Space-Time Approach to the Lattice Dual Resonance Model

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We construct the Veneziano amplitude (in the sense of Koba-Nielsen) from the classical-lattice model with the interaction part $2J_k \delta(t-t_k)$, by the use of the action integral method. We also calculate the one-loop planar amplitude in this model.

§ 1. Introduction

We are able to interpret the Veneziano amplitude as the motion of an infinite number of harmonic oscillators or springs between the pairs of quarks and antiquarks. Nambu and Fubini, Gordon and Veneziano have succeeded in making the factziration theorem for the Veneziano amplitude by using the above concepts. Susskind has explained the origin of the Veneziano amplitude in terms of B–S like equation for the harmonic oscillators between the quark-antiquark pairs.

Further last year (1970), Fairlie and Nielsen proposed an analogue model by using the functional integral method, and obtained the integrand of Koba-Nielsen amplitude. In order to elucidate the connection of Susskind’s or Nielsen’s model with the Veneziano amplitude, Hsue, Sakita and Virasoro employed the functional integral method and derived the dual resonance amplitudes (tree, one-loop planar, and one-loop non-planar). However they have not explained the origin of the interaction part which reproduces the Veneziano amplitude.

Dirac had proposed that the action integral method is valid in quantum mechanics in order to connect the classical description to the quantum description. This method was developed by Feynman, Schwinger and many authors.

The action is defined by

$$c \exp i\mathcal{S} = c \exp i \int_0^T \mathcal{L}(y_j(t), y_j(t), t) dt,$$

where $y_j(t)$’s are the classical solutions of the harmonic oscillator problems. We propose to determine the normalization constant $c$ by the division rule.

Independently in 1969, Miyamoto proposed the validity of the proper-time formulation in the Dual Resonance Model. This formulation has been shown to be equivalent to action integral method by Feynman and Schwinger.

*) We notice that the proper-time formulation is intimately connected with the action integral method of Schwinger and the path integral method of Feynman.
In order to reduce the exponential divergence of a closed loop, we consider the classical-lattice model with the interaction part $\sum k \delta(t - t_i)$ by using the action integral method, which is equivalent to and more transparent than Feynman's path integral method. This model is a natural extension of proper-time formulation of Ref. 11).

In § 2. Preliminary work for the lattice dual resonance model.
In § 3. Action integral method for our model.
In § 4. The Veneziano amplitude in our model.
In § 5. The one-loop planar amplitude.
In Appendix I. we give the solution of the harmonic oscillator problem.
In Appendix II. the relation between the action integral and the S-matrix theory is given.

All authors discuss the integrand of the path integral method and not the integral with respect to proper-time. In Appendix II we derive all the formula consistently from the action integral method.

§ 2. Preliminary work for the lattice dual resonance model

The main purpose of the lattice dual resonance model is to reduce the divergence of closed loop. In this model, however, we can also derive the Veneziano amplitude in the sense of Koba-Nielsen. This model is illustrated in Fig. 1. As is well-known, the Lagrangian for our lattice model has the following form (For simplicity, we consider only one-dimensional problem. Later on we generalize it to the four-dimensional lattice dual resonance model):

![Fig. 1. The lattice model with the interaction term $\sum_{i=1}^{N} k \delta(t - t_i)$, where $q_i$'s are the coordinates in the lattice spaces and $q_i = q_{i+1}$.](image)

$$\mathcal{L} = \sum_{j=1}^{N} \left\{ \frac{m}{2} \dot{q}_j^2 - \frac{g}{2} (q_{j+1} - q_{j})^4 \right\} + \frac{f}{m} \sum_{i=1}^{M} k \delta(t - t_i) \delta_{ij}, \quad (2.1)$$

where $\delta_{ij} = (0, 0, 0)$, $\delta_{ij} = (q_1, \cdots, q_{N-1}, q_N)$ and $t_1 < t_i < \cdots < t_{MN}$. The factor $f$ is determined by the phase convention of the Veneziano amplitude.

