Time-Dependent Variational Approach to the Non-Abelian Pure Gauge Theory

Its Application to Evaluation of the Shear Viscosity of Quantum Gluonic Matter

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The time-dependent variational approach to the pure Yang-Mills gauge theory, especially a color $su(3)$ gauge theory, is formulated in the functional Schrödinger picture with a Gaussian wave functional approximation. The equations of motion for the quantum gauge fields are formulated in the Liouville-von Neumann form. This variational approach is applied in order to derive the transport coefficients, such as the shear viscosity, for the pure gluonic matter by using the linear response theory. As a result, the contribution to the shear viscosity from the quantum gluons is zero up to the lowest order of the coupling $g$ in the quantum gluonic matter.

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§1. Introduction

One of the recent interests for the quark and gluon physics, governed by the quantum chromodynamics (QCD), is to investigate properties of the quark-gluon plasma (QGP) and/or the quark-gluon matter. In the recent progress of the Relativistic Heavy Ion Collider (RHIC) experiments, it is said that the QGP may not be free gas but the strongly interacting quark-gluon matter.¹) The matter composed of quarks and gluons seems to reveal the properties of the liquid, not gas, like the perfect liquid. This conjecture is derived by comparing the obtained experimental data with the phenomenological analysis by using the hydrodynamical simulations without and with rather small shear viscosity, which leads to the near perfect liquid.²)–⁵) In another context, the existence of the lower bound for the shear viscosity is conjectured in the AdS/CFT correspondence.⁶)

Many works to understand properties of the quark and/or gluonic matter were performed recently,⁷)–¹¹) while the transport coefficients, especially the shear viscosity for pure gluonic matter, were evaluated up to the lowest order of the QCD coupling constant $g$ in the early study.¹²) Namely, the shear viscosity $\eta_C$ for the gluonic matter at temperature $T$ can be expressed as⁹),¹²)

$$\eta_C = d_f \frac{T^3}{g^4 \log(1/g^2)} , \quad (1.1)$$

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up to the lowest order of $g$. Here, $d_f$ is numerically determined constant. For the quark matter, the shear viscosity is also evaluated, for example, in the Nambu-Jona-Lasinio (NJL) model\textsuperscript{13} by using the linear response theory,\textsuperscript{14,15} in which a rather small shear viscosity is derived.\textsuperscript{10} The small shear viscosity leads to the short mean free path in general. Thus, it may be shown that the constituents, namely quarks and gluons, of the matter under consideration, are strongly correlated.

However, as was shown by Ref.\textsuperscript{11}, the anomalous contribution to the shear viscosity in the turbulent plasma fields gives the small shear viscosity even in the weak coupling QCD in which the quarks and gluons are weakly correlated. Namely, the shear viscosity $\eta$ can be expressed by including the anomalous contribution $\eta_A$ as\textsuperscript{11}

$$\eta^{-1} = \eta_C^{-1} + \eta_A^{-1}. \tag{1.2}$$

If the anomalous viscosity $\eta_A$ is small, the total shear viscosity $\eta$ becomes small even if the usual shear viscosity $\eta_C$ is large for the small coupling constant $g$ in Eq. (1.1). Thus, the small shear viscosity does not always lead to the strong coupling QCD, namely strongly correlated quark-gluon matter.

In this paper, thus, we consider the pure gluonic matter as the weak coupling system. We concentrate our interest on calculating the shear viscosity for pure quantum gluonic matter without quarks by using the linear response theory. In this paper, the quantum gluon means the quantum fluctuation part around the mean field which leads to Eq. (1.1). Thus, the shear viscosity of the quantum gluonic matter gives the contribution of the next and higher order of $g$ in comparison with Eq. (1.1). One of the purposes in this paper is to investigate the shear viscosity under small QCD coupling $g$ for the quantum gluonic matter, which leads to the additional contribution with the higher order of $g$ to Eq. (1.1) for the shear viscosity.

To deal with the quantum gluons and to investigate the dynamics of the quantum gluons, the time-dependent variational method with the Gaussian functional as a trial wave functional in the functional Schrödinger picture may give a useful tool.\textsuperscript{16} The reason why is that the mean fields and the quantum fluctuations around them can be treated on an equal footing and the higher order contributions for $g$ are automatically included because certain kinds of the Feynman diagrams are taken into account in this variational approach. In this variational approach, the equations of motion for the mean fields and the fluctuation modes around them are derived in a self-consistent manner. Especially, the equations of motion for the quantum gluon fields are formulated in a form of the Liouville-von Neumann equations.

Another merit to use the time-dependent variational method for the pure gluonic matter is that the expectation values for various field operators and their products can easily be calculated because the state or the wave functional is prepared in the process of the variational calculations. When the transport coefficients such as the shear viscosity are calculated by using the linear response theory, the expectation values or thermal averages for the various operators such as the energy-momentum tensor operator are necessary. Thus, the variational approach may be suitable and give a practical method to calculate the transport coefficients.

This paper is organized as follows. In the next section, the time-dependent
variational approach to the pure Yang-Mills theory, especially the $su(3)$ gauge theory as the QCD, is formulated in the Hamiltonian formalism. In §3, the time-dependent variational equations for the quantum gluon fields are reformulated in a form of the Liouville-von Neumann equation for the reduced density matrix of the quantum fluctuation fields at zero and the finite temperatures. In §4, the shear viscosity is evaluated in our framework for pure gluonic matter by using the linear response theory from the viewpoint of weakly coupled QCD or weakly correlated pure gluonic matter. The last section is devoted to a summary and concluding remarks.

§2. Time-dependent variational approach to QCD

In this section, we give a variational method for the pure Yang-Mills gauge theory with color $su(3)$ symmetry in the functional Schrödinger picture with a Gaussian approximation, which is developed in Ref. 16, in a slightly different manner. The trial state is constructed, paying attention to the canonicity condition\textsuperscript{17,18} in our time-dependent variational approach. As a result, the equations of motion for variational functions are obtained as canonical equations of motion in classical mechanics.

2.1. Hamiltonian formalism of pure gauge theory

In this subsection, we summarize the Hamiltonian formalism of the pure Yang-Mills gauge theory for the sake of the definiteness of notations.

Let us start with the following Lagrangian density for the pure gauge theory with the color $su(N)$ symmetry:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{a\mu\nu}^\mu, \\
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c,
\]

(2.1)

where $A_\mu^a$ represents the gauge field and the Greek indices such as $\mu, \nu$ etc. and the Roman indices such as $a, b$ etc. mean the Lorentz and the color indices, respectively. The repeated indices are summed up. Later, we use other Roman indices such as $i, j, \cdots$, which mean the space components of the Lorentz indices, that is 1, 2 and 3. Here, $g$ represents the coupling constant and $f_{abc}$ is the structure constant for the color $su(N)$:

\[
[ T_a, T_b ] = if_{abc}T_c,
\]

(2.2)

where $\{T_a\}$ is the $su(N)$ generators. In the adjoint representation, the $su(N)$ generator can be expressed as $(T_a)_c^b = -if_{abc}$.

