The runaway instability in general relativistic accretion discs

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ABSTRACT

When an accretion disc falls prey to the runaway instability, a large portion of its mass is devoured by the black hole within a few dynamical times. Despite decades of effort, it is still unclear under what conditions such an instability can occur. The technically most advanced relativistic simulations to date were unable to find a clear sign for the onset of the instability. In this work, we present three-dimensional relativistic hydrodynamics simulations of accretion discs around black holes in dynamical space–time. We focus on the configurations that are expected to be particularly prone to the development of this instability. We demonstrate, for the first time, that the fully self-consistent general relativistic evolution does indeed produce a runaway instability.

Key words: accretion, accretion disks – black hole physics – gravitation – instabilities.

1 INTRODUCTION

Thick and massive relativistic accretion discs around black holes (BHs) are thought to form in extreme core-collapse events of massive stars (Woosley 1993; MacFadyen & Woosley 1999; Ott et al. 2011; Sekiguchi & Shibata 2011; Woosley & Heger 2012), and they are a normal outcome for the coalescence of neutron star (NS)–NS (e.g. Ruffert, Rampp & Janka 1997; Rosswog, Ramirez-Ruiz & Davies 2003; Oechslin & Janka 2006; Shibata & Taniguchi 2006; Baiotti, Giacomazzo & Rezzolla 2008; Liu et al. 2008; Kiuchi et al. 2009) and NS–BH binaries (e.g. Rosswog 2005; Shibata & Uryū 2006; Etienne et al. 2009; Shibata et al. 2009; Chawla et al. 2010; Ruffert & Janka 2010; Foucart et al. 2011). Such systems are thought to be candidates for the central engines of gamma-ray bursts (GRBs; Woosley 1993; Popham, Woosley & Fryer 1999; Piran 2004; Lee & Ramirez-Ruiz 2007; Mészáros & Gehrels 2012).

Previous studies of the stability of accretion discs have shown that they can be subject to various types of global instabilities in a number of scenarios (e.g. Abramowicz et al. 1983; Papaloizou & Pringle 1984, 1985; Kojima 1986; Woodward, Tohline & Hachisu 1994; Font & Daigne 2002b; Zanotti, Rezzolla & Font 2003; Kiuchi et al. 2011; Korobkin et al. 2011; Taylor, Miller & Podsiadlowski 2011). Instabilities can result in strongly variable and unstable accretion rates. Abramowicz et al. (1983) discovered the so-called dynamical runaway instability (RI) in thick, self-gravitating accretion discs around BHs. The RI is similar to the dynamical instability in close binary systems that occurs when the more massive binary member overflows its Roche lobe. In this case, the size of the Roche lobe decreases faster than the size of the binary companion, which can ultimately lead to the tidal disruption of the companion and the merger of the binary system. In discs around BHs, a toroidal surface analogous to the Roche lobe can be identified. A meridional cut of this surface exhibits a cusp located at the $L_1$ Lagrange point. If the disc overflows this toroidal Roche surface, then the mass transfer through the cusp will push the cusp outwards, making a larger fraction of the disc matter unstable to accretion. This drives the cusp out even further, and leads to an exponential growth of the mass-transfer rate. As a result, most of the disc gets consumed by the BH within just a few dynamical times.

Abramowicz et al. (1983) found that the development of the RI depends on a wide range of parameters, such as the disc-to-BH mass ratio $M_D/M_{BH}$ and the location of the inner edge of the disc.1 However, their investigation was based on several approximations and simplifications: they used a polytropic equation

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1 Here, the term ‘disc’ refers to initial equilibrium disc configurations. Therefore, the concept of the inner edge for such systems is well defined. Note, however, that disc tend to spread on a viscous time-scale.
of state (EOS) for the disc material, a pseudo-Newtonian potential to model gravity of BH (Paczynsky & Wiita 1980), a disc with constant specific angular momentum and an approximate treatment of the self-gravity of the disc. Subsequent works with more refined approximations found indications of a stabilizing effect due to BH rotation (Wilson 1984; Abramowicz, Karas & Lanza 1998), while a positive radial gradient of specific angular momentum was suggested to strongly suppress the instability (Daigne & Mochkovitch 1997; Abramowicz et al. 1998; Font & Daigne 2002a; Daigne & Font 2004). Moreover, studies using a Newtonian pseudo-potential for the gravity of the BH (Kamna & Chakrabarti 1992; Masuda, Nishida & Eriguchi 1998) and relativistic calculations in a fixed space–time background (Nishida et al. 1996; Font & Daigne 2002b) found that the self-gravity of the disc aggravates the instability. Nevertheless, the distance between the inner edge of the disc and the location of the cusp is probably the most crucial parameter for the development of the RI: for as long as it is too large, the RI is unlikely to occur.

