Dirac Decomposition of Wheeler-DeWitt Equation
in the Bianchi Class A Models

Hidetomo YAMAZAKI* and Tetsuya HARA**

Department of Physics, Kyoto Sangyo University, Kyoto 603-8555, Japan

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The Wheeler-DeWitt equation in the Bianchi Class A cosmological models is expressed generally in terms of a second-order differential equation, like the Klein-Gordon equation. To obtain a positive definite probability density, a new method is investigated, which extends the Dirac Square Root formalism that factorizes the Wheeler-DeWitt equation into a first-order differential equation using the Pauli matrices. The solutions to the Dirac type equation in this method are expressed in terms of a two-component spinor form. The probability density defined by the solution is positive definite, and the conserved current is derived. A newly found spin-like degree of freedom leads to behavior, corresponding to evolution of the universe with an agitated anisotropy oscillation like Zitterbewegung.

§1. Introduction

Arnowitt, Deser and Misner1) (ADM) reformulated general relativity using a canonical formalism, which employs a 3+1 split of the space-time metric, and introduced some constraints through a variational principle. According to the ADM approach, the Einstein equation is derived as the equation of dynamical evolution. The canonical formalism using the ADM approach is a system of dynamics including two constraints: the momentum constraint and the Hamiltonian constraint. The latter governs the time evolution of the system. According to Dirac,2) the procedure of canonical quantization replaces the Hamiltonian constraint with a supplementary condition on the wave function which represents the quantum state of space-time. The Hamiltonian constraint thus becomes the Wheeler-DeWitt (WD) equation, which is fundamental to quantum gravity and determines the quantum state of space-time. The WD equations for specific cosmological models, like the Friedmann-Robertson-Walker model3) and the Bianchi models,4)-6) have been investigated.

The WD equation for the Bianchi models can in general be reduced to a second-order differential equation if the operator-ordering is neglected. The resultant WD equation is similar to a Klein-Gordon type, which has the problem that the probability density is not positive definite. The WD equation also has this kind of problem, as its probability density becomes negative. Negative values of the probability density have been found through numerical calculation by Furusawa,7) who treated the WD equation for the quantum mixmaster model using wave-packet solution.

To avoid the problem of negative probability density, an approach called the

* E-mail: yamazaki@cc.kyoto-su.ac.jp
** E-mail: hara@cc.kyoto-su.ac.jp
Dirac Square Root formalism has been studied. Since the Hamiltonian for Bianchi type models is expressed in terms of a quadratic, a first-order differential equation can be derived by applying a procedure similar to the Dirac method. Mallett applied this procedure to the Friedmann model coupled to a charged scalar field. Kim and Oh mainly studied the Bianchi type-IX model. In their papers, it is postulated the first-order equation which includes an unknown function that depends only on the time parameter, and the wave function is required to be a solution of a two-component column vector. Iterating the first-order equation thus postulated, a differential equation in the unknown function is yielded, so as to be consistent with the WD equation for each component. A differential equation with respect to the time parameter is thus obtained. Indeed, the differential equation is reduced to Riccati’s equation in the case of the Friedmann model with a charged scalar field and the Bernoulli equation in the case of the Bianchi type-IX model. However, it is difficult to obtain exact solutions in both cases.

The purpose of this paper is to propose a new method that factorizes the WD equation into a first-order differential equation, using the Pauli matrices. The resultant differential equation thus obtained is similar to the Dirac equation. As a necessary condition for the Dirac factorization, it is postulated that the resultant Hamiltonian is “self-adjoint”. We apply this method to the WD equation of all the Bianchi Class A vacuum models, except for type-IX model. The solution of the Dirac-type equation is written as a two-component column vector. Using the Dirac-type equation and the complex conjugate, we can define the probability density. Then, we derive the equation of continuity and the conserved current. Here, we restrict the form of Hamiltonian so as to make the probability density positive definite. It is to be noted that our formalism cannot be applied to the Bianchi type-IX model, because the structure of the anisotropic potential for this model is different from that for the other Bianchi Class A models.

The organization of this paper is as follows. In §2, we review the ADM canonical formulation for the vacuum Bianchi Class A models and introduce the WD equation. In §3, we propose a new method in which the WD equation is factorized into a first-order differential equation in terms of the Pauli matrices and apply the method to the Bianchi type-I, II, VI0, VII0 and VIII models. For each of these except for the type-VIII model, the Dirac-type equation is derived and solved. Then, we assume a trial function which is expressed in terms of a two-component spinor. We show that the probability density becomes positive definite and that there is a conserved current. As the time parameter approaches the limit of the singularity of the universe, the WD equations for the Bianchi type-II, VI0, VII0 and VIII models are reduced to that of the type-I model, so that the solutions of the Dirac-type equations approach the solution for the Bianchi type-I model. In §4, we investigate the behavior of the solution of the Dirac-type equation near the singularity in detail. It is shown that from the Heisenberg equation, the quantum behavior of the anisotropic parameters causes quivering motion, and the Hamiltonian system has a ‘spin-like’ degree of freedom. The ‘spin-like’ degree of freedom causes the universe going through a stage of evolution with agitated anisotropy oscillation. In §5, a summary and discussion are given on the results of our formalism and comparisons are made with the Dirac
Square Root formalism. Throughout the paper, the indices $i$, $j$ and $k$ run from 1 to 3, and we use units in which $c = \hbar = 16\pi G = 1$.