The conjugate momentum, $\pi^{a\mu}$, for the field $A_\mu^a$ is defined as

\[
\pi^{a\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu^a} = F_{a\mu}^\mu.
\]

(2.3)

Here, $\dot{A} = \partial A/\partial t$. We introduce the vector notation such as $A_a = (A_1^a, A_2^a, A_3^a)$. Then, the conjugate momentum with a space component can be expressed as

\[
\pi^a = -(F_1^a, F_2^a, F_3^a)
\]
Here, we define the color electric field $E^a$. As is similar to the color electric fields, we define the color magnetic field as

$$B^a = - (F^a_{23}, F^a_{31}, F^a_{12}) = \nabla \times A^a - \frac{1}{2} g f_{abc} A^b \times A^c.$$  

Thus, we define the Hamiltonian density $H_0$ as

$$H_0 = \pi^{a\mu} \dot{A}_\mu - \mathcal{L} = \frac{1}{2} \left[ (E^a)^2 + (B^a)^2 \right] + \pi^a \cdot (\nabla A^a_0 - g f_{abc} A^b A^c_0).$$

As is well known, the gauge theory leads to the constrained system. Namely, the conjugate momentum $\pi^a_0$ is identical to zero, so it is necessary to impose a constraint condition and the consistency condition for the time evolution as

$$\pi^a_0 = F^{a,00} = 0, \quad \dot{\pi}^a_0 = 0.$$  

This fact leads to the Dirac theory of constrained system. In terms of the analytic mechanics, the constrained condition is written as

$$\dot{\pi}^a_0 = \left\{ \pi^a_0, \int d^3 x \mathcal{H} \right\}_P = D \cdot \pi^a = 0,$$

$$D \cdot \pi^a = \nabla \cdot \pi^a - g f_{abc} A^b \cdot \pi^c = \nabla \cdot E^a + ig \cdot i f_{abc} A^b \cdot E^c,$$  

where $\left\{ , \right\}_P$ represents the Poisson bracket. Thus, the Hamiltonian is written as

$$\int d^3 x \mathcal{H}_0 = \int d^3 x \frac{1}{2} \left[ (E^a)^2 + (B^a)^2 \right] + \int d^3 x (\pi^a \cdot \nabla A^a_0 - g f_{abc} \pi^a \cdot A^b A^c_0)$$

$$= \int d^3 x \frac{1}{2} \left[ (E^a)^2 + (B^a)^2 \right],$$

where we used the integration by parts in the second term and the constrained condition (2.7). Thus, hereafter, we use the Hamiltonian density as

$$\mathcal{H}_0 = \frac{1}{2} \left[ (E^a)^2 + (B^a)^2 \right].$$

For the later convenience, we introduce the following variable, $G$:

$$G = G^a T^a = \left( \nabla \cdot E^a + ig \cdot i f_{abc} A^b \cdot E^c \right) T^a = \nabla \cdot E + ig[A^i, E^i],$$

where $A^i = A^i_a T^a$ and so on, and $G^a = D \cdot \pi^a$. Thus, it is understood that $G$ is nothing but the infinitesimal generator of the gauge transformation.
2.2. Variational approach to pure gauge theory in quantum field theory

In this subsection, we formulate the time-dependent variational method for the
pure Yang-Mills gauge theory by using the functional Schrödinger picture
within the Gaussian approximation. We formulate our variational method in the
canonical form by the help of the canonical variable or canonicity conditions.17,18)

The time-dependent variational principle is formulated as

$$\delta \int dt \langle \Phi | i \frac{\partial}{\partial t} - \int d^3x \mathcal{H} | \Phi \rangle = 0,$$

(2.12)

where $\mathcal{H}$ represents the Hamiltonian density under consideration. In the functional
Schrödinger picture, the commutation relation $[A_i^a(x), E_j^b(y)] = i \delta_{ij} \delta_{ab} \delta^3(x - y)$
leads to

$$E_i^a(x) | \Phi \rangle = -i \frac{\delta}{\delta A_i^a(x)} | \Phi \rangle.$$

(2.13)

It is restricted ourselves that the trial state $| \Phi \rangle$ or the trial wave functional
$\Phi(A^a) = \langle A^a | \Phi \rangle$ has the following Gaussian form as

$$\Phi(A^a) = \mathcal{N}^{-1} \exp(i \langle E | A - \mathcal{A} \rangle) \text{exp} \left( -\langle A - \mathcal{A} | \frac{1}{4G} | A - \mathcal{A} \rangle \right).$$

(2.14)

Here, we used abbreviated notations as

$$\langle E | A \rangle = \int d^3x E_i^a(x,t) \cdot A^a(x),$$

$$\langle A | \frac{1}{4G} | A \rangle = \int \int d^3x d^3y A_i^a(x) \frac{1}{4G} G^{-1} G_{ij}^{ab}(x,y,t) A_j^b(y).$$

(2.15)

Here, $\mathcal{A}_i^a(x,t), E_i^a(x,t), G_{ij}^{ab}(x,y,t)$ and $\Sigma_{ij}^{ab}(x,y,t)$ are the variational functions
which are determined by the time-dependent variational principle. The reason
why the form (2.14) is adopted is that the canonicity conditions for $(\mathcal{A}_i^a, E_i^a)$ and
$(G_{ij}^{ab}, \Sigma_{ij}^{ab})$ are automatically satisfied:

$$\langle \Phi | i \frac{\delta}{\delta A_i^a} | \Phi \rangle = E_i^a, \quad \langle \Phi | i \frac{\delta}{\delta E_i^a} | \Phi \rangle = 0,$$

$$\langle \Phi | i \frac{\delta}{\delta G_{ij}^{ab}} | \Phi \rangle = 0, \quad \langle \Phi | i \frac{\delta}{\delta \Sigma_{ij}^{ab}} | \Phi \rangle = -G_{ij}^{ab}.$$  

(2.16)

Thus, our time-dependent variational method is formulated as a canonical form.

In the functional Schrödinger picture, the expectation values are easily calculated
as follows:

$$\langle \Phi | A_i^a(x) | \Phi \rangle = \mathcal{A}_i^a(x,t),$$

$$\langle \Phi | E_i^a(x) | \Phi \rangle = E_i^a(x,t),$$

$$\langle \Phi | A_i^a(x) A_j^b(y) | \Phi \rangle = \mathcal{A}_i^a(x,t) \mathcal{A}_j^b(y,t) + G_{ij}^{ab}(x,y,t),$$

$$\langle \Phi | E_i^a(x) E_j^b(y) | \Phi \rangle = E_i^a(x,t) E_j^b(y,t) + \frac{1}{4G} G^{-1} G_{ij}^{ab}(x,y,t) + 4(\Sigma G \Sigma)_{ij}^{ab}(x,y,t),$$

$$\langle \Phi | A^a(x) \cdot E^b(x) | \Phi \rangle = \mathcal{A}_i^a(x,t) \cdot \mathcal{E}_j^b(x,t) + 2(G \Sigma)_{ij}^{ab}(x,x,t).$$

(2.17)
Thus, it is understood that the $\overrightarrow{A}_i^a$ represent the classical fields of gauge fields and the diagonal component of $G_{ij}^{ab}$, that is, $G_{ii}^{aa}$, where indices $i$ and $a$ are no sum, is a quantum fluctuations around the classical field $\overrightarrow{A}_i^a$. Thus, in this functional Schrödinger picture, the two-point function $G_{ij}^{ab}(x, y, t)$ plays a role of the gauge-particle propagator.

It should be noted here that the trial state (2.14) does not have the gauge symmetry, that is, $\mathcal{G}[\Phi] \neq 0$. Thus, we impose the gauge invariance by introducing the Lagrange multiplier. From (2.8), the constraint $D \cdot \pi^a = 0$ is recast into another form $\mathcal{G}^a = 0$ from (2.11), where $\mathcal{G}^a$ is the generator of the gauge transformation. Thus, we introduce the effective Hamiltonian density $\mathcal{H}$ by considering the gauge invariance in the space of the trial states as:

$$\mathcal{H} = \mathcal{H}_0 - \omega^a(x)\mathcal{G}^a(x), \quad (2.18)$$

where $\omega^a(x)$ represents a Lagrange multiplier, which insure the constraint $\mathcal{G}^a(x) = 0$. Thus, we use the above Hamiltonian density in order to determine the time dependences of the variational functions $\overrightarrow{A}_i^a(x, t)$, $\mathcal{E}_i^a(x, t)$, $G_{ij}^{ab}(x, y, t)$ and $\Sigma_{ij}^{ab}(x, y, t)$.