Recently, Montero, Font & Shibata (2010) performed the first fully general relativistic simulations of thick accretion discs around BHs in axisymmetry for a few dynamical times. For the particular models they studied, they found no signature of a RI during the simulated time. However, the inner surface of their disc models was located away from the Roche surfaces (Montero, private communication), so that the instability might not have had sufficiently favourable conditions or sufficient time to develop within the time period of disc evolution that was considered. Therefore, their results do not rule out the existence of the RI in disc models where the inner edge is located closer to the cusp.

In Korobkin et al. (2011), some of us have analysed the stability of slender and moderately slender accretion discs around BHs with $M_{\text{d}}/M_{\text{BH}}$ in the range of $0.11, 0.24$ using three-dimensional (3D) numerical simulations in full general relativity (GR). Although we did observe the development of several non-axisymmetric instabilities in our models, we found no traces of the RI. This result is, perhaps, not surprising since the inner radii of our disc models were located significantly away from the Roche surface (see Korobkin et al. 2011 for more details).

In a similar study, Kiuchi et al. (2011) performed simulations of self-gravitating discs around BHs in full GR with the aim of obtaining the gravitational wave signal from the non-axisymmetric Papaloizou–Pringle instability (PPI). They considered four disc models with constant and non-constant specific angular momentum and disc-to-BH mass ratios of 0.06 and 0.10. No RI was observed in their simulations, but no information was given regarding the relative location of the inner edge of the disc and the Roche surface. Therefore, it is difficult to judge if their models were sufficiently susceptible to develop the RI.

In order to understand whether the RI can occur at all in the most general case, one first has to explore whether it can develop in configurations that are particularly prone to the instability. Such systems contain discs that exactly fill their Roche lobes, have significant fractions of the BH masses, constant specific angular momentum profiles and non-rotating BHs. It is left to future studies to explore under which circumstances such configurations would form in nature. Our aim here is to explore whether the RI can occur in principle if the fully dynamical general relativistic effects are taken into account properly. Our 3D general relativistic simulations indeed confirm that the RI occurs in these cases.

This paper is organized as follows. In Section 2, we describe the numerical methods used in our simulations. In Section 3, we describe the initial disc model. In Section 4, we present our results, and in Section 5 present our conclusions for future work. Unless otherwise noted, throughout the paper we measure distances in the units of the Schwarzschild radius \( r_s = 2GM_{\text{BH}}(0)/c^2 \) of the BH at \( t = 0 \), while the rest of the quantities are measured in cgs units.

## 2 NUMERICAL METHODS

The numerical time-evolution in our study is performed with the GR hydrodynamics code \textsc{thor} (Zink, Schnetter & Tiglio 2008) and the space–time evolution code \textsc{quilt} (Pazos et al. 2007). These two codes are based on and communicate with each other via the open-source \textsc{cactus} computational infrastructure (Goode et al. 2003; Schnetter et al. 2007), and use the \textsc{carpet} mesh refinement and multiblock driver (Schnetter, Hawley & Hawke 2004; Schnetter et al. 2006). Initial models of self-gravitating discs around BHs are constructed using an improved version of the \textsc{rns} code (Stergioulas 2007). We use a multiblock grid with an adapted resolution which is similar to the one used in Korobkin et al. (2011). The grid resolution is \( \Delta_{\text{min}} \approx 0.022 r_g \) near the BH, \( \Delta_{\text{max}} \approx 0.2 r_g \) at the disc density maximum, and \( \Delta_{\text{out}} \approx 5.0 r_g \) at the outer boundary. For more details of the numerical methods as well as the grid structure used in our simulations, we refer the reader to Korobkin et al. (2011).

