditional probability functions expressed in terms of the commonly used scaled variable \( \nu \) or, what is the same, \( \gamma_u = u/\sqrt{\bar{\sigma}} = \delta(z)/\sqrt{2\bar{\sigma}_M} \), where \( \gamma_u \) is the same as in equation (10) but applied to the entire universe, i.e. for \( M \to \infty (\sigma_M \to 0) \) and \( \delta \to 0 \). Due to the lower limit in mass in the Tinker08 MFs, the cumulative unconditional probability should be constructed by integrating the MF from each mass \( M \) at a given redshift, corresponding to the argument \( \gamma_u \), to infinity. We define the following function:

\[
P(\gamma_u; z) \equiv 1 - \int_{0}^{\infty} \frac{M}{\bar{\rho}} \frac{dN}{dM} d\bar{M} = \int_{0}^{M(\nu_u; z)} p(M, \delta_\epsilon(z)|\infty, 0) dM, \tag{22}
\]

where \( dN/d\bar{M} \), given by equation (15), is the halo MF, and the mass \( M(\nu_u; z) \) is an intrinsic function of \( \gamma_u \) for a given \( z \). As long as the integral in equation (22) is correctly normalized, \( P(\gamma_u; z) \) represents the cumulative probability to find an isolated region with overdensity \( \delta_\epsilon(z) \) (collapsed region) and mass \( \leq M \) (or \( \gamma_u \) less than the value corresponding to \( M \) for the given \( z \)). When we apply equation (12) to the Gaussian (PS) case, we obtain \( P_{PS}(\gamma_u; z) = 1 - 2F(\delta(z)_{\epsilon}, \infty, 0) \). Since in such a case \( F \) is a function of \( \gamma_u \) only, the dependence of \( P_{PS}(\gamma_u; z) \) on \( z \) vanishes (\( P_{PS} \) depends on \( z \) but through \( \nu_u \)); this is the well known fact that for a Gaussian field, the halo multiplicity function is universal when expressed in terms of the scaled variable \( \nu \) (or \( \gamma_u \)).

By using the Tinker08 halo MF in the first equality of equation (22), the cumulative probability function \( P(\gamma_u; z) \) can be obtained numerically. The result is shown in the top left panel of Fig. 8 down to the limit mass at each epoch (solid black lines for \( z = 0, 1, 2, 3, 4 \)). The corresponding result for the Gaussian (PS) case, with \( \delta_\epsilon \) rescaled by a factor 0.86 in order to fit the high-mass end of the Tinker08 MF, is also plotted (red dot–dashed curves). It is interesting to note that for the Tinker08 MF, the dependence of \( P(\gamma_u; z) \) on \( z \) is negligible, which suggests that \( P(\gamma_u) \) as inferred from the simulations deviates only slightly from a universal function. In what follows, we will consider that the function \( P(\gamma_u) \) is the same at all redshifts, omitting the explicit argument \( z \).

In order to make \( P(\gamma_u) \) physically meaningful as a cumulative probability, we need to know its entire profile, i.e. down to \( \gamma_u = 0 \) (\( M = 0 \)). Then, we proceed first by smoothly extrapolating the Tinker08 \( P(\gamma_u) \) applying a cubic polynomial calculated with the conditions to be 0 when \( \gamma_u = 0 \), and fitting the low limit of the numerical curve up to the second derivative. The result is shown by the dashed black curve in the left upper panel of Fig. 8. The plot highlights the difference between the Tinker08 and the PS cases in the \( P(\gamma_u) \) shape when \( \gamma_u \leq 0.5 \). While the latter reaches the origin following a straight line, the first one shows a curved shape. The physical reason of such a difference can be due to the lack of normalization in the Tinker08 MF; such a lack of normalization can arise from the presence of diffuse matter in simulations unaccounted in the MF (see below).

In the following, we discuss critical issues that appear when a confrontation between analytical approaches (e.g. ES and CF) and the N-body simulations is carried out. Then, taking into account these issues, we propose an economical strategy that makes compatible for the CF the cumulative probability \( P(\gamma_u) \) inferred from simulations and extrapolated to low masses. In the analytic formalisms, the linear overdense region of mass \( M \) is linked to a collapsed halo of the same mass (mass conservation; see discussion in Section 2.2), and the progenitor distributions at each redshift take into account all the available mass as part of collapsed haloes. The question is whether the simulations account for the mass in the same way.