The expectation value of the Hamiltonian can be expressed as the following simple form:

$$\langle H \rangle = \langle \Phi | \int d^3 x [\mathcal{H}_0 - \omega^a(x)\mathcal{G}^a(x)] | \Phi \rangle$$

$$= (\langle H_0 \rangle - \int d^3 x \omega^a(x)\langle \mathcal{G}^a(x) \rangle)$$

$$\langle H_0 \rangle = \int d^3 x \left( \frac{1}{2} \overrightarrow{B}^a(x) \cdot \overrightarrow{B}^a(x) + \frac{1}{2} \overrightarrow{E}^a(x) \cdot \overrightarrow{E}^a(x) + \frac{1}{8} \mathrm{Tr}(x|G^{-1}|x) \right.$$  

$$\left. + 2 \mathrm{Tr}(\Sigma \mathcal{G} \Sigma|x|) + \frac{1}{2} \mathrm{Tr}(x|KG|x) + \frac{g^2}{8} (\mathrm{Tr}[S^i T^a(x|G|x)])^2 + \frac{g^2}{4} \mathrm{Tr} [S^i T^a(x|G|x) S^j T^a(x|G|x)] \right)$$

$$\langle \mathcal{G}^a(x) \rangle = \nabla \cdot \overrightarrow{E}^a(x) - ig \mathrm{Tr}(x|T^a[S^i \cdot G]|x) + ig \cdot if_{abc} \overrightarrow{A}^b \cdot \overrightarrow{E}^c,$$  

$$\langle \mathcal{G}^a(x) \rangle = \nabla \cdot \overrightarrow{B}^a(x) - ig \mathrm{Tr}(x|T^a[S^i \cdot G]|x) + ig \cdot if_{abc} \overrightarrow{A}^b \cdot \overrightarrow{B}^c,$$  

where we define

$$\overrightarrow{B}_i^a = \epsilon_{ijk} \partial_j \overrightarrow{A}_k^a - \frac{1}{2} g f^{abc} \epsilon_{ijk} \overrightarrow{A}_j^b \overrightarrow{A}_k^c,$$

$$(S^i)_{jk} = i \epsilon_{ijk}, \quad (T^a)_{bc} = -i f^{abc},$$

$$K = (-i S \cdot D)^2 - g S \cdot \overrightarrow{B},$$

$$D = \nabla - ig \overrightarrow{A}, \quad \overrightarrow{A} = \overrightarrow{A}_i^a T^a, \quad \overrightarrow{B} = \overrightarrow{B}^a T^a.$$

(2.20)

Here, we can use the abbreviated notation such as $\langle x|G|y \rangle = G_{ij}^{ab}(x, y, t)$. In the above representation, the $S$ represent the spin 1 matrices whose spatial component with $i$ is $S^i$.

2.3. Variational equations and their solutions in the time-independent case

The equations of motion for the variational functions are derived from the time-dependent variational principle in Eq. (2.12) with the Hamiltonian density (2.18).
The results are summarized in the form of canonical equations of motion as

$$\dot{\mathbf{A}}^a(x, t) = \frac{\delta \langle H \rangle}{\delta \mathbf{E}^a(x, t)}, \quad \dot{\mathbf{E}}^a(x, t) = -\frac{\delta \langle H \rangle}{\delta \mathbf{A}^a(x, t)}; \quad (2.21a)$$

$$\dot{G}_{ij}^{ab}(x, y, t) = \frac{\delta \langle H \rangle}{\delta \Sigma_{ij}^{ab}(x, y, t)}, \quad \dot{\Sigma}_{ij}^{ab}(x, y, t) = -\frac{\delta \langle H \rangle}{\delta G_{ij}^{ab}(x, y, t)}; \quad (2.21b)$$

where $\langle H \rangle$ is given in Eq. (2.19).

In the time-independent case, the above equations of motion in Eq. (2.21a) allow the following solutions within the lowest order of $g$ as

$$\mathbf{A}^a(x) = 0, \quad \mathbf{E}^a(x) = -\nabla^a(x). \quad (2.22)$$

In the above solutions, the mean field or the classical field $\mathbf{A}^t(x)$ is identical to zero, namely, under this situation, only the quantum field is dealt with in this matter system. Further, in the lowest order of $g$, the time-independent equations (2.21b) present the following solutions under $\mathbf{A}^a = 0$:

$$G_{ij}^{ab}(x, y) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} G_{ij}^{ab}(\mathbf{k}),$$

$$G_{ij}^{ab}(\mathbf{k}) = \delta^{ab} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) G_k, \quad G_k = \frac{1}{2|\mathbf{k}|}, \quad (2.23a)$$

$$\Sigma_{ij}^{ab}(x, y) = \int \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} e^{-i\mathbf{k}' \cdot \mathbf{y}} \langle \mathbf{k}' | a | \Sigma_{ij} | b \rangle \delta^{ab}(\mathbf{k}' - \mathbf{k} - \mathbf{q}),$$

$$\langle \mathbf{k}' | a | \Sigma_{ij} | b \rangle = \frac{1}{2} \left( \delta_{il} - \frac{k_i' k_l'}{k'^2} \right) \left( \delta_{lj} - \frac{k_j' k_l'}{k'^2} \right) g_{ij}^{abc} G_k G_k', \quad (2.23b)$$

$$\omega^a(\mathbf{q}) = \int d^3x \omega_a(x) e^{-i\mathbf{q} \cdot \mathbf{x}}. \quad (2.23c)$$

Thus, $G_{ij}^{ab}(\mathbf{k})$ has only the transverse component. This feature is plausible for the gauge-particle propagation.

2.4. Thouless-Valatin correction

In the Hamiltonian density (2·18), the constrained term $\omega^a G^a$ is introduced. This treatment resembles that of the nuclear rotation.\(^{21}\) In the nuclear many-body theory, the collective rotational motion of the axially symmetric deformed nuclei for $z$-axis is described in the same way used in this section. If the nuclear rotation occurs in the perpendicular to the $x$ axis, the state $|\Psi(t)\rangle = e^{-iEt} |\tilde{\Psi}\rangle$ is replaced as

$$|\Psi_\omega(t)\rangle = e^{-i\omega t \tilde{J}_z} e^{-iE_w t} |\tilde{\Psi}\rangle, \quad (2.24)$$

where the angular momentum operator $\tilde{J}_z$ and the angular velocity $\omega$ are introduced. Then, the Schrödinger equation, $i\hbar \partial_t |\Psi_\omega(t)\rangle = \hat{H} |\Psi_\omega(t)\rangle$ is recast into

$$(\hat{H} - \omega \hat{J}_z) |\Psi\rangle = E_\omega |\Psi\rangle, \quad (2.25)$$
where \( \hat{H} \) is the original nuclear Hamiltonian. It is known that we have to get rid of the effect of nuclear rotation from the total energy as

\[
E = \langle \Psi | \hat{H} | \Psi \rangle - \frac{\langle \Psi | \hat{J}_z^2 | \Psi \rangle}{2I} = \langle \Psi | \hat{H} | \Psi \rangle - \Delta E_{TV} ,
\]

where \( I \) is the moment of inertia and is defined as

\[
I = \lim_{\omega \to 0} \frac{\langle \Psi_\omega | \hat{J}_z | \Psi_\omega \rangle}{\omega} .
\]