§2. Canonical quantization formulation for the Bianchi Class A models

The Einstein-Hilbert action is given by

$$I = \int \sqrt{-^{(4)}g}^{(4)}R d^4x,$$

where $^{(4)}g$ and $^{(4)}R$ are the determinant of the space-time metric and the curvature, respectively. Here the index (4) indicates a 4-geometry quantity. The metric for the Bianchi models is generally given by

$$ds^2 = -N^2 dt^2 + e^{2\alpha} e^{2\beta_{ij}} \chi^i \chi^j,$$

where $N$ (the lapse function), $\alpha$ and $\beta_{ij}$ are functions of $t$ only. The quantity $e^{2\alpha}$ plays the role of a scale factor, so that the universe realizes the initial singularity as $\alpha \to -\infty$, $\beta_{ij}$ are the anisotropic parameters of the universe, which form a traceless matrix, and the $\chi^i$ are 1-forms. According to Misner, the anisotropic parameters $\beta_{ij}$ are expressed in terms of two parameters, $\beta_+$ and $\beta_-$ (for the Bianchi Class A and Bianchi type-V models) as

$$\beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+).$$

The 1-forms obey

$$d\chi^i = \frac{1}{2} C^i_{jk} \chi^j \wedge \chi^k,$$

where the $C^i_{jk}$ are the structure constants.

The Bianchi models are characterized by the structure constants and are classified into two groups according to them, the so-called Class A and B models. It should be noted that, as was first indicated by Hawking, it cannot be reduced to the correct field equations by the ADM canonical formulation in the Bianchi Class B models except for the Bianchi type-V model. The reason is that the spatial divergence term does not vanish. Hence, we consider only the Bianchi Class A models, because we use a canonical formulation. The characteristic of the Class A models is that the structure constants obey the condition

$$C^i_{ji} = 0.$$

We must therefore select the 1-forms so as to satisfy Eq. (2.5).

After carrying out the procedure for the ADM decomposition, we use the coordinates $\alpha$, $\beta_+$ and $\beta_-$ and the corresponding canonical momenta, as $p_\alpha$, $p_+$ and $p_-$. From the variation of $N$, one obtains the Hamiltonian constraint. The 3-geometry curvature $^{(3)}R$ is obtained from the structure constants. According to Misner, the relation of the curvature $^{(3)}R$ to the anisotropic potential $V(\beta_+, \beta_-)$ is introduced by

$$^{(3)}R = \frac{3}{2} e^{2\alpha} (1 - V(\beta_+, \beta_-)).$$
As a result, the Hamiltonian becomes
\[ H = -p_\alpha^2 + p_+^2 + p_-^2 + e^{4\alpha}(V - 1). \] (2.7)

In the canonical quantization procedure, we use the Schrödinger representation of the operators \( p_\alpha, p_+, \) and \( p_- :\)
\[ \hat{p}_\alpha = -i \frac{\partial}{\partial \alpha}, \quad \hat{p}_+ = -i \frac{\partial}{\partial \beta_+}, \quad \hat{p}_- = -i \frac{\partial}{\partial \beta_-}. \] (2.8)

Because the Hamiltonian constraint is a first-class constraint, the quantized Hamiltonian constraint must be satisfied by the supplementary condition
\[ \hat{H}\Psi = 0, \] (2.9)

\[ \hat{H} \equiv \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + e^{4\alpha}(V - 1), \] (2.10)

where \( \Psi = \Psi(\alpha, \beta_+, \beta_-) \) is a state vector of the system, called the “wave function of the universe”. This is the Wheeler-DeWitt equation for the Bianchi models.

§3. Dirac factorization for the Wheeler-DeWitt equation

According to Furusawa’s discussion, \(^7\) the probability density of the Klein-Gordon type,
\[ \rho(\alpha, \beta_+, \beta_-) = \frac{i}{2} (\Psi^* \partial_\alpha \Psi - \Psi \partial_\alpha \Psi^*) \]
becomes negative for some values of \( \alpha \). Negative values of the probability density are inconsistent with physical theories, and this is the defect of the WD equation. To investigate and solve this problem, we adopt procedure similar to that used by Dirac to factorize the Klein-Gordon equation in order to obtain a positive definite probability density;\(^*\) that is, we attempt to factorize the WD equation into a first-order differential equation. We refer to the first-order equation as a Dirac-type equation. As a necessary condition of Dirac factorization, it is postulated that the resultant Hamiltonian is “self-adjoint”, because one may attempt various methods of factorization into a first-order equation, and there are no rules for such factorization. We suppose that the WD equation for the Bianchi models is replaced by the equation of the Dirac-type
\[ \mathcal{H}\Psi \equiv \left( \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v \right) \Psi = 0, \] (3.1)

where \( v \equiv \sqrt{V - 1} \), and the Pauli matrices \( \sigma_i \) are expressed as
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

\(^*\) In principle, if we write \( \hat{h} = \left[ \hat{p}_+^2 + \hat{p}_-^2 + e^{4\alpha}(V - 1) \right]^{1/2} \), the scalar function \( \Psi \) which satisfies \( i \frac{\partial}{\partial \alpha} \Psi = \hat{h} \Psi \) could be used to obtain a positive definite probability.\(^{15}\) However, the method to solve the equation is not well understood, except in the case of \( V = 1 \).
We here rewrite the Dirac-type equation (3.1) in the Schrödinger form

\[ i \frac{\partial}{\partial \alpha} \Psi = \left[ \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v \right] \Psi. \tag{3.2} \]

The self-adjointness of the Hamiltonian is easily proved. We have

\[ \mathcal{H}^\dagger = \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v = \mathcal{H}, \tag{3.3} \]

where the momentum operators are self-adjoint and commute with the Pauli matrices.