First, in the simulations the range of halo masses is limited; there is no information below the halo mass resolution limit. If one extrapolates the fitted halo MF to lower masses and integrate the obtained multiplicity function from 0 to infinity, then one finds a deficit. This implies that the fitted MF should change its slope (be steeper) at smaller masses, and/or that some fraction of the mass is actually not in the virialized haloes. The latter brings to consideration a second issue: in the simulations a non-negligible fraction of mass is indeed diffuse (not in haloes).

The presence of diffuse matter is due to several reasons.

(i) A halo is counted only if it contains tens of particles; groups with less particles are part of what is considered as diffuse matter.
(ii) Several authors have shown that a significant fraction of the gravitationally bounded particles is actually further away from the spherical virial radius (e.g. Prada et al. 2006; Cuesta et al. 2008; Anderhalden & Diemand 2011; Lacerna & Padilla 2011). It is not easy to determine the halo radius that contains all the bounded...
particles, and likely this radius presents a large variation from case to case, depending on environment, epoch, previous assembly history, mass, etc. Let us call $\psi$ the average mass fraction of matter gravitationally bounded to haloes but not accounted in the conventional spherical virial mass.

(iii) Due to true dynamical processes, e.g. when haloes collide, some fraction of the particles are ejected from the merged system (Wang et al. 2011); let $\varphi$ be the overall fraction of such ejected mass.

The diffuse mass produced by all of these effects may eventually fall on to the growing haloes. Therefore, at difference of the analytical formalisms, the mass growth of haloes in simulations happens also in the form of diffuse accretion, which is expected to reduce the minor merger rates as compared to the analytical formalisms. In the current simulations, at least 30 percent of the $z = 0$ halo masses came in diffuse accretion (cf. Genel et al. 2011; Wang et al. 2011).

(iv) It was suggested also a background of diffuse matter, remnant of the cut-off in the mass power spectrum due to the relativistic free-streaming damping (e.g. $\sim 10^{-6} \, M_{\odot}$ for the neutralino); at very high redshifts, most of this matter due to the cut-off seems to be diffuse (Angulo & White 2010), but at the redshifts of significant halo mass assembly, the effects mentioned above dominate over this.

The diffuse matter in simulations implies that the mass originally linked to a given overdensity is less than that defined inside the spherical virial radius. Due to these effects, the conventional virial masses deplete then depleted on average by the fraction $\varphi + \psi$.

Now, considering the issues discussed above and that in the CF the diffuse matter is not taken into account, we need to make as compatible as possible the cumulative probability $P(\gamma_{\ast})$ inferred from simulations with the one (correctly normalized from 0 to infinity) to be used in the CF. In the case of point (ii), the solution is simple: a mass rescaling allows us to recover the mass inside the collapsed halo proper of the analytical formalism. With this correction the analytic approach is justified. The ejected matter (point iii) may be considered a simple extension of this approximation. After several experiments, we find that an economical way is using a strategy based on the following ideas: (a) a mass rescaling allows us to pass from a model (Tinker08) $P(\gamma_{\ast})$ to another $P(\gamma_{\ast})$, where the complications of the ejected matter and incorrect halo mass definition in simulations are avoided in such a way that the masses are now roughly compatible with the ones of the analytic case and $P(\gamma_{\ast})$ can be handled with the CF; (b) a good empirical criterion is to choose $P(\gamma_{\ast})$ with a reduced curvature when extrapolated to low masses in agreement with the shape of $P_{PS}(\gamma_{\ast})$. Finally, the choice of the mass rescaling factor is justified by the result, i.e. the MAHs and MRs being in agreement with simulations.