Using the Euler's equation we obtain the equations of the lattice motions,

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = 0; \quad \ddot{q}_j = -\frac{g}{m} (2q_j - q_{j+1} - q_{j-1}) - \frac{f}{m} \sum_{i=1}^{M} k \delta(t - t_i) \delta_{jN}. \quad (2.2)$$

To solve Eq. (2.2), it is convenient to proceed to normal mode. Using the properties of a cyclic matrix, we can rewrite Eq. (2.1) into the following form (see Appendix I):
where
\[
\omega_j = \sqrt{4(g/m)} \sin(j\pi/N) \quad \text{and} \quad j=1, \ldots, N.
\]

From the Euler equation, we obtain
\[
\ddot{y}_j = -\omega_j^2 y_j + \frac{f}{m} \sum_{i=1}^{sM} k_i \delta(t-t_i).
\]

As the solution of Eq. (2.4), we obtain the following one:
\[
y_j(t) = A \cos \omega_j t + B \sin \omega_j t + \frac{f}{m} \sum_{i=1}^{sM} k_i \int_{t_0}^{t} \frac{d\omega \exp(-i\omega_j(t-t_0))}{\omega^2 - \omega_j^2}.
\]

As the boundary conditions for the action integral method, we choose the two initial and final values of \(y_j\)'s at \(t_0\) and \(T(>t_0)\). Further for the last term of Eq. (2.5), we choose the same contour as Feynman propagator as shown in Fig. 2. Finally, we obtain the following expression for Eq. (2.5):
\[
y_j(t) = \frac{1}{\sin \omega_j(T-t_0)} (y_j(t_0) \sin \omega_j(T-t) + y_j(T) \sin \omega_j(t-t_0))
\]
\[
+ \sum_{i=1}^{sM} \frac{f k_i}{2 \omega_j m} \left( \exp(-i \omega_j(t-t_i)) \theta(t-t_i) + \exp(i \omega_j(t-t_i)) \theta(t_i-t) \right),
\]

when \(if\) in Eq. (2.5) has been replaced by \(f\).

§ 3. Action integral method in the lattice model

In order to apply the action integral method to our problem, we first evaluate this action integral which is an integral of the Lagrangian equation (2.3). We substitute the classical solution (2.6) into Eq. (2.3) (the original Lagrangian (\(\hbar=c=1\))).

\[
iS = i \int_{t_0}^{T} L dt
\]
\[\begin{align*}
&= \sum_j \left[ \frac{m}{2} \omega_j(y^*_j(T) + y^*_j(t_0)) \frac{\cos \omega_j(T - t_0)}{\sin \omega_j(T - t_0)} - \frac{m}{2} \omega_j(y_j(T)y_j(t_0)) \\
&\quad + y_j(t_0) y_j(T) \cdot \frac{1}{\sin \omega_j(T - t_0)} + \frac{1}{2} \{ y_j(t_0) \int \sum \{ y_j(T) \ exp(-i \omega_j(t_n - t_0)) \\
&\quad + y_j(T) \int \sum \{ k_n \ exp(i \omega_j(T - t_n)) \} + i \int \sum \{ 4 + 2 \ exp(-2i \omega_j(T - t_n)) \\
&\quad + 2 \ exp(2i \omega_j(t_n - t_0)) + \sum_{1 \leq n \leq m} \{ 2 \ k_n \ \exp(i \omega_j(t_n - t_m)) \} \\
&\quad + 2 \ exp(-i \omega_j(2T - t_n - t_m)) + 2 \ \exp(\{ \omega_j(2t_n - t_m) \}) \} \right]
\end{align*}\]

where \( \tilde{y}(T) \) and \( \tilde{y}(t_0) \) correspond to zero modes. From this part we can derive the energy momentum conservation which plays the important role.

Now we must prove two theorems.