With the help of relations (2.8), the Dirac-type equation (3.1) can be rewritten as

\[ -i \frac{\partial \Psi}{\partial \alpha} - i\sigma_1 \frac{\partial \Psi}{\partial \beta_+} - i\sigma_2 \frac{\partial \Psi}{\partial \beta_-} + \sigma_3 e^{2\alpha} v \Psi = 0, \tag{3.4} \]

and the corresponding adjoint equation is

\[ i \frac{\partial \Psi^\dagger}{\partial \alpha} + i \frac{\partial \Psi^\dagger}{\partial \beta_+} \sigma_1 + i \frac{\partial \Psi^\dagger}{\partial \beta_-} \sigma_2 + \Psi^\dagger \sigma_3 e^{2\alpha} v \Psi = 0. \tag{3.5} \]

Multiplying Eq. (3.4) by \( \Psi^\dagger \) from the left-hand side, we have

\[ -i \Psi^\dagger \frac{\partial \Psi}{\partial \alpha} - i\Psi^\dagger \sigma_1 \frac{\partial \Psi}{\partial \beta_+} - i\Psi^\dagger \sigma_2 \frac{\partial \Psi}{\partial \beta_-} + \Psi^\dagger \sigma_3 e^{2\alpha} v \Psi = 0. \tag{3.6} \]

Subtracting Eq. (3.6) from Eq. (3.5) multiplied by \( \Psi \) from the right-hand side, we obtain

\[ \frac{\partial}{\partial \alpha} \Psi^\dagger \Psi + \frac{\partial}{\partial \beta_+} \left( \Psi^\dagger \sigma_1 \Psi \right) + \frac{\partial}{\partial \beta_-} \left( \Psi^\dagger \sigma_2 \Psi \right) = 0. \tag{3.7} \]

Setting the probability density \( \rho \) and the current vectors \( j_+ \) and \( j_- \) as

\[ \rho = \Psi^\dagger \Psi, \quad j_+ = \Psi^\dagger \sigma_1 \Psi, \quad j_- = \Psi^\dagger \sigma_2 \Psi, \tag{3.8} \]

we get the equation of continuity,

\[ \frac{\partial \rho}{\partial \alpha} + \frac{\partial j_+}{\partial \beta_+} + \frac{\partial j_-}{\partial \beta_-} = 0. \tag{3.9} \]

Then, one sees that the probability density is positive definite. We apply our formalism to solve the Bianchi type-I, II, VI\(_0\), VII\(_0\) and VIII models below.

It is to be noted that this factorization does not correspond exactly to the WD equation. By applying the operator \( (\hat{p}_\alpha - \sigma_1 \hat{p}_+ - \sigma_2 \hat{p}_- - \sigma_3 e^{2\alpha} v) \) to Eq. (3.1), it becomes, using the relation (2.8),

\[ \left[ \hat{p}_\alpha^2 - \hat{p}_+^2 - \hat{p}_-^2 - e^{4\alpha} v^2 - e^{2\alpha} \left( 2i\sigma_3 v + \sigma_1 \frac{\partial v}{\partial \beta_-} - \sigma_2 \frac{\partial v}{\partial \beta_+} \right) \right] \Psi = 0. \tag{3.10} \]

Regarding the extra terms as a new constraint on the wave function as

\[ \left( 2i\sigma_3 v + \sigma_1 \frac{\partial v}{\partial \beta_-} - \sigma_2 \frac{\partial v}{\partial \beta_+} \right) \Psi = 0, \tag{3.11} \]
and solving the constraint, we cannot obtain solutions that satisfy the Dirac-type equation, except with \( v = 0 \). For this reason, we changed the constraint to

\[
\langle \Psi | \left( 2i\sigma_3 v + \sigma_1 \frac{\partial v}{\partial \beta_-} - \sigma_2 \frac{\partial v}{\partial \beta_+} \right) | \Psi \rangle = 0, \tag{3.12}
\]

but again failed to find solutions.

Because of finding no solutions when the above conditions are imposed, we consider only the Dirac-type equation without such constraint conditions on the wave function. Then the solutions of the Dirac-type equation that do not satisfy the constraint of (3.11) and/or (3.12) are not solutions of WD equation. The situation here is the same as that in which the solutions of the Dirac equation do not satisfy the Klein-Gordon equation when there is a potential which is a function of space-time coordinates.