Following this strategy, we explore now the effects of a simple constant mass rescaling of the Tinker08 halo MF on $P(\gamma_{\ast})$. The panels of Fig. 8 show the cases of virial mass rescaling by factors $f = 1.1$, $1.2$, $1.3$, $1.4$ and $1.5$, respectively. It is noteworthy that for $f = 1.3$–$1.4$, the extrapolation of $P(\gamma_{\ast})$ to low $\gamma_{\ast}$ values is close to a straight line and, in general, this probability function approaches to $P_{PS}(\gamma_{\ast})$. Moreover, from the papers mentioned above, it can be said that an average halo mass correction by $\varphi + \psi \sim 30$–$40$ percent (assumed to be mass independent) is within the uncertainties. We conclude that this exercise, changing the shape of $P(\gamma_{\ast})$ obtained from simulations so that it is closer to the shape of the analytic (PS) Gaussian probability, implies a mass rescaling of $f \approx 1.3$, in rough agreement with measures of gravitationally bounded halo mass and diffuse matter considerations in simulations. We do not attempt to follow a more rigorous exercise, for instance introducing a mass dependence in $f$, because of the large uncertainties involved in the simulation analysis, and because at this point we require only an indicative mass correction that encompasses all the complexity implied in passing from the linear overdensity to a virialized structure.

4.2 Mass aggregation histories and merger rates according to the mass function from simulations

The function $P(\gamma_{\ast})$ given by equation (22) and obtained from the Tinker08 halo MF is the cumulative unconditional probability to find an isolated region with overdensity $\delta_{\ast}$ and mass $\leq M$ inside the entire universe, being in this case the cosmic $\delta_{\ast} = \delta_{c}/\sqrt{2\sigma_{M}}$. Now, in order to apply the CF, we need to pass from such unconditional probability to a conditional probability. Taking into account the generalized PS ansatz given by equation (5) and according to the definition of $\gamma$ in equation (10), the simplest random field compatible with the previous reasoning is

$$F(M', \delta' | M, \delta) \equiv \frac{1 - P(\gamma)}{2}. \quad (23)$$

In this way, even if the statistics is not any more Gaussian, equation (5) by construction is satisfied and the involved mathematics is simple. From Fig. 8 one can see that our inferred $P(\gamma)$ from the Tinker08 MFs is different from the one corresponding to a Gaussian field (see for a similar conclusion Cole et al. 2008; Neistein et al. 2010).

The probability $p(M', \delta' | M, \delta) \, dM'$ to find an isolated region with overdensity $\delta'$ and mass between $M'$ and $M' + dM'$ inside an isolated region with overdensity $\delta$ and mass $M$, according to equations (12) and (23), is

$$p(M', \delta' | M, \delta) \, dM' = \frac{dP(\gamma)}{d\gamma} \, dM'. \quad (24)$$

We can apply now the Monte Carlo method to find the halo MAH and MRs according to the procedure defined in Section 2.2.

For $2 \times 10^{9}$ Monte Carlo realizations, we calculate the average MAHs and MRs by using the cumulative probability $P(\gamma)$ inferred from the Tinker08 MF (without any mass rescaling, i.e. $f = 1$). The left-hand panels of Fig. 9 show the average MAH (upper panel, solid line) and MRs (lower panel, same line) according to the Millennium Simulations (Fakhouri et al. 2010) is shown with the dashed line. The predicted MRs are higher at all epochs, but more at lower redshifts, than those measured in the simulations by Fakhouri & Ma (2008) and Fakhouri et al. (2010; thick ticks in the $dM/dz$ axis correspond to the $z = 0$ MRs as reported by these authors). This produces also too fast a mass growth of haloes as seen in the upper panel. On the ground of these results, we conclude that the cumulative probability inferred from the original Tinker08 halo MF leads to halo MAHs and MRs in disagreement with those measured in numerical simulations.

In the right-hand panels of Fig. 9 we present the same as in the left-hand panels but now for $P(\gamma)$ inferred from the Tinker08 halo MF with the virial mass rescaled by the factors $f = 1.2$, $1.3$, and $1.4$ (see Fig. 8). The solid curves are for $f = 1.3$ and the left(right upper(lower) curves are for $f = 1.2(f = 1.4$ in the top (bottom) panel). The predicted average halo MAH and MRs are now close to those measured in the simulations (certainly within the scatter and uncertainties) for the case $f = 1.3$. It is remarkable that the value $f = 1.3$ is close to that one which provides a $P(\gamma_{\ast})$ with a shape similar to the shape of $P_{PS}(\gamma_{\ast})$ according to
the discussion in Section 4.1. These two findings are completely independent between them. Besides, the value $f \approx 1.3$ seems to be consistent with measures of the diffuse matter component in the simulations (see for references Section 4.1). Note that the MAH and MRs measured from the simulations could change if the virial mass is rescaled; however, we do not expect significant changes because in the case of the MAH and MR the physics has to do with mass ratios.