### 3 INITIAL DISC MODEL

In this study, we focus on two initial disc models, denoted A and B. Table 1 lists the parameters of these models, while Fig. 1 illustrates the structure of initial model A. We construct our discs using a polytropic EOS \( p = K \rho^{\gamma} \) with \( \gamma = 4/3 \), while the time evolution is performed with the two-parameter ideal gas EOS (with the same \( \gamma \)), which allows for shock heating. We do not include additional physics such as magnetic fields, nuclear or neutrino processes. Therefore, the results can be rescaled using one free parameter such as the initial BH mass. In particular, the cgs parameters listed in Table 1 assume \( M_{\text{BH}} = 5 M_{\odot} \). The disc-to-BH mass ratio \( M_{\text{disc}}/M_{\text{BH}} \approx 0.21 \) for model A and \( \approx 0.16 \) for model B. Both discs have a constant specific angular momentum profile. The black contour in Fig. 1 corresponds to the Roche surface, while the green ones show the contours of constant pressure. The disc in model

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific angular momentum ( \ell' (M_{\text{BH}}) )</td>
<td>3.91</td>
<td>3.84</td>
</tr>
<tr>
<td>Polytropic constant ( K (10^{12} \text{ cm}^3 \text{ g}^{-1/3} \text{ s}^{-2}) )</td>
<td>5.35</td>
<td>5.72</td>
</tr>
<tr>
<td>Maximum density ( \rho_{\text{max}} (10^{-2} \text{ g} \text{ cm}^{-3}) )</td>
<td>1.63</td>
<td>1.33</td>
</tr>
<tr>
<td>Disc-to-BH mass ratio ( M_{\text{d}}/M_{\text{BH}} )</td>
<td>0.2097</td>
<td>0.1628</td>
</tr>
<tr>
<td>Kinetic to potential energy ( T'/U' )</td>
<td>0.2666</td>
<td>0.2469</td>
</tr>
<tr>
<td>Inner radius ( r_{\text{in}} (r_g) )</td>
<td>2.082</td>
<td>2.127</td>
</tr>
<tr>
<td>Location of the cusp ( r_l [r_g] )</td>
<td>2.024</td>
<td>2.148</td>
</tr>
<tr>
<td>Outer radius ( r_{\text{out}} (r_g) )</td>
<td>12.08</td>
<td>12.37</td>
</tr>
<tr>
<td>Central radius ( r_{\text{max}} (r_g) )</td>
<td>4.687</td>
<td>4.342</td>
</tr>
<tr>
<td>Orbital frequency at ( r_{\text{max}}, \Omega_{\text{max}} (s^{-1}) )</td>
<td>1414</td>
<td>1581</td>
</tr>
<tr>
<td>Orbital period at ( r_{\text{max}}, T_{\text{max}} (s) )</td>
<td>( 4.44 \times 10^{-3} )</td>
<td>( 3.96 \times 10^{-3} )</td>
</tr>
<tr>
<td>Potential gap ( \Delta W := W_{\text{in}} - W_{\text{L}} )</td>
<td>(-1.2 \times 10^{-4} )</td>
<td>+0.01</td>
</tr>
</tbody>
</table>
4 RESULTS

Models A and B both develop the RI. Therefore, in the following, we concentrate on the results for model A, while model B will be discussed later in this section. The top panel of Fig. 2 shows the time evolution of the cusp radius and the radial coordinate of the location of the maximum disc density $r_{\text{max}}$. Because of initial metric perturbations induced by matching the initial data to a vacuum BH metric near the horizon (cf. discussion in Korobkin et al. 2011), the BH mass and cusp radius settle to a new, $\sim 3$ per cent smaller value within $t \lesssim 0.5 t_{\text{orb}}$. The smaller gravitational pull of the BH leads to a rapid increase of $r_{\text{max}}$ by $\sim 9$ per cent within the same time interval. Since the cusp is now located at a smaller radius, the disc is less likely to become subject to the RI. However, the initial metric perturbations induce oscillations in the disc, which are particularly evident in the evolution of $r_{\text{max}}$ at $t \lesssim 6 t_{\text{orb}}$. These oscillations lead to occasional crossing of the Roche lobe by the disc, resulting in a steady and slow accretion of the disc material on to the BH at $t \lesssim 9 t_{\text{orb}}$. This is visible in the evolution of the disc mass and angular momentum shown in the centre panel of Fig. 2.

The radius $r_{\text{max}}$ gradually decreases due to this slow accretion up to the time when the inner disc radius becomes as small as the cusp radius. This occurs at $t \sim 9 t_{\text{orb}}$. At that point, the cusp radius starts increasing, leading to an acceleration of the accretion. By the end of our simulation ($t = 18 t_{\text{orb}}$), $\sim 75$ per cent of the initial disc mass has been accreted on to the BH. Such an accelerated accretion due to dynamical migration of the cusp towards the disc during which most of the disc material is swallowed by the BH is exactly the defining property of the RI. Thus, our simulations show that the RI can indeed occur, at least for the most susceptible models, which are considered here.