3.1. The Bianchi type-I model

The metric of the Bianchi type-I model is given in Eq. (2.2), in which the 1-forms are given by

\[
\chi^1 = dx^1, \quad \chi^2 = dx^2, \quad \chi^3 = dx^3. \tag{3.13}
\]

The structure constants in the model obviously imply

\[
C^i_{\ jk} = 0. \tag{3.14}
\]

It leads to the zero curvature of 3-dimensional hypersurfaces, \(^3R_1 = 0\), which implies a spatially flat structure. From this, we find that the anisotropic potential becomes \( V_1 = 1 \). Then, the WD equation is reduced to

\[
\left( -\hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2 \right) \Psi = 0. \tag{3.15}
\]

We postulate here that the Dirac-type equation for the Bianchi type-I model holds:

\[
\mathcal{H}_1 \Psi_1(\alpha, \beta_+, \beta_-) = (\hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_-) \Psi = 0. \tag{3.16}
\]

The solutions to this equation are known.\(^5,6\) They are plane wave solutions possessing the two-component spinor form

\[
\Psi_1(\alpha, \beta_+, \beta_-) = e^{i(k_+\beta_++k_-\beta_-+\omega\alpha)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.17}
\]

where \( \omega \) and \( k_\pm \) are arbitrary real constants. Substituting the solution (3.17) into the Dirac-type equation (3.16), we get the components

\[
\Psi_1 = \frac{k_+-ik_-}{\omega} \psi_2 \quad \text{and} \quad \Psi_2 = \frac{k_++ik_-}{\omega} \psi_1. \tag{3.18}
\]

This obviously implies \( |\psi_1|^2 = |\psi_2|^2 \), and the relation between \( \omega \) and \( k_\pm \) is

\[
\omega = \pm \sqrt{k_+^2 + k_-^2}. \tag{3.19}
\]
Thus, the parameter $\omega$ corresponding to the frequency of the plane wave depends on the parameters $k_+$ and $k_-$ corresponding to the wavenumber, which vary continuously.

The probability density is

$$\rho_I(\alpha, \beta_+, \beta_-) = \Psi_I^\dagger \Psi_I = |\psi_1|^2 + |\psi_2|^2 = 2|\psi_1|^2,$$

which is positive definite. The current vectors $j_+$ and $j_-$ are

$$j_+ = \psi_2^* \psi_1 + \psi_1^* \psi_2$$

and

$$j_- = i\psi_2^* \psi_1 - i\psi_1^* \psi_2,$$

which take some constant values. Therefore, the equation of continuity (3.9) holds, and there is a conserved current. From the parameters (2.3) and the results of Eqs. (3.21) and (3.22), we find that the evolution of the quantized universe is whether one of the three spatial axes is expanding and two axes are contracting or one axis is contracting and two axes are expanding.

As mentioned above, it should be noted here that since the WD equation and the Dirac-type equation for the Bianchi type-II, VI$_0$, VII$_0$ and VIII models are reduced to those of the Bianchi type-I model in the singularity limit ($\alpha \rightarrow -\infty$), all models have the same property that the behavior near the singularity becomes that of the Bianchi type-I solution. We discuss the quantized Hamiltonian system of the Bianchi type-I model in §4.

3.2. The Bianchi type-II model

The metric of the Bianchi type-II model is expressed as Eq. (2.2), and the 1-forms are given by

$$\chi^1 = dx^1 - x^3 dx^2, \quad \chi^2 = dx^2, \quad \chi^3 = dx^3.$$  

(3.23)

Substituting the 1-forms (3.23) into Eq. (2.4), we get the structure constants:

$$C_{23}^1 = -C_{32}^1 = 1.$$  

(3.24)

The other components are zero. The curvature is

$$^{(3)}R_{II} = -\frac{1}{2} e^{2\alpha + 4\beta_+ + 4\sqrt{3}\beta_-},$$  

(3.25)

and the anisotropic potential $V_{II}(\beta_+, \beta_-)$ is obtained from the relation (2.6) as

$$V_{II} - 1 = \frac{1}{3} e^{4\beta_+ + 4\sqrt{3}\beta_-}.$$  

(3.26)

Then, the WD equation is given by

$$\left( -\hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2 + \frac{1}{3} e^{4\alpha + 4\beta_+ + 4\sqrt{3}\beta_-} \right) \Psi = 0.$$  

(3.27)
We postulate here that the factorized Hamiltonian can be expressed as
\[ \mathcal{H}_{II} = \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \frac{\sigma_3}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \tag{3.28} \]
and that the Dirac-type equation holds:
\[ \mathcal{H}_{II} \Psi_{II} = 0. \tag{3.29} \]

With the help of the relation (2.8), this becomes
\[ i \frac{\partial \Psi_{II}}{\partial \alpha} = \begin{pmatrix} \frac{1}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} & -i \frac{\partial}{\partial \beta_+} - \frac{\partial}{\partial \beta_-} \\ -i \frac{\partial}{\partial \beta_+} + \frac{\partial}{\partial \beta_-} & - \frac{1}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \end{pmatrix} \Psi_{II}. \tag{3.30} \]

Now we look for a solution to this equation. We consider a trial function that gives a two-component spinor form and note that the form of the Dirac-type equation is
\[ \Psi_{II} = \exp \left[ - \frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.31} \]
where the spinor components \( \psi_1 \) and \( \psi_2 \) are arbitrary constants. Substituting the trial function (3.31) into Eq. (3.30), we get the relation of the spinor components
\[ \psi_2 = -\psi_1. \tag{3.32} \]