This energy correction, \( \Delta E_{TV} \), in Eq. (2.26) is well known as the Thouless-Valatin correction. \(^{23}\)

Thus, in the approach to the pure Yang-Mills theory, it is first pointed out that the same correction term is necessary in Ref. 16). For the pure Yang-Mills theory in the treatment of the variational method, the Thouless-Valatin correction term can be expressed as

\[
\Delta E_{TV} = \int \int d^3x d^3y \langle \Phi | G^a(x) G^b(y) | \Phi \rangle \langle a x | \frac{1}{2I} | b y \rangle ,
\]

where the moment of inertia for the gauge rotation, \( I^{ab}(x, y) = \langle a x | I | b y \rangle \), is defined as

\[
I^{ab}(x, y) = \lim_{\omega^{b(y)} \to 0} \langle \Phi | G^{a(x)}(x) | \Phi \rangle \omega^{b(y)} .
\]

It is first shown that, in Ref. 16), the above Thouless-Valatin correction term and the contribution of the moment of inertia for the gauge rotation play essential roles in order to reproduce the one-loop running coupling constant. The validity of our time-dependent variational approach owes the fact that the one-loop running coupling constant is exactly reproduced in the lowest order approximation of \( g \) under \( \vec{A} = 0 \) developed in Ref. 16).

§3. Time-dependent variational equations for quantum gauge fields

In this section, we present the equations of motion for the quantum fluctuations around the classical field configurations \( \vec{A}^a \) and \( \vec{E}^a \), namely, \( G_{ij}^{ab} \) and \( \Sigma_{ij}^{ab} \) for quantum gauge fields, in a slightly different forms from Eq. (2.21b). We can formulate the equations of motion for quantum gauge fields as the Liouville-von Neumann equation.

3.1. Liouville-von Neumann equation for quantum gauge fields

First, the reduced density matrix \( \mathcal{M} \) is introduced as is similar to the Hartree-Bogoliubov theory for many-body physics in the boson systems. We define the reduced density matrix \(^{24), 25}\) for the quantum gauge fields as

\[
\mathcal{M}_{ij}^{ab}(x, y, t) = \begin{pmatrix}
-\frac{i}{2} \langle \hat{A}^a_i(x, t) \hat{E}^b_j(y, t) \rangle & \frac{1}{2} \langle \hat{A}^a_i(x, t) \hat{A}^b_j(y, t) \rangle \\
\frac{1}{2} \langle \hat{E}^a_i(x, t) \hat{E}^b_j(y, t) \rangle & \frac{i}{2} \langle \hat{E}^a_i(x, t) \hat{A}^b_j(y, t) \rangle - \frac{1}{2} \langle \hat{E}^a_i(x, t) \hat{E}^b_j(y, t) \rangle
goal-precision
From the above eigenvalue equation, we can derive the following equation:

\[
\begin{pmatrix}
-2i(G\Sigma)_{ij}^{ab}(x, y, t) & G_{ij}^{ab}(x, y, t) \\
(G^{-1})_{ij}^{ab}(x, y, t) + 4(\Sigma G\Sigma)_{ij}^{ab}(x, y, t) & 2i(\Sigma G)_{ij}^{ab}(x, y, t)
\end{pmatrix},
\]

where \( \langle \cdots \rangle \) represents the expectation values for the state \(|\Phi(t)\rangle\) in (2-14) as is shown in Eq. (2.17). In the latter, the expectation values are replaced into the thermal averages. Here, \( \hat{A}^{a}_i(x, t) \) means the quantum fluctuations around the classical configuration \( \bar{A}^{a}_i \). Thus, this reduced density matrix \( \mathcal{M} \) can be regarded as the one consisting of the quantum gauge fields.

By the help of the Heisenberg equations of motion in the Heisenberg picture, the time evolution of the reduced density matrix is easily derived. As a result, we can obtain the following Liouville-von Neumann type equation of motion for the reduced density matrix composed of the quantum gauge fields as

\[
i\mathcal{M}_{ij}^{ab}(x, y, t) = [\tilde{\mathcal{H}}, \mathcal{M}]_{ij}^{ab}(x, y, t),
\]

\[
\tilde{\mathcal{H}}_{ij}^{ab}(x, y, t) = \begin{pmatrix}
\lambda_{ij}^{ab}(x) & \delta_{ij}\delta_{ab} \\
\Gamma_{ij}^{ab}(x, t) & \lambda_{ij}^{ab}(x)
\end{pmatrix} \delta^3(x - y),
\]

\[
\Gamma_{ij}^{ab}(x, t) = K_{ij}^{ab} + g^2(S_{k}T^c(x)|\mathcal{G}|x)S_{k}T^c_{ij}^{ab} + \frac{g^2}{2}(S_{k}T^c)^{ab}_{ij} \text{Tr}[S_{k}T^c(x)|\mathcal{G}|x],
\]

\[
\lambda_{ij}^{ab}(x) = -ig\omega^p(x)f^{pab}\delta_{ij} = g(\omega^p(x)T^p)^{ab}_{ij},
\]

where \( K_{ij}^{ab}, S_{k} \) and \( T^c \) have been defined in Eq. (2.20). Here, \( \tilde{\mathcal{H}} \) is the Hamiltonian matrix which governs the time evolution of the reduced density matrix.

At this stage, it is important to indicate that the square of the reduced density matrix \( \mathcal{M} \) is easily obtained from the second line of Eq. (3.1) as

\[
\mathcal{M}^2 = \begin{pmatrix}
\frac{1}{4} & 0 \\
0 & \frac{1}{4}
\end{pmatrix}.
\]

Thus, the eigenvalues of the reduced density matrix are \( \pm 1/2 \) as is similar to the case of the linear sigma model.\textsuperscript{24,25}

The eigenvector for the eigenvalue 1/2 can be expressed as

\[
|x|1/2_{nai} = \begin{pmatrix}
u_n^{a}(x, t) \\
v_n^{a}(x, t)
\end{pmatrix},
\]

where \( n \) represents a certain quantum number. Then the following eigenvalue equation should be satisfied:

\[
\int d^3y \mathcal{M}_{ij}^{ab}(x, y, t) \begin{pmatrix}
u_n^{b}(y, t) \\
v_n^{b}(y, t)
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
u_n^{a}(y, t) \\
v_n^{a}(y, t)
\end{pmatrix}.
\]

From the above eigenvalue equation, we can derive the following equation:

\[
\int d^3y \mathcal{M}_{ij}^{ab}(x, y, t) \begin{pmatrix}
u_n^{b}(y, t) \\
v_n^{b}(y, t)
\end{pmatrix} = -\frac{1}{2} \begin{pmatrix}
u_n^{a}(y, t) \\
v_n^{a}(y, t)
\end{pmatrix}.
\]
Thus, we conclude that the $t(u^*_n, -v^*_n)$ is the eigenvector for the reduced density matrix $\mathcal{M}$ with the eigenvalue $-1/2$, which is expressed as $\langle x | -1/2_n ai \rangle$. Further, we can derive the following:

$$\int d^3y \mathcal{M}_{ij}^{ab}(x, y, t) \left( \begin{array}{c} v^b_{nj}(y, t) \\ u^b_{nj}(y, t) \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} v^a_{ni}(y, t) \\ u^a_{ni}(y, t) \end{array} \right).$$  \tag{3.7}$$

Thus, we can introduce another vector as

$$\langle x | 1/2_n ai \rangle = \left( \begin{array}{c} v^a_{ni}(x, t) \\ u^a_{ni}(x, t) \end{array} \right).$$  \tag{3.8}$$

Then, we can express

$$\langle 1/2_n b,j | x \rangle = (v^*_{nj}(x, t), u^*_{nj}(x, t)).$$

3.2. Spectral decomposition of reduced density matrix

In the previous subsection, it is learned that the reduced density matrix $\mathcal{M}$ has the eigenvalues $\pm 1/2$. In this subsection, the eigenvalue equation is summarized taking into account the extension to the finite temperature systems. Further, by using the eigenstates for $\mathcal{M}$, the reduced density matrix is expressed in the form of the spectral decomposition.