It is interesting to note that the development of the axisymmetric RI triggers non-axisymmetric deformations of the disc. The bottom panel of Fig. 2 shows the normalized amplitude of non-axisymmetric $m = 1, 2, 3$ deformations (see Korobkin et al. 2011 for the exact definition of the amplitude). The $m = 1$ deformation grows exponentially starting at $\sim 7 t_{\text{orb}}$ until $\sim 14 t_{\text{orb}}$, reaches its peak value of $\sim 0.027$, at which point it saturates. This deformation develops due to the so-called Papaloizou and Pringle instability (Papaloizou & Pringle 1984), enhanced by an eccentric motion of the BH (as described in Korobkin et al. 2011). The deformations corresponding to the other values of $m$ do not show a strong growth and remain below $\sim 10^{-3}$. Since both the PPI and RI develop roughly at the same time in our simulations, they could, in principle, interact non-linearly, when sufficiently large amplitudes are reached. In particular, the PPI-induced deformations might be responsible for the apparent saturation of the RI towards the end of our simulations. Such deformations redistribute angular momentum outwards, which, in turn, inhibits the development of the RI (Daigne & Font 2004) (while another factor responsible for the eventual saturation of the RI is, of course, the depletion of the disc mass). Nevertheless, the rather moderate amplitude of the $m = 1$ mode of $\sim 0.027$ is unlikely to drastically affect the evolution of the RI, especially during the early evolution. Neither can it be held responsible for the onset of the RI itself, since the accretion stream overflowing the cusp and triggering the RI is only slightly non-axisymmetric. A more detailed analysis of the non-linear interaction between the PPI and the RI is required to clarify its role.

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**Figure 1.** The structure of the initial disc model A. Red and green lines show the contours of the effective potential $W = \log |u|$ and of the disc pressure, respectively. Here, $u_r$ is the covariant $t$-component of the four-velocity of a test particle on a circular orbit with the given specific angular momentum for model A. The values of $W$ are given in geometrized units $G = c = 1$. Contours of the pressure are equally spaced by 0.5 in decimal logarithmic scale, starting from the maximum pressure (green dot on the $r$-axis). The blue contour corresponds to $W = 0$, and the black contour marks the location of the toroidal Roche lobe with the cusp. The disc fills the Roche lobe almost up to the cusp. The grey area marks the location of the BH event horizon.

**Figure 2.** Evolution of the disc. Top: radial location of the cusp and the disc density maximum, normalized to their initial values. Centre: evolution of the disc rest mass and angular momentum for models A and B in full GR as well as for model A in the Cowling approximation. For clarity, the deviations from 1 of the mass and angular momentum in Cowling are multiplied by a factor of 10. Bottom: normalized amplitudes of non-axisymmetric $m = 1, 2, 3$ deformations in model A and $m = 1$ deformation in model B.
Fig. 3 shows four snapshots of the disc density in the meridional plane, for model A, corresponding to the times $t/t_{\text{orb}} = 0.5, 6, 12$ and 18. The snapshot at $t = 0.5 t_{\text{orb}}$ shows the disc in a perturbed state after the passage of the initial metric perturbation. As the disc moves away from the BH, several surface waves can be seen propagating to the outer side of the disc in an axisymmetric manner. As noted above, such waves have sufficiently high amplitude to overfill the Roche lobe and support a small amount of accretion. The next snapshot at $t = 6 t_{\text{orb}}$ shows a wider accretion stream and an extended structure of the outer parts of the disc, caused by heating by the shocks formed during the radial disc oscillations. Such shock heating efficiently damps out the radial disc oscillation and causes the disc to heat up and expand, similarly to what was observed in Korobkin et al. (2011). This thermal expansion is confined to the outer regions of the disc. Indeed, if the expansion significantly affected the inner regions, the disc would overflow its Roche lobe and additional streams from the upper and lower faces would emerge. Since this is not observed in Fig. 3, we conclude that the inner side of the disc is not affected. The last two snapshots show the disc during the rapid accretion phase at $t = 12 t_{\text{orb}}$ and $18 t_{\text{orb}}$. By $t = 18 t_{\text{orb}}$, the accretion stream is very wide, but the high-density parts have moved closer to the BH and occupy a smaller volume, indicating a smaller disc mass.