3.1°) The action is defined as follows:
\[\langle \{ y_j(T) \} \ | \ \{ y_j(t_0) \} \rangle \]
\[= \exp \left\{ i \ \int \ \sum \ \{ y_j(T), y_j(t_0), t \}, \{ y_j(T), y_j(t_0), t \}, t \ dt \right\}
\]
where \{ y_j(t) \} 's are the abbreviation for \( y_1(t), y_2(t), \cdots, y_N(t) \). Then we get
\[i \ \frac{\partial}{\partial T} \langle \{ y_j(T) \} \ | \ \{ y_j(t_0) \} \rangle = H \langle \{ y_j(T) \} \ | \ \{ y_j(t_0) \} \rangle,
\]
noting that \{ y_j \} and \{ y_j \} depend on \( T \).

3.2°) The division rule for the free action
\[\langle \{ y_j(T) \} \ | \ \{ y_j(t_0) \} \rangle = \int \prod_{j=1}^N \left\{ y_j(T) \langle \{ y_j(T) \} \ | \ \{ y_j(t) \} \right\} \right\}
\]
\[\begin{align*}
&\quad \left\{ y_j(T) \langle \{ y_j(T) \} \ | \ \{ y_j(t) \} \right\} \right\}
\end{align*}\]

\[= n(T, t) n(t, t_0) \sqrt{\prod_{j=1}^N \sin \omega_j(T - t) \sin \omega_j(T - t_0)} \frac{1}{\prod_{j=1}^N (m \omega_j/2) (\sqrt{\pi} \ e^{i \pi/4})^N}
\]
\begin{align}
\times \exp \left\{ i \sum_{j=1}^{N} \frac{m \omega_j^2}{2} \left[ (y_j^*(T) + y_j^*(t_0)) \frac{\cos \omega_j (T-t_0)}{\sin \omega_j (T-t_0)} + 2 (y_j(T)y_j(t_0)) \frac{1}{\sin \omega_j (T-t_0)} \right] \right\},
\end{align}

where \( n(T, t) \) and \( n(t, t_0) \) are the normalization constants. Therefore we can assign the normalization constants to

\begin{align}
n(T, t_0) &= \sqrt{\prod_{j=1}^{N} \frac{1}{\sin \omega_j (T-t_0)} \left( \frac{m \omega_j^2}{2} \right) \left( \sqrt{\pi} e^{i(\pi/4)} \right)^{-N}}, \\
n(t, t_0) &= \sqrt{\prod_{j=1}^{N} \frac{1}{\sin \omega_j (t-t_0)} \left( \frac{m \omega_j^2}{2} \right) \left( \sqrt{\pi} e^{i(\pi/4)} \right)^{-N}}.
\end{align}

Thus we can show that the division rule holds:

\begin{align}
\langle \{y_1(T)\} | \{y_1(t_0)\} \rangle &= \int \prod_{j=1}^{N} dy_j(t) \langle \{y_1(T)\} | \{y_1(t)\} \rangle \langle \{y_1(t)\} | \{y_1(t_0)\} \rangle.
\end{align}

We expect that these two theorems hold for a general dynamical system and the quantal effect is included in \( n(T, t) \).

\section*{§ 4. The Veneziano amplitude in our model}

In order to derive Veneziano amplitude (i.e. \(|z_i - z_j|^{2|k_j|^2}\)), we must obtain the transition amplitude which is an expectation value between zeroth eigenfunctions of the harmonic oscillator (say, ground states).\(^*\)