Considering the extension to finite temperature systems, the eigenvalue equations are described with the abstract representation as

$$\mathcal{M}_{ij}^{ab} | \sigma f_n, b, j \rangle = \sigma f_n | \sigma f_n, a, i \rangle, \quad \mathcal{M}_{ij}^{ab\dagger} | \sigma f_n^\dagger, b, j \rangle = \sigma f_n | \sigma f_n^\dagger, a, i \rangle,$$  \tag{3.9}$$

where $f = 1/2$ and $\sigma = \pm$ at zero temperature developed in the previous subsection. From the second equation in (3.9), we obtain

$$\langle \sigma f_n^\dagger, b, j | \mathcal{M}_{ji}^{ba} = \sigma f_n \langle \sigma f_n^\dagger, a, i \rangle.$$  \tag{3.10}$$

By using Eqs. (3.9) and (3.10), we can easily derive the following orthogonal relations as

$$\sum_{a,i} \langle \sigma' f_{n'}^\dagger, a, i | \sigma f_n, a, i \rangle = \delta_{\sigma' \sigma} \delta_{n' n},$$  \tag{3.11}$$

where the normalization condition of $| \sigma f_n, a, i \rangle$ is taken into account. By using the completeness relation $\int d^3x | x \rangle \langle x | = 1$ and the expression such as (3.4) with $\sigma = \sigma' = 1$ or $\sigma = 1$ and $\sigma' = -1$, the above condition (3.11) is rewritten as

$$\sum_{a,i} \int d^3x (v^*_{n'i}(x)u^a_{ni}(x) + u^*_{n'i}(x)v^a_{ni}(x)) = \delta_{nn'},$$

$$\sum_{a,i} \int d^3x (u^*_{n'i}(x)v^a_{ni}(x) - v^*_{n'i}(x)u^a_{ni}(x)) = 0,$$

$$\sum_{a,i} \int d^3x (u^*_{n'i}(x)v^a_{ni}(x) - v^*_{n'i}(x)u^a_{ni}(x)) = 0.$$  \tag{3.12}$$
Thus, the reduced density matrix $\mathcal{M}$ itself can be expressed in terms of the eigenstates of $\mathcal{M}$ as follows:

$$M_{ij}^{ab}(x, y, t) = \sum_{n(\sigma > 0)}^n \left[ \left( u_{ni}^{a}(y, t) \begin{pmatrix} v_{nj}^{a}(y) & u_{nj}^{b}(y) \end{pmatrix} \right. \right. \\
\left. \left. + \left( v_{ni}^{a}(y, t) \begin{pmatrix} -u_{nj}^{b}(y) & v_{nj}^{b}(y) \end{pmatrix} \right. \right. \right. \\
\left. \left. \right. \right. \\
\left. \left. \right. \right. \left. \right. \mathcal{H}_{ij}^{ab}(x, y, t) \right] \ . \ (3.13)$$

We can easily verify that the above $\mathcal{M}$ satisfy the eigenvalue equations (3.5) and (3.6) with $f_n$ instead of $1/2$.

Next, let us determine the eigenvalue $f_n$ in the system at finite temperature. In the time-independent case, from Eq. (3.2a), it is seen that the reduced density matrix and the Hamiltonian matrix commute each other. Thus, there exist the simultaneous eigenstates for $\mathcal{M}$ and $\tilde{\mathcal{H}}$ whose eigenvalues are $\sigma f_n$ and $\sigma E_n$, respectively. Namely,

$$\int d^3y \tilde{\mathcal{H}}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{nj}^{b}(y, t) \\ -v_{nj}^{a}(y, t) \end{pmatrix} = E_n \begin{pmatrix} u_{ni}^{a}(y, t) \\ v_{ni}^{a}(y, t) \end{pmatrix} \ ,$$

$$\int d^3y \tilde{\mathcal{H}}_{ij}^{ab}(x, y, t) \begin{pmatrix} u_{nj}^{a}(y, t) \\ -v_{nj}^{b}(y, t) \end{pmatrix} = -E_n \begin{pmatrix} u_{ni}^{a}(y, t) \\ -v_{ni}^{a}(y, t) \end{pmatrix} \ . \ (3.14)$$

Thus, we can derive $\text{Tr}(\tilde{\mathcal{H}}\mathcal{M}) = \sum_m E_m$. Here, the Helmholtz free energy is defined as

$$F = \langle \tilde{\mathcal{H}} \rangle - TS \ ,$$

$$\langle \tilde{\mathcal{H}} \rangle = \text{Tr}(\tilde{\mathcal{H}}\mathcal{M}) = \sum_m E_m \ ,$$

$$S = \sum_{m\sigma} \left[ (1 + n_{m\sigma}) \ln(1 + n_{m\sigma}) - n_{m\sigma} \ln n_{m\sigma} \right] \ , \ (3.15)$$

where $T$ is temperature. Thus, the minimization condition is imposed:

$$\delta F = \text{Tr}(\tilde{\mathcal{H}}\delta \mathcal{M}) - T \delta S = 0 \ . \ (3.16)$$

Here, we assume that $\sigma f_m$ depends on $n_{m\sigma}$ linearly. Under this assumption, $\delta \mathcal{M}/\delta n_{m\sigma} = t(u, v)(v^*, u^*)$ for $\sigma = +$ and $\delta \mathcal{M}/\delta n_{m\sigma} = t(u^*, v^*)(-v, u)$ for $\sigma = -$ are obtained respectively. Thus, we obtain

$$\frac{\delta F}{\delta n_{m\sigma}} = E_m - T \ln \frac{1 + n_{m\sigma}}{n_{m\sigma}} = 0 \ ,$$

i.e., $n_{m\sigma} = \frac{1}{eE_m/T - 1} \ , \ (3.17)$

where $\sigma = \pm$. Thus, we omit the suffix $\sigma$ in $n_{m\sigma}$. Of course, the eigenvalue $f$ is reduced to $1/2$ when $T \to 0$. Thus, finally, we obtain

$$f_m = n_m + \frac{1}{2} \ , \quad n_m = \frac{1}{eE_m/T - 1} \ . \ (3.18)$$
From the (1,2)-component of the reduced density matrix in Eq. (3.1), the two-point function $G$ can be decomposed of each spectral function as

$$G_{ab}^{ij}(x,y,t) = \sum_n f_n \left[ u_{n_i}^a(x) u_{n_j}^b(y) + u_{n_i}^{a*}(x) u_{n_j}^b(y) \right].$$  (3.19)

Hereafter, we take the quantum number $n$ as momentum $k$. Then, we obtain

$$G_{ab}^{ij}(x,y,t) = \int \frac{d^3k}{(2\pi)^3} f_k \left[ u_{k_i}^a(x) u_{k_j}^b(y) + u_{k_i}^{a*}(x) u_{k_j}^b(y) \right].$$  (3.20)

At zero temperature, we have already derived the expression of $G$ in Eq. (2.23). In the above expression in (3.20), $f_k = 1/2$ and $uu^*$ is also expressed. By using the knowledge of the zero temperature case, the above expression in (3.20) can be recast into

$$G_{ab}^{ij}(x,y,t) = \delta_{ab} \int \frac{d^3k}{(2\pi)^3} G^T_k \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) e^{i k \cdot (x-y)},$$

$$G^T_k = (2n_k + 1) \frac{1}{2|k|}.  (3.21)$$

From the eigenvalue equation (3.14), the energy eigenvalue for the quantum gauge fields is easily obtained. To check the validity of this treatment, we put $\omega^a(x) = 0$. Then from (3.14), we obtain

$$E_k \delta_{ij} \delta^{ab} = \Gamma^{1/2}_{ij}^a(k) ,$$

$$\Gamma^{1/2}_{ij}^a(k) = \delta^{ab}|k| \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right)$$  (3.22)

up to the order of $g$, which is a well-known result.