Here we have used the condition that the Dirac-type equation (3.29) for the Bianchi type-II model is reduced to that for the Bianchi type-I model in the singularity limit \( (\alpha \to -\infty) \). The trial function (3.31) would include such asymptotic behavior. The trial function involving the plane wave part is expressed as
\[ \Psi_{II} = e^{i(k_+ \beta_+ + k_- \beta_- - \omega \alpha)} \exp \left[ - \frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \right] \begin{pmatrix} \psi_1 \\ -\psi_1 \end{pmatrix}. \tag{3.33} \]

Substituting the solution (3.33) into Eq. (3.30), we have
\[ \begin{pmatrix} -\omega & k_+ - ik_- \\ k_+ + ik_- & -\omega \end{pmatrix} \begin{pmatrix} \psi_1 \\ -\psi_1 \end{pmatrix} = 0. \tag{3.34} \]

Therefore, the coefficients must satisfy
\[ k_+ = -\omega, \quad k_- = 0. \tag{3.35} \]

Consequently, the solution can be expressed in terms of the product of the amplitude and oscillation parts, restricted by Eq. (3.35):
\[ \Psi_{II} = e^{-i\omega(\beta_+ + \alpha)} \exp \left[ - \frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \right] \begin{pmatrix} \psi_1 \\ -\psi_1 \end{pmatrix}. \tag{3.36} \]
The probability density is obtained, by using the wave function (3.36), as
\[ \rho_{\Pi}(\alpha, \beta_+, \beta_-) = \Psi_{\Pi}^\dagger \Psi_{\Pi} = 2|\psi_1|^2 \exp \left[-\frac{1}{3}e^{2\alpha+2\beta_+ + 2\sqrt{3}\beta_-} \right]. \] (3.37)

This is always positive definite. One sees that this probability density goes to zero as \( \beta_+ + \sqrt{3}\beta_- \to \infty \), while it approaches a certain constant as \( \beta_+ + \sqrt{3}\beta_- \to -\infty \).

The current vectors \( j_+ \) and \( j_- \) given by Eq. (3.8) are
\[ j_+ = -2|\psi_1|^2 \exp \left[-\frac{1}{3}e^{2\alpha+2\beta_+ + 2\sqrt{3}\beta_-} \right], \] (3.38)
and
\[ j_- = 0. \] (3.39)

Hence, the equation of continuity (3.9) holds, and there is a conserved current. Moreover, it is shown that the wave part propagates in the direction of the \( \beta_+ \) axis only, in accordance with Eq. (3.35).

3.3. The Bianchi type-\( V\!<\!VI \) model

The metric of the Bianchi type-\( V\!<\!VI \) model is expressed as Eq. (2.2), and the 1-forms are given by\(^{16}\)
\[ \chi^1 = \cosh x^3 dx^1 - \sinh x^3 dx^2, \quad \chi^2 = -\sinh x^3 dx^1 + \cosh x^3 dx^2, \quad \chi^3 = dx^3. \] (3.40)

The structure constants are
\[ C_{23}^1 = -C_{32}^1 = 1, \quad C_{31}^2 = -C_{13}^2 = -1, \] (3.41)
with the rest zero. The anisotropic potential becomes
\[ V_{VI_0} - 1 = \frac{4}{3}e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) + 1). \] (3.42)

Then, the WD equation for the Bianchi type-\( V\!<\!VI \) model is given by
\[ \left[-\hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2 + \frac{4}{3}e^{4\alpha+4\beta_+} (\cosh(4\sqrt{3}\beta_-) + 1) \right] \Psi = 0. \] (3.43)

We postulate that the Dirac-type equation for the Bianchi type-\( V\!<\!VI \) model can be expressed as
\[ \mathcal{H}_{VI_0} \Psi_{VI_0} = \left[\hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \frac{2}{\sqrt{3}}\sigma_3 e^{2\alpha+2\beta_+} \left(e^{2\sqrt{3}\beta_-} + e^{-2\sqrt{3}\beta_-} \right) \right] \Psi_{VI_0} = 0. \] (3.44)

As in the case of the Bianchi type-II model, we assume a trial function of the form
\[ \Psi_{VI_0} = \exp \left[-\frac{1}{3}e^{2\alpha+2\beta_+} \left(e^{2\sqrt{3}\beta_-} - e^{-2\sqrt{3}\beta_-} \right) \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \] (3.45)
where the components $\psi_1$ and $\psi_2$ are arbitrary constants. Substituting the trial function (3.45) into the Dirac-type equation (3.44), we get

$$\psi_2 = -\psi_1.$$  \hfill (3.46)

From the condition that the wave function (3.45) reduces to the plane wave solution in the singularity limit, we obtain

$$k_+ = -\omega, \quad k_- = 0.$$  \hfill (3.47)

The wave function involving the plane wave part is finally obtained as

$$\Psi_{VI_0} = e^{-i\omega(\beta_+ + \alpha)} \exp \left[ -\frac{1}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} - e^{-2\sqrt{3}\beta_-} \right) \right] \begin{pmatrix} \psi_1 \\ -\psi_1 \end{pmatrix}. \hfill (3.48)$$

The probability density becomes

$$\rho_{VI_0}(\alpha, \beta_+, \beta_-) = \Psi_{VI_0}^\dagger \Psi_{VI_0} = 2|\psi_1|^2 \exp \left[ -\frac{2}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} - e^{-2\sqrt{3}\beta_-} \right) \right], \hfill (3.49)$$

which is positive definite. The current vectors are

$$j_+ = -2|\psi_1|^2 \exp \left[ -\frac{2}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} - e^{-2\sqrt{3}\beta_-} \right) \right], \quad j_- = 0.$$  \hfill (3.50)

We see that the equation of continuity holds and there is a conserved current. The plane wave part of the wave function (3.48) propagates in the direction of the $\beta_+$ axis only.