To secure our findings against the possibility of an error in the treatment of hydrodynamics, we have complemented the fully dynamical GR simulation of model A with one in the Cowling approximation (i.e. on a fixed metric background). Such a model should undergo at most a slow and steady accretion, in a way similar to the early ($t \lesssim 6 t_{\text{orb}}$) evolution of the fully dynamical model. There should be no unstable growth of accretion on a fixed metric background. As we can see from the time evolution of the disc mass and angular momentum shown in the centre panel of Fig. 2, this is indeed the case.

Interestingly, model A does not develop the PPI in our Cowling simulation, where amplitudes of non-axisymmetric deformations remain below $\sim 10^{-5}$ throughout the evolution. Since the PPI in Cowling approximation relies on the existence of the non-accreting inner edge of the disc (Blaes 1987; Hawley 1991), the absence of PPI is probably caused by the accretion at a small rate through the cusp, which can completely freeze the development of PPI modes in Cowling approximation (Blaes 1987; Hawley 1991). This is different in the case of a dynamical metric, where the PPI can be enhanced by the motion of the BH (Korobkin et al. 2011). For this reason, accretion fails to suppress the PPI in model A with a dynamical metric (this is also observed in fig. 1 of Kiuchi et al. 2011 for the models with constant specific angular momentum).

Compared to model A, in model B the disc overflows its Roche lobe by a small amount $\Delta W = 0.01$. In this case, the exponential accretion should commence immediately and continue until a significant fraction of the disc is accreted on to the BH within a few
dynamical times. This is what we observe in our simulation, as can be seen from the evolution of the disc mass and angular momentum shown in the centre panel of Fig. 2. Notice, however, that in this case the development of the instability happens more slowly, because of the lower disc-to-BH mass ratio. The amplitude of non-axisymmetric deformations is also lower. In particular, the $m = 1$ mode exhibits slow and irregular growth and stays below $\sim 10^{-3}$ throughout the simulation (cf. the centre panel of Fig. 2). In the absence of any significant non-axisymmetric deformation, the accretion of a substantial fraction of the disc mass on to the BH within a few dynamical times can only be attributed to the development of the RI.

As mentioned in Section 1, Montero et al. (2010) presented axisymmetric simulations of two disc models around BHs with a constant specific angular momentum. Their simulations were also fully general relativistic with dynamical space–time evolution, similar to our simulations. The absence of the RI in Montero et al. (2010) must be attributed to differences in the initial disc models (compared to our configurations) and specifically to the fact that their initial models did not exactly fill the Roche lobe. Because of this, the instability is less likely to occur within the limited simulation time. The same argument may explain the absence of the RI in Kiuchi et al. (2011). If confirmed, this would further underline the importance of the distance between the cusp and the inner disc surface for the development of RI.

5 CONCLUSION

In this study, we have, for the first time, demonstrated that the RI in self-gravitating accretion discs does indeed occur in fully dynamical general relativistic evolutions. We have selected two models that are particularly prone to the development of such an instability. Our models have an appreciable disc-to-BH mass ratio (0.21 for model A and 0.16 for model B, see Table 1), a constant profile of specific angular momentum and a non-rotating BH. Moreover, and perhaps most importantly for the development of the RI, our disc model A almost exactly fills its Roche lobe, while model B slightly overfills it.

Our simulations show that both models develop the RI, exhibiting unstable accretion of the disc matter on to the BH within just a few dynamical times. More than half of the disc mass is absorbed by the BH by the end of our simulations, which were terminated only because we ran out of available computing time.

Our results demonstrate that the RI does indeed occur, at least in the models considered here. Future research will need to investigate how this depends on the parameters of the initial disc models, such as the disc-to-BH mass ratio, the gradient of the specific angular momentum of the disc and the BH spin, in order to establish the astrophysical significance of the instability.

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REFERENCES

Pazos E., Dorband E. N., Nagar A., Palenzuela C., Schnetter E., Tiglio M., 2007, Classical Quantum Gravity, 24, 341

3 The RI was also not observed in full GR simulations of the accretions discs formed in the aftermath of the merger of NSs with their binary compact companions (e.g. Rezzolla et al. 2010). We believe the same rationale applies also here.
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