Finally, it should be noted that the gauge-particle has only transverse component as is realized in Eq. (2.23) with factor $(\delta_{ij} - k_i k_j/k^2)$. Since the reduced density matrix $M$ includes $G$ and $\Sigma$ composed of the quantum gauge field, $M$ also contains the factor $(\delta_{ij} - k_i k_j/k^2)$. Thus, we define the following projection operator $\hat{P}$:

$$\frac{\langle k' | \hat{P}_{ij} | k \rangle}{\langle k' | k \rangle} = \delta_{ij} - \frac{k_i k_j}{k^2} .$$  (3.23)

Then, $\hat{P}$ certainly has a property of the projection operator, namely, $\hat{P}^2 = \hat{P}$. As is easily shown, the following relations are satisfied:

$$G = \hat{P} G \hat{P} ,$$

$$\Sigma = \hat{P} \Sigma \hat{P} ,$$

$$M = \hat{P} M \hat{P} = \hat{P} M = M \hat{P} .$$  (3.24)

Since there always exists the projection factor to the transverse component, $(\delta_{ij} - k_i k_j/k^2)$, the inverse of the two-point function $G$ should be regarded as the two point function which satisfies the following relation:

$$G^{-1} G = \hat{P} .$$  (3.25)
§4. Transport coefficients of quantum gluonic matter

Hereafter, we deal with the color $su(3)$ pure gauge theory, namely the QCD without quarks. In this section, we present the expression of the transport coefficients in our variational approach by using the Kubo formula\textsuperscript{14,15} for the quantum gluonic matter with the color $su(3)$ symmetry.

4.1. Kubo formula based on the variational approach

By taking into account an external source field $\hat{A}$ and its conjugate force $F(t)$, the Hamiltonian $\hat{H}$ is modified from $\hat{H}_0$ to

$$\hat{H}(t) = \hat{H}_0 - \hat{A}F(t).$$

(4.1)

If the external force $F(t)$ is adopted as $F(t) = Fe^{-i\omega t}$, then, an observable $\hat{B}$ at time $t$, $\langle \hat{B} \rangle_t = \langle \hat{B} \rangle_{eq} + \chi_{BA}(\omega)Fe^{-i\omega t}$,

(4.2)

where $\langle \cdot \cdot \cdot \rangle_{eq}$ denotes the thermal average with respect to the equilibrium state. Here, $\chi_{BA}(\omega)$ is called the complex admittance and is defined as

$$\chi_{BA}(\omega) = -\lim_{\epsilon \to +0} \frac{i}{\hbar} \int_0^\infty dt \langle [ \hat{A}, \hat{B}(t) ] \rangle_{eq} e^{i\omega t - \epsilon t},$$

(4.3)

where

$$\varphi_{BA}(t) = -\frac{i}{\hbar} \langle [ \hat{A}, \hat{B}(t) ] \rangle_{eq}$$

(4.4)

is a quantum response function. By using the integration by parts and an property of the equilibrium state, namely, $\langle \dot{A}B \rangle_{eq} = -\langle AB \rangle_{eq}$, the complex admittance is recast into

$$\chi_{BA}(\omega) = \frac{1}{\hbar\omega} \lim_{\epsilon \to +0} \int_0^\infty dt e^{i\omega t - \epsilon t} \langle [ \hat{B}(t), \dot{A}(0) ] \rangle_{eq}$$

$$- \frac{1}{\hbar\omega} \lim_{\epsilon \to +0} \int_0^\infty dt e^{-\epsilon t} \langle [ \dot{A}(t), \hat{B}(0) ] \rangle_{eq}.$$  

(4.5)

Following the general theory,\textsuperscript{15} the transport coefficients are obtained by adopting both operators $\hat{A}$ and $\hat{B}$ being currents $J(t)$ as

$$\chi_{BA}(\omega, k = 0) = \frac{1}{\omega} \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r e^{i\omega t - \epsilon t} \langle [ J(r, t), J(0, 0) ] \rangle_{eq}$$

$$- \frac{1}{\omega} \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r e^{-\epsilon t} \langle [ J(r, t), J(0, 0) ] \rangle_{eq},$$

(4.6)
where we return to the natural unit, $\hbar = 1$. Here, current $J(t)$ is defined as

$$J(t) = \int d^3r J(r, t) e^{-ik \cdot r}$$  (4.7)

for an application to the gluonic matter in mind.

4.2. Shear viscosity in the quantum gluonic matter

In the gluonic matter, the dependence of the coupling constant $g$ for the usual, not anomalous, shear viscosity is given as (1.1) in the lowest order of $g$.

In this paper, since we deal with the quantum gluonic field described by the variables $G$ and $\Sigma$ with $A = 0$, so we can evaluate the usual shear viscosity $\eta_C$ with higher order of $g$ compared with those developed in the previous papers. Hereafter, we denote $\eta_C$ as $\eta$ simply.

In order to calculate the shear viscosity, the energy-momentum tensor for the pure gluonic field is necessary. In the symmetric representation, the energy-momentum tensor $T_{\mu \nu}$ is obtained as

$$T_{\mu \nu} = F_{\alpha \rho}^\mu F_{\alpha \rho}^\nu - \frac{1}{4} \delta_{\mu \nu} F_{\alpha \rho}^\alpha F_{\rho \sigma}^\sigma .$$  (4.8)

Then, $T_{\mu \nu}$ can be expressed in terms of the color electric and color magnetic fields as

$$T_{00}(r) = \frac{1}{2} (E^a(r) \cdot E^a(r) + B^a(r) \cdot B^a(r)) ,$$

$$T_{0i}(r) = -\epsilon_{ijk} E^a_j(r) B^a_k(r) ,$$

$$T_{ij}(r) = -E^a_i(r) E^a_j(r) - B^a_i(r) B^a_j(r) + \frac{1}{2} \delta_{ij} (E^a(r) \cdot E^a(r) + B^a(r) \cdot B^a(r)) .$$  (4.9)

The shear viscosity $\eta(\omega)$ is obtained by taking the current $J$ as $T_{xy} = T_{12}$ in Eq. (4.6):

$$\eta(\omega) = \frac{i}{\omega} [\Pi^R(\omega) - \Pi^R(0)] ,$$

$$\Pi^R(\omega) = -i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3r e^{i\omega t - \epsilon t} \langle [ T_{12}(r, t) , T_{12}(0, 0) ] \rangle_{eq} .$$  (4.10)

By taking a limit $\omega \to 0$, the shear viscosity $\eta(0)$ has a simple form as

$$\eta(0) = -\frac{d}{d\omega} \text{Im} \Pi^R(\omega) \bigg|_{\omega \to 0} .$$  (4.11)

Here, the thermal average $\langle \cdots \rangle_{eq}$ can be replaced to the expectation value given in Eq. (3.21) for $G$ and (2.23b) for $\Sigma$ with $G^T_k$ instead of $G_k$ in our variational approach at finite temperature. Thus, it is necessary for calculating the shear viscosity to evaluate the thermal average $\langle [ T_{12}(r, t) , T_{12}(0, 0) ] \rangle$ in our framework.