\begin{align}
\text{Transition amplitude} &= \int \prod_{j=1}^{N} dy_j(T) dy_j(t_0) d\bar{y}(T) d\bar{y}(t_0) n(T, t_0) \bar{n}(T, t_0) \\
&\times \left( \frac{m \omega_j}{\sqrt{\pi}} \right)^{1/2} \exp \left( -\sum_{j=1}^{N} \frac{m}{2} \omega_j (y_j^*(T) + y_j^*(t_0)) \exp \left( \frac{i}{2} \sum_{j=1}^{N} \omega_j (T-t_0) \right) \right) \exp(iS) \\
&= (2\pi) \delta (\sum_{j=1}^{N} k_j) \left( \frac{2}{m} \right)^{1/2} \left( \sqrt{\pi} e^{i(\pi/4)} \right)^{N} n(T, t_0) \bar{n}(T, t_0) \\
&\times \left\{ \prod_{j=1}^{N} \frac{m \omega_j}{4 \sin \omega_j (T-t_0)} \left( 1 - \exp \left( i \omega_j (T-t_0) \right) \right) \right\}^{-1/2} \\
&\times \left\{ \prod_{j=1}^{N} \frac{m \omega_j}{4 \sin \omega_j (T-t_0)} \left( -1 - \exp \left( i \omega_j (T-t_0) \right) \right) \right\}^{-1/2} \left( \sqrt{\pi} e^{i(\pi/4)} \right)^{N} n(T, t_0) \bar{n}(T, t_0) \\
&\times \exp \left\{ -\sum_{k=1}^{N} \frac{f_{k_j}^2}{16m \omega_j} \left[ \sum_{l=1}^{k_j} k_l^3 \left( e^{-2H_{k_j}(T-t_0)} + e^{-2H_{k_j}(T-t_0)} + 2e^{-2H_{k_j}(T-t_0)} \right) \right] \right\},
\end{align}

\(^*\) Harmonic oscillator eigenfunction \( \phi_n = (m \omega/\pi)^{1/4} H_n(\xi) / \sqrt{2^n n!} e^{-\xi^2/2} \), where \( \xi = \sqrt{m \omega} x \) and \( H_n(\xi) = (-1)^n e^{\xi^2} (dn/d\xi^n) e^{-\xi} \).
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\[
+ 2 \sum_{1 \leq t < s \leq M} k_t k_m \left( e^{-i\omega_j(t_1 + t_m - t_0)} + e^{-i\omega_j(t_2 - t_1 + t_m)} + e^{-i\omega_j(t - t_0)} + e^{-i\omega_j(t - t_0 + t_1 - t_m)} \right)
\]

\[
- \frac{f^2}{16 m \omega_j} \left[ \sum k_t^2 \left( 4 + 2e^{-i\omega_j(t_1 - t_2)} + 2e^{i\omega_j(t_0 - t_1)} \right) \right]
\]

\[
+ 2 \sum_{1 \leq t < s \leq M} k_t k_m \left( 8 - 4e^{i\omega_j(t_1 - t_m)} + 2e^{-i\omega_j(2t_1 - t_m)} + 2e^{i\omega_j(2t_0 - t_1 - t_m)} \right)
\}

where we assign the normalization constant to

\[
n(T, t_0) = \sqrt{\prod_{j=1}^{N} \frac{1}{\sin \omega_j(T - t_0)} \frac{m \omega_j}{2} (\sqrt{\pi} e^{i(\omega_j)^{-N}})}
\]

and

\[
n(T, t_0) = \sqrt{\left( \frac{m}{2} \right)} (\sqrt{\pi} e^{i(\omega_j)^{-1}}).
\]

\[
T = \infty
\]

\[
t_0 = \infty
\]

Fig. 4. The space-time diagram.

Now for the boundary condition we choose the following conditions (\(\omega \rightarrow \omega - i\varepsilon\)):

\[
T \rightarrow +\infty, \quad t_0 \rightarrow -\infty,
\]

which correspond to the conditions of \(S\)-matrix theory. Therefore we can obtain the transition amplitude\(^(*)\) such as