For completeness, we have applied the same procedure to the fitting MF given by Sheth & Tormen (1999). We have obtained practically the same results as for the Tinker08 MF, which for economy we do not repeat here.

Summarizing, through a mass rescaling that makes compatible the masses between simulations and the analytic framework, we have derived from the halo Tinker08 MF a conditional field probability function given by equation (23) particularly handy in the sense that it is compatible with the PS ansatz (equation 5), in spite of the fact that it deviates from the one of a Gaussian density fluctuation field. Starting from such probability, our CF allows us to calculate the halo MAHs and MRs. The obtained average MAHs and MRs are consistent with numerical simulations when the virial mass in the input Tinker08 halo MF is rescaled.

For economy, we plotted results regarding the average MAH and MRs only for one descendant mass, $10^{13} \, M_\odot$. The conclusions are similar for other masses. Fig. 10 shows the value of $f$ required to agree with the Fakhouri et al. (2010) average MAHs at the redshift one-half (solid line) and one-tenth (dashed line) of the mass for $z = 0$ descendants of $10^{11}$, $10^{12}$, $10^{13}$, $10^{14}$ and $10^{15} \, M_\odot$. As is seen, the mass rescaling factor is roughly the same for masses below $\sim 10^{14} \, M_\odot$, something in between 1.30 and 1.34, in order to agree with the MAHs down to one-tenth of the $z = 0$ descendant mass. For larger masses, this factor should be slightly larger. We conclude that, taking into account a mass rescaling of 1.3, the CF allows for a reasonable good description of the halo MFs, and the average MAHs and MRs of haloes $\lesssim 10^{14} \, M_\odot$ obtained in numerical simulations. Probably, a better tuning of this description can be attained by varying $f$ with mass; however, this task becomes difficult due to subtle aspects, for example, the way the averaging is carried out when calculating the average MAHs and MRs. We keep such an exploration for future works.
In the analytical formalisms there is no mass dependent on mass (moving barrier; \( f \approx 1.3 \)). Remarkably, this rescaling makes the halo mass compatible with the mass used in the analytical formalisms having in mind that in simulations not all the mass is in haloes. On the ground of the CF, the conditional probability function is obtained from the unconditional one by making the former compatible with the PS ansatz, in spite of that the shape of the latter deviates from the one corresponding to a Gaussian statistics. Such a situation allows us to easily use the CF and our Monte Carlo algorithm for calculating the halo MAHs and MRs. The obtained average MAHs and MRs depend critically on the mass rescaling factor. If this factor, assumed constant, is \( f \approx 1.3 \), then the agreement with the average MAH and MRs in simulations becomes remarkable, at least for descendant haloes up to \( \sim 10^{14} \) \( M_\odot \). It is encouraging that the rescaling of the virial mass in the Tinker08 halo MF by a factor \( f \approx 1.3 \) allows both for a well behaved overall cumulative field unconditional probability function and for halo MAHs and MRs in agreement with simulations. It is important to remark that the value of \( f \) depends on the resolution limit of numerical simulations.

(iv) Dark matter in simulations exhibits a rather complex structure. Not all the mass is counted in the haloes defined up to the conventional virial radius, but a diffuse component is present because of at least three reasons: (a) the mass resolution limit on halo identification, (b) the not counting of gravitationally bounded particles that are further away from the virial radius, and (c) the mass ejection generated by mergers. The contribution (a) has to do with the low mass extension of the MF. Due to (b) and (c) the mass accounted in the virialized halo is smaller than in the original overdense region, producing this normalization defect in the MF. The disagreement between the simulation and analytic MFs seems to lie on these last two effects as well as on a departure from the Gaussian distribution of the density fluctuations. The conditional probability function inferred from simulations with our approach and its use in the CF allow for an estimate of about 30 per cent for the diffuse mass fraction related to effects (b) and (c), and show that the conditional probability corresponds to a density fluctuation field (slightly) deviated from Gaussianity. In agreement with the previous argument, the Tinker08 MF does not include all the dark matter present in the cosmological boxes and consequently it is not normalizable. Tinker08 indeed acknowledge that this is the case.