However, we need the operator at time $t$, namely, $T_{12}(r, t)$. In order to derive this operator at $t$, we only need to evaluate the operators $E^a$ and $B^a$ at time $t$.
because the expression of $T_{12}(r, t)$ has the same dependence with respect to $E^a$ and $B^a$, that is,

$$T_{12}(r, t) = -E_1^a(r, t)E_2^a(r, t) - B_1^a(r, t)B_2^a(r, t) .$$  \hspace{1cm} (4.12)

The operators $E^a(r, t)$ and $B^a(r, t)$ are obtained as a result of the time evolution governed by the Hamiltonian $\int d^3x H_0$ with (2.10):

$$E_i^a(r, t) = e^{iH_0 t}E_i^a(r)e^{-iH_0 t} , \quad B_i^a(r, t) = e^{iH_0 t}B_i^a(r)e^{-iH_0 t} ,$$  \hspace{1cm} (4.13)

$$H_0 = \int d^3r \mathcal{H}_0 = \int d^3r \frac{1}{2}[(E^a(r))^2 + (B^a(r))^2] .$$  \hspace{1cm} (4.14)

Here, we can derive

$$[ H_0 , E_i^a(r) ] = -i\varepsilon_{ijk}\partial_k B_{j}^a(r) - i\frac{1}{2}g\varepsilon_{ijk}f^{abc}(B_j^b(r)A_k^c(r) + A_k^c(r)B_j^b(r))
= -i(S \cdot \mathbf{p})_{ik}B_{j}^a(r) + O(g) ,$$

$$[ H_0 , B_i^a(r) ] = i\varepsilon_{ijk}\partial_k E_{j}^a(r) + i\frac{1}{2}g\varepsilon_{ijk}f^{abc}(B_j^b(r)A_k^c(r) + A_k^c(r)E_j^b(r))
= i(S \cdot \mathbf{p})_{ik}E_{j}^a(r) + O(g) ,$$

$$\hat{\mathbf{p}} = -i\frac{\partial}{\partial r} ,$$

where $S$ is defined in (2.20). Thus, up to the lowest order of $g$ in our quantum gluonic matter, we can derive the operators at time $t$ as

$$E_i^a(r, t) = \hat{C}_{ij}^{ab}(t)E_j^b(r) + \hat{D}_{ij}^{ab}(t)B_j^a(r) + O(g) ,$$

$$B_i^a(r, t) = \hat{C}_{ij}^{ab}(t)B_j^b(r) - \hat{D}_{ij}^{ab}(t)E_j^a(r) + O(g) ,$$

$$\hat{C}_{ij}^{ab}(t) = \delta^{ab}[\cos(t(S \cdot \hat{\mathbf{p}}))]_{ij} , \quad \hat{D}_{ij}^{ab}(t) = \delta^{ab}[\sin(t(S \cdot \hat{\mathbf{p}}))]_{ij} .$$

Thus, we can derive the energy-momentum tensor operator at time $t$ in Eq. (4.12).

From (4.10), we need the $\langle [T_{12}(r, t), T_{12}(r', 0)] \rangle$ to estimate the shear viscosity in quantum gluonic matter. After lengthy but straightforward calculation, we can derive the following form up to the lowest order of $g$:

$$\langle [ T_{12}(r, t) , T_{12}(r', 0) ] \rangle = \tilde{M}_{12}^{ab}(t) \cdot 2i\delta^3(r - r') \cdot \left[ \varepsilon_{j1l}\tilde{\Xi}_{l2}^{ab}(r, r', t) + \varepsilon_{j2l}\tilde{\Xi}_{l1}^{ab}(r, r', t) 
+ \varepsilon_{i1l}\tilde{\Xi}_{2j}^{ab}(r', r, t) + \varepsilon_{i2l}\tilde{\Xi}_{1j}^{ab}(r', r, t) \right]$$

$$- \tilde{N}_{12}^{ab}(t) \cdot i\delta^3(r - r') \cdot \left[ \varepsilon_{j1l}\tilde{T}_{l2}^{ab}(r, r', t) + \varepsilon_{j2l}\tilde{T}_{l1}^{ab}(r, r', t) 
- \varepsilon_{i1l}\tilde{T}_{2j}^{ab}(r', r, t) - \varepsilon_{i2l}\tilde{T}_{1j}^{ab}(r', r, t) \right] + O(g^2) ,$$  \hspace{1cm} (4.17)

where we define

$$\tilde{M}_{I,j}^{ab}(t) = \tilde{C}_{ii}^{ab}(t)\hat{C}_{jj}^{ab}(t) + \tilde{D}_{ii}^{ab}(t)\hat{D}_{jj}^{ab}(t) ,$$

$$\tilde{N}_{I,j}^{ab}(t) = \tilde{C}_{i1}^{ab}(t)\hat{C}_{j2}^{ab}(t) + \tilde{D}_{i1}^{ab}(t)\hat{D}_{j2}^{ab}(t) .$$
\[ \hat{N}^{ab}_{ij}(t) = \hat{C}^{ca}_{i}(t)\hat{D}^{cb}_{j}(t) - \hat{D}^{ca}_{i}(t)\hat{C}^{cb}_{j}(t), \]
\[ \Xi^{ab}_{ij}(r, r', t) = \epsilon_{imk}\partial_m^{(G\Sigma)}_{kij}(r, r', t) + \epsilon_{jnkn}\partial_m^{(G\Sigma)}_{ijk}(r, r', t) + O(g^2) \]
\[ = -(\mathbf{S} \cdot \hat{\mathbf{p}})_{ik}(G\Sigma)^{ab}_{kij}(r, r', t) - \mathbf{G}_{ik}^{ab}(r, r', t)(\mathbf{S} \cdot \hat{\mathbf{p}})_{ij} + O(g^2) \]
\[ = \frac{1}{2} \left[ \langle \mathbf{E}^a_i(r) B^b_j(r') \rangle + \langle B^a_i(r) \mathbf{E}^b_j(r') \rangle \right] + O(g^2) , \]
\[ \hat{\Gamma}^{ab}_{ij}(r, r', t) = -\frac{1}{4}(G^{-1})^{ab}_{ij}(r, r', t) + \epsilon_{ipm}\epsilon_{jkn}\partial_p^{(G\Sigma)} G_{mn}^{ab}(r, r', t) \]
\[ = \frac{1}{4}(G^{-1})^{ab}_{ij}(r, r', t) + (\mathbf{S} \cdot \hat{\mathbf{p}})_{im} G_{mn}^{ab}(r, r', t) + O(g^2) \]
\[ = \langle B^a_i(r) B^b_j(r') \rangle - \langle B^a_i(r) \mathbf{E}^b_j(r') \rangle + O(g^2) , \]
\[ \Xi^{ab}_{ij} = 0 \quad \text{for any } i, j, k, l, m \text{ and } n . \quad (4.19) \]