Veneziano amplitude = \(\frac{2\pi}{\sigma^{(4)}} \sum k_t \exp \left\{ -\frac{f^2}{16 m \omega_j} (4 \sum k_t^2) \right\} \]

\[
+ 2 \cdot 4 \sum_{1 \leq t < s \leq M} k_t k_m \left( 2 - e^{i\omega_j(t_1 - t_m)} \right)
\]

\[
= \frac{2\pi}{\sigma^{(4)}} \sum k_t \exp \left\{ +\frac{f^2}{2 m \omega_j} \sum_{1 \leq t < s \leq M} k_t k_m e^{i\omega_j(t_1 - t_m)} \right\}
\]

\[
\times \exp \left\{ -\frac{f^2}{2 m \omega_j} \sum_{1 \leq t < s \leq M} k_t k_m \right\}
\]

\(^{(*)}\) \exp[\Re\{Eq. (4.4)\}] = \prod_{1 \leq t < s \leq M} |z_t - z_m|^2 k_t k_m.

\(^{**}\) If we sandwich the exp(\(i\mathcal{S}\)) between \(H_1(\gamma_j(T))\) and \(H_3(\gamma(t_0))\), we obtain the following conclusion:

\[
\{\text{Eq. (4.4)}\} \times \left\{ \prod_{j=1}^{N} \frac{\sin \omega_j(T - t_0)}{m \omega_j} \right\} \left\{ \begin{array}{c} t_0 = -\infty \\ t_0 = +\infty \end{array} \right\} = 0.
\]
Now we can divide the above result by $\langle \exp \left( -\sum \frac{1}{n} \right) \rangle$. This fact is seen in Ref. 6. Then the above result corresponds to the Koba-Nielsen kernel (as $\omega_j = n$, $n = 1, 2, 3, 4, \infty$ and in the four-dimensional lattice model). We can assign the factor $f^2$ to $-4m (f = i2\sqrt{m})$ in order to obtain the correct phase. The imaginary part corresponds to decay width. This formulation may be useful for real reactions.

\section{The One-Loop Diagram Amplitude}

In this section, we will obtain the one-loop diagram amplitude. It is sufficient to insert the $\delta$-function in the connecting part.

One-loop amplitude $= \int \prod_{j=1}^{N} dy_j(T) dy_j(t_0) d\bar{y}(T) d\bar{y}(t_0) n(T, t_0) \bar{n}(T, t_0)$

$\times \delta (y_j(T) - y_j(t_0)) \delta (\bar{y}(T) - \bar{y}(t_0)) \exp(iS)$

$= \int \prod_{j=1}^{N} dy_j(T) dy(T) n(T, t_0) \bar{n}(T, t_0) \exp(iS)$

$= (2\pi)^{\delta(4)} \left( \prod_{j=1}^{N} \frac{m\omega_j/2}{\sin \omega_j(T-t_0)/2} \sqrt{m/2} \left( \frac{1}{\sqrt{m/2}} \right) \left( \sqrt{\pi} e^{\delta(\pi/4)} \right)^{(N+1)} \right)$

$\times \sqrt{\prod_{j=1}^{N} m\omega_j \tan \omega_j(T-t_0) / 2} \left( \sqrt{\pi} e^{\delta(\pi/4)} \right)^{N}$

$\times \exp \left\{ -\frac{N}{16m\omega_j} \left[ \frac{f^2}{(1 + \exp(-i\omega_j(T-t_0)))} + \frac{f^2}{(1 - \exp(-i\omega_j(T-t_0)))} \right] \right\}$

$\times \left( \sum_{k=1}^{M} k^2 \exp(-2i\omega_j(t_2-t_0)) + \exp(-2i\omega_j(T-t_0)) + 2 \exp(i\omega_j(T-t_0)) \right)$
In the four-dimensional lattice model, we obtain

\[ A = (2\pi)^{4\omega} \left( \sum_{k_{\mathbf{m}}} k_{\mathbf{m}} \right) \prod_{j=1}^{N} \left[ \exp \left( -i \omega_j (T - t_0) / 2 \right) \right] \left( \sqrt{\pi} e^{i(\pi/4)} \right)^4 \]

\[ \times \prod_{j=1}^{N} \prod_{1 \leq l \leq L_{l<}^{<} < M} \left( \phi_{l_{\mathbf{m}}} \right) e^{i2k_{\mathbf{m}}/\lambda}, \]