5 CONCLUSIONS AND DISCUSSION

We develop an approach able to connect the overdensities of the density fluctuation field with the abundance and assembly history of the virialized haloes. Starting from a very general basis, our CF concerns the inventory of the isolated overdense regions given the field conditional probability function of the density fluctuation field. This formalism allows us to calculate at the same time and in a simple way: the halo MF at any \( z \), the halo MAHs, and the halo MR histories. After that, our main goal is to identify a strategy that allows us to use the CF for describing optimally the numerical simulation results regarding these three halo statistical and evolutionary features. In the way of attaining this goal, we arrived at some important conclusions.

(i) For the Gaussian density field (the generalized PS ansatz, equation (5), applies), the predicted average halo MAH and MRs agree roughly with those measured in current cosmological (ΛCDM) simulations, but the halo MF, as previously found, is overabundant at intermediate masses and deficient at high masses as compared to simulations, e.g. with the Tinker08 MF.

(ii) The introduction of \( \delta_c \) depending on mass (moving barrier; Sheth et al. 2001) instead of the constant one improves the halo MF as compared to simulations, but dramatically spoils the average MAH and MRs. Nonetheless, the major problem with the mass-dependent \( \delta_c \) is the excessive sensitivity of the MAH and MRs to the redshift step used in the Monte Carlo merger tree construction.

(iii) A cumulative field unconditional probability function can be inferred from the halo MF measured in simulations (Tinker08), this being almost the same (universal) at different redshifts when expressed in terms of \( \delta_c / \sigma \); its definition has to be complemented by adequately extrapolating it to low masses. This inference is particularly suitable regarding the shape when the halo virial mass is rescaled by \( f \approx 1.3 \). Remarkably, this rescaling makes the halo mass compatible with the mass used in the analytical formalisms having in mind that in simulations not all the mass is in haloes. On the ground of the CF, the conditional probability function is obtained from the unconditional one by making the former compatible with the PS ansatz, in spite of that the shape of the latter deviates from the one corresponding to a Gaussian statistics. Such a situation allows us to easily use the CF and our Monte Carlo algorithm for calculating the halo MAHs and MRs. The obtained average MAHs and MRs depend critically on the mass rescaling factor. If this factor, assumed constant, is \( f \approx 1.3 \), then the agreement with the average MAH and MRs in simulations becomes remarkable, at least for descendant haloes up to \( \sim 10^{14} \) \( M_\odot \). It is encouraging that the rescaling of the virial mass in the Tinker08 halo MF by a factor \( f \approx 1.3 \) allows both for a well behaved overall cumulative field unconditional probability function and for halo MAHs and MRs in agreement with simulations. It is important to remark that the value of \( f \) depends on the resolution limit of numerical simulations.

A consistent description of the cosmological simulation results regarding the halo MF, MAHs and MRs through an analytical (statistical) formalism has been achieved. The interesting finding is that such a description could not be attained for a Gaussian density fluctuation field, as well as without taking into account the complex distribution of dark matter in simulations. In this sense, our results urge for a revision of two questions.

(i) The correct accounting in simulations of the mass in virialized haloes and of diffuse matter. In the analytical formalisms there is an exact equality between the mass of a linear overdensity and that of the collapsed halo used for the counting in the MF, while in the numerical simulations this is not the case because of effects (b) and (c) mentioned above. Therefore, when confronting the halo MF calculated in the analytical formalisms with that one measured in simulations, these issues should be taken into account. Our approach is a first approximation to this complex problem. Further exploration is necessary.
The correct introduction of the statistics in analytical formalisms. The hierarchical structure formation implies that in the same density field coexist collapsed structures at small scales with regions in the linear regime at large scales. Then, even if the primordial statistics is Gaussian, at the time the analytical approach is applied the density field at the scales of interest could already have deviated from Gaussian initial conditions (e.g. Coles & Jones 1991), as numerical simulations suggest.

In any case, the CF and the heuristic approach presented here allow us to overcome some of the difficulties related to these questions, and attain a consistent description of the halo MF, MAHs and MRs measured in cosmological numerical simulations.

ACKNOWLEDGEMENTS

We thank the anonymous referee for his/her comments and suggestions which helped us in improving this paper. CF is grateful to Dr. Flavio Firmani from the University of Victoria (BC, Ca) for the encouraging support to carry out this work. VA acknowledges CONACyT grant 167332-F for partial funding.

REFERENCES


This paper has been typeset from a LaTeX file prepared by the author.