3.4. The Bianchi type-VII$_0$ model

The metric of the Bianchi type-VII$_0$ model is expressed as Eq. (2.2), and the 1-forms are given by $^{(16)}$

$$\chi^1 = \cos x^1 dx^2 + \sin x^1 dx^3, \quad \chi^2 = -\sin x^1 dx^2 + \cos x^1 dx^3, \quad \chi^3 = dx^1. \hfill (3.51)$$

The structure constants are

$$C^{1}_{23} = -C^{1}_{32} = -1, \quad C^{2}_{31} = -C^{2}_{13} = -1,$$  \hfill (3.52)

with the rest zero. The anisotropic potential is

$$V_{VII_0} - 1 = \frac{4}{3} e^{4\beta_+} \left( \cosh(4\sqrt{3}\beta_-) - 1 \right). \hfill (3.53)$$

Then, the WD equation for the Bianchi type-VII$_0$ model is given by

$$\left[ \hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2 + \frac{4}{3} e^{4\alpha + 4\beta_+} \left( \cosh(4\sqrt{3}\beta_-) - 1 \right) \right] \Psi = 0.$$

Here, we postulate that the Dirac-type equation with a factorized Hamiltonian is expressed as

$$\mathcal{H}_{VI_0} \Psi_{VII_0} \equiv \left[ \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \frac{2}{\sqrt{3}} \sigma_3 e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} - e^{-2\sqrt{3}\beta_-} \right) \right] \Psi_{VII_0} = 0.$$  \hfill (3.55)
As in the case of previous models, we can find the wave function

$$\Psi_{\text{VII}_0} = e^{-i\omega(\beta_+ + \alpha)} \exp \left[ -\frac{1}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} + e^{-2\sqrt{3}\beta_-} \right) \right] \begin{pmatrix} \psi_1 \\ -\psi'_1 \end{pmatrix}. \quad (3.56)$$

The probability density then becomes

$$\rho_{\text{VII}_0}(\alpha, \beta_+ + \beta_-, \beta_-) = \Psi_{\text{VII}_0}^\dagger \Psi_{\text{VII}_0} = 2 |\psi_1|^2 \exp \left[ -\frac{2}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} + e^{-2\sqrt{3}\beta_-} \right) \right], \quad (3.57)$$

which is positive definite. The current vectors are

$$j_+ = 2 |\psi_1|^2 \exp \left[ -\frac{2}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3}\beta_-} + e^{-2\sqrt{3}\beta_-} \right) \right], \quad j_- = 0. \quad (3.58)$$

We thus see that the equation of continuity holds and there is a conserved current. The plane wave part of the wave function (3.56) propagates in the direction of the $\beta_+$ axis only. In the limit $\beta_+ \to -\infty$, the value of the probability density approaches a certain constant, while it converges to zero as $\beta_+ \to \infty$ or $\beta_- \to \pm \infty$.

### 3.5. The Bianchi type-VIII model

The metric of the Bianchi type-VIII model is expressed as Eq. (2.2), and the 1-forms are given by

\[
\begin{align*}
\chi^1 &= \cosh x^2 \cos x^3 dx^1 - \sin x^3 dx^2, \\
\chi^2 &= \cosh x^2 \sin x^3 dx^1 + \cos x^3 dx^2, \\
\chi^3 &= \sinh x^2 dx^1 + dx^3.
\end{align*}
\]

The structure constants are

$$C_{23}^1 = -C_{32}^1 = -1, \quad C_{31}^2 = -C_{13}^2 = -1, \quad C_{12}^3 = -C_{21}^3 = 1, \quad (3.60)$$

with the rest zero. The anisotropic potential becomes

$$V_{\text{VIII}} - 1 = \frac{1}{3} \left( e^{-8\beta_+} - 2 e^{4\beta_+} + 4 e^{-2\beta_+} \cosh(2\sqrt{3}\beta_-) + 2 e^{4\beta_+} \cosh(4\sqrt{3}\beta_-) \right). \quad (3.61)$$

However, since the anisotropic potential (3.61) is a very complicated expression, we cannot determine the solution of the Dirac-type equation in the form of Eq. (3.1).

### §4. The behavior near the singularity

It is interesting to consider the behavior of the wave function near the singularity. Misner$^4$ investigated the Bianchi type-IX (mixmaster) model, which is represented by a classical dynamical system in which an imaginary particle moves in the anisotropic potential of an approximated regular triangle. This can be regarded as a billiard dynamical system in the $\beta_+ - \beta_-$ plane. For this reason, Misner’s picture is very helpful to study the universe in terms of the anisotropic parameters.

The vacuum Bianchi type-I universe is the so-called Kasner universe. The evolution of the model in the classical region represents an anisotropically expanding
universe in which two axes are expanding and the other is contracting. In the quantum region, a quadratic Hamiltonian is derived using the ADM approach, and the evolution of the quantized universe is represented by a WD equation like a Klein-Gordon type. Here, there is the problem that the probability density could become negative as in the case of the Klein-Gordon equation.