where

\[ \phi_{l_{\mathbf{m}}} = \exp \left[ - \frac{1}{2\omega_j} \left( 8 + \exp \left( -2i\omega_j (T - t_i) \right) + \exp \left( -2i\omega_j (T - t_m) \right) \right) \right] \]

\[ + \exp \left( 2i\omega_j (t_0 - t_i) \right) + \exp \left( -2i\omega_j (T - t_m) \right) - 4 \exp \left( i\omega_j (t_i - t_m) \right) \]

\[ + 2 \exp \left( -i\omega_j (2T - t_i - t_m) \right) + 2 \exp \left( i\omega_j (2t_0 - t_i - t_m) \right) \]

\[ \times \exp \left[ \frac{(1 + \exp \left( -i\omega_j (T - t_0) \right))}{(1 - \exp \left( -i\omega_j (T - t_0) \right))} \right] \left( - \frac{1}{2} \exp \left( -2i\omega_j (t_i - t_0) \right) \right) \]

\[ - \frac{1}{2} \exp \left( -2i\omega_j (T - t_0) \right) + 2 \exp \left( -i\omega_j (T - t_0) \right) \]

\[ + \exp \left( -i\omega_j (t_i + t_m - 2t_0) \right) + \exp \left( -i\omega_j (2T - t_i - t_m) \right) \]

\[ + \exp \left( -i\omega_j (T - t_0 + t_i - t_m) \right) + \exp \left( -i\omega_j (T - t_0 - t_i + t_m) \right) \].

(5.1)**

If we replace the proper-time factor \( \exp(-i(T - t_0)) \) such as

\[ \exp(-i(T - t_0)) = \omega = \omega_1 \omega_2 \cdots \omega_M, \]

then we obtain the following relation:

\[ \exp(-it_i) = \omega_1 = \omega, \]

\[ \text{(*) We use the momentum conservation, } \sum_{l=1}^{M} k_{l} = 0. \]

\[ \text{(**) Of course, we can divide the divergence-like term } \left( -\sum (1/n) \right) \text{ in Eq. (5.2).} \]
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\[ \exp(-i(t_2-t_1)) = z_2/z_1 = \frac{1}{x_1}, \]

\[ \exp(-i(T-t_{2m})) = z_1/z_N = w/x_{2M}, \]

\[ \exp(-i[(T-t_s)+(t_s-t_{s-1})+\cdots+(t_1-t_s)]) \]

\[ = \frac{w}{x_{2M}} \frac{1}{x_{2M-1}} \frac{1}{x_1} w = w. \] (5.3)

Therefore we obtain the integrand of the one-loop diagram.

One-loop amplitude = \((-i)^n \int dt_n \int dt_{n-1} \cdots \int dt_2 \frac{2^n}{\ln w} e^{-\frac{2}{w}w^m}. A, \)

where

\[ \ln w = -iT \quad \text{or} \quad e^{-iT} = w. \quad w_j = 1, 2, 3, \cdots N. \]

The difficulty of the divergence problem which usually occurs in a loop amplitude is avoided in Eq. (5.1), because of the finite numbers of the mode. This technique may be useful in the calculation of the form factor in the dual resonance model.

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Appendix I

The solution of the harmonic oscillator problem

Solving original equation, we may put as follows:

\[ q_j = A_N \sum_{k=1}^{N} e^{-iR_j x_k}, \quad \delta_{j,N} = B_N \sum_{k=1}^{N} e^{-iR_j x_k}, \] (A.I.1)

where \( A_N \) and \( B_N \) are the normalization constants, and \( R_j \) is determined by the periodic condition \( q_j = q_{j+N} \); \( R_j = (2\pi/N)j = a j \), where \( a \) is a lattice constant.