Next, let us consider the second term in (4.17). At zero temperature, the equation of motion for \( G \) can be derived from (2.19) or (2.21) by \( \delta \langle H \rangle / \delta G = 0 \) which leads to
\[ -\frac{1}{8} G^{-2} + \frac{1}{2} K = 0 \quad (4.20) \]
in the lowest order of \( g \). Thus, the solution of \( G \) is written as
\[ G = \frac{1}{2\sqrt{K}} = \frac{1}{2\mathbf{S} \cdot \hat{\mathbf{p}}} , \quad (4.21) \]
where \( K = (\mathbf{S} \cdot \hat{\mathbf{p}})^2 \) with \( \mathbf{A} = \mathbf{0} \) in (2.20). Using the above fact, from (4.18), the following is derived up to the lowest order of \( g \):
\[ \hat{\Gamma}^{ab}_{ij} = -\left[ \frac{1}{4} G^{-1} - (\mathbf{S} \cdot \hat{\mathbf{p}}) G(\mathbf{S} \cdot \hat{\mathbf{p}}) \right]_{ij}^{ab} = 0 . \quad (4.22) \]
As a result, the shear viscosity for the pure quantum gluonic matter is zero at zero temperature up to the order of \( g^1 \):
\[ \eta(\omega) = 0 \quad (4.23) \]
up to the order of \( g \) at zero temperature.
At finite temperature, we have the solution for $G$ and $\Sigma$ in Eqs. (3.21) and (2.23b) with $G_k^T$ instead of $G_k$. Thus, we obtain $\mathcal{Y}$ in the lowest order approximation as

$$\mathcal{Y}_{ij}^{ab}(r, r', t) = \delta^{ab} \frac{d^3 k}{(2\pi)^3} e^{i k \cdot (r - r')} \cdot 2|k| \cdot \frac{n_k (n_k + 1)}{2n_k + 1} \left( \delta_{ij} - \frac{k_i k_j}{|k|^2} \right)$$

$$= \delta^{ab} \frac{d^3 k}{(2\pi)^3} e^{i k \cdot (r - r')} \cdot \frac{|k|}{\sinh \left( \frac{E_k}{T} \right)} \left( \delta_{ij} - \frac{k_i k_j}{|k|^2} \right). \tag{4.24}$$

From Eq. (4.10), the complex admittance for the shear viscosity $\eta(\omega)$ is written as

$$\Pi^R(\omega) = -i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3 r \ e^{i \omega t - \epsilon t} \langle [T_{12}(r, t), T_{12}(0, 0)] \rangle$$

$$= i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3 r \ e^{i \omega t - \epsilon t} \dot{N}_{12ij}^{ab}(t) (i \partial_i \delta^3(r))$$

$$\times [\epsilon_{j11} T_{12}^{ab}(r, 0, t) + \epsilon_{j21} T_{12}^{ab}(r, 0, t) - \epsilon_{i11} T_{12}^{ab}(0, r, t) - \epsilon_{i21} T_{12}^{ab}(0, r, t)]$$

$$= i \lim_{\epsilon \to +0} \int_0^\infty dt \int d^3 r \ e^{i \omega t - \epsilon t} \dot{N}_{12ij}^{ab}(t) \left[ \delta^3(r) \right.$$\n
$$\times [\epsilon_{j11} T_{12}^{ab}(r, 0, t) + \epsilon_{j21} T_{12}^{ab}(r, 0, t) + \epsilon_{i11} T_{12}^{ab}(0, r, t) + \epsilon_{i21} T_{12}^{ab}(0, r, t)] \bigg], \tag{4.25}$$

where the integration by parts has been carried out from the second line to the third line and we define $\mathcal{T}_{i, ij}(r, r', t)$ as

$$\mathcal{T}_{i, ij}(r, r', t) = \delta^{ab} \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot (r - r')} \frac{k_i |k|}{\sinh \left( \frac{E_k}{T} \right)} \left( \delta_{ij} - \frac{k_i k_j}{|k|^2} \right). \tag{4.26}$$

Here, up to the lowest order of $g$ in this quantum gluonic matter, the operator $\dot{N}_{12ij}^{ab}(t)$ is written in Eq. (4.18) with (4.16). Thus, we can expand $\dot{N}_{12ij}^{ab}(t)$ as

$$\dot{N}_{12ij}^{ab}(t) = \delta^{ab} \sum_{n=1} [f(2n - 1)t^{2n-1}(S \cdot \hat{p})^{2n-1}]_{12ij}$$

$$= \delta^{ab} \left[ \delta_{1i} t(S \cdot \hat{p})_{2j} - \delta_{2j} t(S \cdot \hat{p})_{1i} + O(\hat{p}^3) \right], \tag{4.27}$$

where $f(x)$’s are expansion matrices with numerical factors for this expansion and $\hat{p} = -i \partial / \partial r$. Thus,\n
$$\int d^3 r \dot{N}_{12ij}^{ab}(t) \left[ \delta^3(r) \mathcal{T}_{i, km}^{ab}(r, 0, t) \right] = 0 \tag{4.28}$$

for any $i$, $j$, $k$ and $m$ because the integrand is the total derivative with respect to $r$. Thus, we can obtain the relation $\Pi^R(\omega) = 0$. Finally, from Eqs. (4.25) and (4.28), we conclude that the shear viscosity $\eta(\omega)$ in Eq. (4.10) is as follows:

$$\eta(\omega) = 0 \tag{4.29}$$
up to the lowest order of $g$ in the quantum gluonic matter. Here, the terms of the next order of $\hat{C}^{ab}_{ij}(t)$ and $\hat{D}^{ab}_{ij}(t)$ in $\hat{N}^{ab}_{12ij}(t)$ include the coupling constant $g$ without the spatial derivative. Thus, Eq. (4.28) is not satisfied in the order of $g$. Therefore, the result (4.29) is valid up to the order of $g^0$ while (4.23) is valid up to the order of $g^1$ at zero temperature because the mechanism to vanish the value of shear viscosity is different.

§5. Summary and concluding remarks

In this paper, the time-dependent variational method for the pure Yang-Mills gauge theory is formulated in the functional Schrödinger picture with the Gaussian trial wave functional. In this variational method, the classical mean fields and the quantum fluctuations around them are treated self-consistently and both degrees of freedom are coupled with each other. Further, the equations of motion for the quantum fluctuations around the mean fields were reformulated in a form of the Liouville-von Neumann equation for the reduced density matrix which was introduced in the Hartree-Bogoliubov approximation developed in the many-body problems for boson systems.

This variational method developed in this paper was applied to the pure quantum gluonic matter system in order to evaluate the shear viscosity, which is one of the transport coefficients of the gluonic matter in the system with the color $su(3)$ symmetry, namely, the QCD without quarks. As a result, it was shown that there is no contribution of the quantum gluons to the shear viscosity in the pure gluonic matter up to the lowest order of the QCD coupling $g$ at finite temperature. Namely, up to the order of $g^0$, the contribution of the quantum gluons to the shear viscosity is nothing. At zero temperature, adding to the order of $g^0$, there is no contribution up to the order of $g$ due to the equations of motion. Thus, for small $g$, namely, from the viewpoint of the weak coupling QCD, the shear viscosity in quantum gluonic matter may be small because the quantum gluons contribute to the shear viscosity from the order of $g$ or higher at finite temperature.

Recently, Matsui and Matsuo have given the transport equations for the Wigner distribution function and the anomalous distribution function which are coupled with each other to determine the dynamics of the meson fields and the fluctuations around them in the linear sigma model, as is similar to our formalism. In order to compare the theoretical analysis with the experimental results, the information of the gluon distribution function may be necessary as was discussed in Ref. 26 in the case of the linear sigma model. It is one of important further problems to investigate the gluon distribution function governed by the transport equation derived by this formalism such as the extended Boltzmann equations. Our approach should be related to another approach in terms of the extended Boltzmann equations, in which the improvement of our time-dependent variational approach should be investigated in comparison to the usual Boltzmann equation approach. These are left as future problems.

It may be interesting to investigate the higher order contribution to the shear viscosity because the approximation used in this paper corresponds to the Hartree-
Bogoliubov like approximation. The random phase approximation (RPA) is missing in the treatment in this paper. It may be necessary to extend our treatment to including the RPA like modes as was developed for the linear sigma model.\(^{27}\) Further, it may be also interesting to investigate the behavior of other transport coefficients in this framework developed in this paper. They are future problems.

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