Let us consider to change the point of view of our formalism. Near the singularity ($\alpha \to -\infty$), the Dirac-type equations for the Bianchi Class A models are reduced to that of Bianchi type-I,

$$i \hbar \frac{\partial \Psi}{\partial \alpha} = (\sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_-) \Psi \equiv \hat{h}' \Psi. \quad (4.1)$$

This dynamical system describes a massless imaginary particle moving in a constant (zero) potential. The equations of motion in the Heisenberg representation $\hat{\beta}_+ \hbar$ and $\hat{\beta}_- \hbar$ for $\hat{\beta}_+$ and $\hat{\beta}_-$ are

$$\frac{\partial \hat{\beta}_+ \hbar}{\partial \alpha} = i \left[ \hat{h}', \hat{\beta}_+ \hbar \right] = i e^{i \alpha \hat{h}'} \sigma_1 \left[ \hat{p}_+, \hat{\beta}_+ \right] e^{-i \alpha \hat{h}'} = \hat{\sigma}_1 \hbar \quad (4.2)$$

and

$$\frac{\partial \hat{\beta}_- \hbar}{\partial \alpha} = i \left[ \hat{h}', \hat{\beta}_- \hbar \right] = i e^{i \alpha \hat{h}'} \sigma_2 \left[ \hat{p}_-, \hat{\beta}_- \right] e^{-i \alpha \hat{h}'} = \hat{\sigma}_2 \hbar, \quad (4.3)$$

where the commutative relation $[\hat{\sigma}_b, \hat{p}_a] = i \delta_{ab} (a, b = +, -)$ and notation $\hat{A}_H = e^{i \alpha \hat{h}'} \hat{A} e^{-i \alpha \hat{h}'}$ are used.

To see the motion further, we note that the equation for $\hat{\sigma}_1 \hbar$ is

$$\frac{\partial \hat{\sigma}_1 \hbar}{\partial \alpha} = i \left[ \hat{h}', \hat{\sigma}_1 \hbar \right] = i(\hat{h}' \hat{\sigma}_1 \hbar + \hat{\sigma}_1 \hbar \hat{h}') - 2i \hat{\sigma}_1 \hbar \hat{h}' = 2i \hat{p}_+ - 2i \hat{\sigma}_1 \hbar \hat{h}'. \quad (4.4)$$

Because $\hat{p}_+$ and $\hat{h}'$ are independent of $\alpha$, the above equation can be integrated to yield

$$\hat{\sigma}_1 \hbar(\alpha) = \left( \hat{\sigma}_1 \hbar(0) - \frac{\hat{p}_+}{\hat{h}'} \right) e^{-2i \hat{h}' \alpha} + \frac{\hat{p}_+}{\hat{h}'} \hat{h}'. \quad (4.5)$$

Then the solution for $\hat{\beta}_+ \hbar$ is given by

$$\hat{\beta}_+ \hbar(\alpha) = \hat{\beta}_+ \hbar(0) + \frac{\hat{p}_+}{\hat{h}'} \alpha + i \left( \hat{\sigma}_1 \hbar(0) - \frac{\hat{p}_+}{\hat{h}'} \right) e^{-2i \hat{h}' \alpha} \frac{2 \hat{h}'}{\hat{h}'}. \quad (4.6)$$

where $\hat{\beta}_+ \hbar(0)$ and $\hat{\sigma}_1 \hbar(0)$ are constants. The last term of the above equation is the oscillating part.

The oscillation of the last term in Eq. (4.6) is similar to Zitterbewegung\(^{18}\) (rapidly oscillating motion) of an electron, so that the imaginary particle displays a trembling motion that corresponds to the evolution of the universe. This behavior in quantum mechanics with a factorized Hamiltonian is different from that for the WD equation.

Here, we set the orbital angular momentum $\hat{m}'$ as

$$\hat{m}' = \hat{\beta}_+ \hbar - \hat{\beta}_- \hbar. \quad (4.7)$$
The Heisenberg equation of motion for the orbital angular momentum is

\[
\frac{d\hat{m}'_H}{d\alpha} = i \left[ \hat{h}', \hat{m}'_H \right] = \hat{\sigma}_1 H \hat{p}_- - \hat{\sigma}_2 H \hat{p}_+ .
\] (4.8)

Furthermore, we have

\[
\frac{d\hat{\sigma}_3 H}{d\alpha} = i \left[ \hat{h}', \hat{\sigma}_3 H \right] = 2 \hat{\sigma}_2 H \hat{p}_+ - 2 \hat{\sigma}_1 H \hat{p}_- .
\] (4.9)

The quantity \(\hat{m}' + \frac{1}{2} \hat{\sigma}_3\) is a constant of motion. We can interpret this result as the massless imaginary particle having a ‘spin-like’ angular momentum \(\frac{1}{2} \hat{\sigma}_3\). The Dirac-type equation (4.1) is similar to the Weyl equation, which has a two-dimensional spatial part in this case. We know that the Weyl equation describes the behavior of a massless neutrino with a half spin. In the analogy of this characteristic with the Weyl equation, then, the massless imaginary particle would have a half spin. Accordingly, we believe that the Dirac-type equation for the Bianchi type-I model describes the universe as a massless imaginary particle with a ‘spin-like’ degree of freedom. Thus, with these ‘spin-like’ degrees of freedom the universe experiences an early quantum stage of evolution with anisotropy oscillation. However, the proper physical interpretation of the ‘spin-like’ degree of freedom is not yet clear. In conclusion, we found that the Dirac-type equation for the Bianchi type-I model exhibits interesting quantum behavior. This contradicts Misner’s conclusion that the status of the wave function remains classical near the initial singularity.  