Substituting Eq. (A.I.1) into Eq. (1.2), we can obtain the following solution:

\[ x_k = \frac{1}{\sin \omega_k (T-t_0)} \{ x_k(t_0) \sin \omega_j (T-t) + x_k(T) \sin \omega_j (t-t_0) \} \]

\[ + \sum_{i=1}^{M} \frac{k_i}{2\omega_km} (e^{-i\omega_k(t-t_0)} \theta (t-t_0) + e^{i\omega_k(t-t_0)} \theta (t_0-t)) e^{-iR_j} \left( \frac{B_N}{A_N} \right), \] (A.I.2)
where
\[ \omega_k^2 = \frac{4g}{m} \sin^2 \frac{ka}{2}, \]
\[ x_k(t_0) = \frac{1}{\sqrt{N}} \sum_{f=1}^{N} q_f(t_0) e^{i k R_f^1}, \]
\[ x_k(T) = \frac{1}{\sqrt{N}} \sum_{f=1}^{N} q_f(T) e^{i k R_f^1}. \]

Compared with Eq. (1.6) and Eq. (A.1.2), we know that the function \( x_k \) is the normal mode solution \( y_j \) and the factor \( \exp(-i k R_j^1) \) is the transform matrix elements.

In order to express the coordinate \( q_j \) in the lattice in terms of \( x_k \) (or \( y \)), we have to substitute Eq. (A.1.2) into Eq. (A.1.1)
\[ q_j = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-i k R_j^1} x_k, \]  
\[ = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{-i k R_j^1} \left\{ \frac{1}{\sin \omega_k(T-t_0)} \left[ \frac{1}{\sqrt{N}} \sum_{l=1}^{N} q_l(t_0) e^{i k R_l^1} \times \sin \omega_k(T-t) + \frac{1}{N} \sum_{l=1}^{N} q_l(T) e^{i k R_l^1} \sin \omega_k(t-t_0) \right] \right\} . \]  
\[ + \sum_{l=1}^{M} \frac{k_j}{2 \omega_{k} m} \left( e^{-i \omega_k(t-t_0)} \theta(t-t_0) + e^{i \omega_k(t-t_0)} \theta(t_0-t) \right) \} . \]  

**Appendix II**

*The relation between the action integral and the S-matrix theory*

II.1) In S-matrix theory we use perturbation theory.

For our aim, we write
\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I, \]  
\[ i \partial_{\tau} U(\tau) = \mathcal{H} U(\tau). \]  

To obtain the expansion of the wave function in powers of \( \mathcal{H}_I \), we go into the interaction representation. The related operator is defined by
\[ V(\tau) = U_{0}^{-1}(\tau) \mathcal{H}_I U(\tau), \]  
where
\[ U_{0}(\tau) = \exp(-i \tau \mathcal{H}_0). \]  

The function \( V(\tau) \) is determined by
\[ i \partial_{\tau} V(\tau) = U_{0}^{-1}(\tau) \mathcal{H}_I U_{0}(\tau) V(\tau) \]  
(A.2.5)
Combining Eqs. (A·II·5) and (A·II·6) in the integral equation and using the interaction, we obtain

\[ V(\tau) = \sum_{n=0}^{\infty} (-i)^n \int d\tau_{n-1} U_{0}^{-1}(\tau_{n-1}) \mathcal{H}_{1}(\tau_{n-1}) \int d\tau_{n-1} \cdots \]

\[ \times \int d\tau_{1} U_{0}^{-1}(\tau_{1}) \mathcal{H}_{1} V_{0}(\tau_{1}). \]  

(A·II·7)

Now we apply Eq. (A·II·7) to the Nambu-Miyamoto equation,

\[ \left( \partial^{2} - N + V + \frac{e}{\partial \tau} \right) \psi(x, n\tau) = 0, \]

(A·II·8)

where \( N = \sum n\alpha_{n}^{+} a_{n} \),

\[ V(x) = \sum_{n=1}^{\infty} \frac{\partial}{\partial r} \left( a_{n}^{(r)} e^{irr} + a_{n}^{+(r)} e^{-irr} \right). \]

We obtain the Veneziano amplitude after changing the interaction representation,

\[ S(-\infty, \infty) = T \exp \left\{ -i \int V(p, \tau) d\tau \right\}, \]

(A·II·9)

where

\[ V(p, \tau) = \int \frac{d^{3}k}{\sqrt{(2\pi)^{3}2k_{0}}} \exp \left\{ i \left( (p - k)^{2} - \rho^{2} \right) \tau \right\} A_{k} e^{i\phi_{p}(\tau)} + \text{h.c.} \]

For the one-loop amplitude, we also obtain as follows:

\[ \text{trace} \left\{ \frac{1}{\varepsilon - H} V \right\} = \sum_{n=1}^{\infty} L_{n} \]

\[ = -\sum_{n=1}^{\infty} (-i)^{n} \int_{-\infty}^{0} d\tau_{n} \int_{-\infty}^{\tau_{n}} d\tau_{n-1} \cdots \int_{-\infty}^{\tau_{1}} d\tau_{1} \exp \left\{ i\varepsilon \tau_{n} \right\} \text{trace} \]

\[ \times \left[ V_{p}(\tau_{n}) V_{p}(\tau_{n-1}) \cdots V_{p}(\tau_{1}) \right]. \]  

(A·II·10)

II. 2) On the other hand, in the action integral method, we start from the following Lagrangian:

\[ \mathcal{L} = \frac{m}{2} \sum_{f=1}^{N} (\dot{y}_{f}^{2} - \omega_{f}^{2} y_{f}^{2}) + \int \frac{d^{3}k}{\sqrt{(2\pi)^{3}2k_{0}}} A_{k} e^{i\phi_{p}(0, \tau)} + \text{h.c.} \]  

(A·II·11)

Of course in the Schrödinger representation, we have to drop out the \( \tau \)-dependence in the Hamiltonian. As §2, we calculate the action integral;
According to action integral method developed by Feynman, the transition probability is obtained such as

\[ \langle \psi(T) | e^{iS} | \psi(t_0) \rangle = \langle \psi(T) | 1 | \psi(t_0) \rangle S_0 + \langle \psi(T) | iS_1 | \psi(t_0) \rangle + \cdots, \]  

(A·II·13)

where \( iS_0 = i \int \mathcal{L}_0 dt \) (\( \mathcal{L}_0 \): free Lagrangian) and \( iS_1 = i \int \mathcal{L}_1 dt \) (\( \mathcal{L}_1 \): interaction Lagrangian). We rewrite down for the n-th order

\[
(-1)^n \int_{t_n}^{t_0} \int_{t_{n-1}}^{t_n} \ldots \int_{t_2}^{t_3} \int_{t_1}^{t_2} dt_n \ldots dt_{n-1} \int_{t_0}^{t_{n-1}} dt_{n-2} \ldots \int_{t_0}^{t_1} dt_1 \times \langle \psi(T) | e^{ik\phi(t_n)} e^{ik\phi(t_{n-1})} \cdots e^{ik\phi(t_1)} | \psi(t_0) \rangle \times \langle \psi(T) | \psi(T - \Delta t) \psi(T - 2\Delta t) \cdots | \psi(t_0) \rangle = \int_{t_n}^{t_0} dt_n \int_{t_{n-1}}^{t_n} dt_{n-1} \ldots \int_{t_0}^{t_1} dt_1 \times \prod_{1 \leq i < j \leq n} \left( 1 - \frac{Z_{i,j}}{Z_i} \right) e^{iS_1},
\]  

(A·II·14)

where \( \langle e^{iS_0} \rangle \) means the free action integral. This term vanishes in the expectation value calculation.

References

10) Y. Nambu, Prog. Theor. Phys. 5 (1950), 82.

Note added in proof:

In Eq. (2-1) we have dropped the free action part, \( \exp \{ (p^2 + m^2) dt \} \), which corresponds to propagator. Thus we have to correct this term in Eq. (4-4).