§5. Conclusions and discussion

We have investigated the first-order equation using the Pauli matrices by factorizing the WD equation for all of the vacuum Bianchi Class A models, except for the type-IX model. We have derived a solution expressed in terms of two-component spinors for the Bianchi type-I, II, VI\(_{0}\) and VII\(_{0}\) models. It has been shown that the probability density becomes positive definite and the equation of continuity holds with a conserved current. However, we could not find a solution for the Bianchi type-VIII model because the form of anisotropic potential is very complicated. In the Bianchi type-II model, it is possible to factorize the quadratic Hamiltonian of the WD equation into self-adjoint Hamiltonians that differ from the Hamiltonian form of Eq. (3.28). However, we cannot extract physically interesting solutions to the Dirac-type equations with such Hamiltonians. Similar results were obtained for the Bianchi type-VI\(_{0}\) and type-VII\(_{0}\) models.

The Dirac-type equations for the Bianchi type-II, VI\(_{0}\) and VII\(_{0}\) model reduce to that for the Bianchi type-I model near the singularity (\(\alpha \to -\infty\)). The quantized universe near the singularity can be expressed as a wave solution with a two-component spinor, and the anisotropic parameters \(\hat{\beta}_+\) and \(\hat{\beta}_-\) display Zitterbewegung motion. This Zitterbewegung is related to the ‘spin-like’ degree of freedom, and thus the ‘spin-like’ degree of freedom exhibits behavior that corresponds to a phenomenon in which the universe goes through an early quantum stage of evolution with anisotropy oscillation. In the context of this Dirac-type equation, the Zitterbewegung is an in-
interesting subject to investigate problems related to the big bang, inflation, and the initial singularity.

As mentioned in §1, our method cannot be applied to the Bianchi type-IX model, because the anisotropic potential of the Bianchi type-IX $V_{IX}$ is given by

$$V_{IX} = \frac{1}{3}e^{-8\beta_+} - \frac{4}{3}e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + \frac{2}{3}e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1) + 1 > 0. \quad (5.1)$$

The quantity $v^2 = V_{IX} - 1$ takes negative values as well as positive values here. This means that the quantity $v = \sqrt{V_{IX} - 1}$, which appears in the factorized theory, turns out to be imaginary as well as real. Therefore, we cannot factorize the quadratic Hamiltonian into a fixed form. Hence, we could not use the Pauli matrices, but the Dirac matrices. To remedy this defect, we note that the anisotropic potential $V_{IX}$ itself is non-negative, and we carry out the Dirac procedure of factorization with anticommuting $\gamma^\mu$ ($\mu = 0, 1, 2, 3$) and $\gamma_5$ matrices. One possibility is to make the factorization in the form

$$\left(\gamma^0 p_\alpha + \gamma^1 p_+ + \gamma^2 p_- + \gamma^3 v' e^{2\alpha} + \gamma_5 e^{2\alpha}\right)\Psi = 0, \quad (5.2)$$

where we set $v' = \sqrt{V_{IX}}$. However, as we have seen, because the anisotropic potential $V_{IX}$ is very complicated, we cannot determine the solution to the Dirac-type equation (5.2) with the factorized Hamiltonian.

We now briefly compare our formalism with the Dirac Square Root (DSR) formalism. In the DSR, it is postulated that the factorized Dirac-type equation for the Bianchi type-I model can be expressed as

$$\left(i \frac{\partial}{\partial \alpha} - i\sigma_1 \frac{\partial}{\partial \beta_+} - i\sigma_2 \frac{\partial}{\partial \beta_-}\right)\Psi(\alpha, \beta_+, \beta_-) = 0. \quad (5.3)$$

The Dirac-type equation (5.3) then implies that the WD equation for the Bianchi type-I model is

$$\left(-\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2}\right)\Psi(\alpha, \beta_+, \beta_-) = 0, \quad (5.4)$$

for each component of the wave function. Following Mallett, it is postulated that the Dirac-type equation can be expressed as

$$\left(i \frac{\partial}{\partial \alpha} - i\sigma_1 \frac{\partial}{\partial \beta_+} - i\sigma_2 \frac{\partial}{\partial \beta_-} + iW(\alpha)\right)\Psi(\alpha, \beta_+, \beta_-) = 0, \quad (5.5)$$

where $W(\alpha)$ is an unknown function depending on $\alpha$ only.

In a paper by Kim and Oh, the first-order equation (5.5) is applied to the Bianchi type-IX model. However, it seems that it is not satisfactory to apply the DSR formalism to the WD equation including an anisotropic potential term like (5.1) and to factorize into the form of the Dirac-type equation (5.5), because the differential operators of $\beta_+$ and $\beta_-$ do not act on the unknown function $W(\alpha)$, although the original WD equation has a potential written in terms of $\beta_+$ and $\beta_-$. By contrast, we apply our formalism to the Bianchi Class A models because it can be used to factorize the WD equation including the anisotropic potential term into the Dirac-type equation (3.1). We would like to regard this as an extended DSR formalism.
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