The stability and masses of disc galaxies

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Received 1981 November 12; in original form 1981 September 2

Summary. Using \textit{N}-body experiments we investigate the global stability of a series of models designed to match the observed photometric and kinematic properties of disc galaxies. The models, therefore, have an exponential surface density profile and rotation curves which are flat at large radii. We find a simple delineator of stability to bar-like modes for a cold disc: $v_m/(\alpha M_D G)^{1/2} = 1.1$, where $v_m$ is the maximum rotational velocity, $\alpha^{-1}$ is the scale length of the exponential disc and $M_D$ is the total disc mass. This is to be compared to a self-gravitating exponential disc for which $v_m/(\alpha M_D G)^{1/2} = 0.63$, thus a hot ‘halo’ component is required to increase this ratio in a cool disc and provide stability to bar formation. This criterion has been found to apply independent of the concentration of the halo (bulge) component. In addition, the results appear to be insensitive to a substantial increase in random motions in the central part of the disc and also to changes in the shape of the rotation curve or the adopted surface density profile. The results from the models are compared with available photometric and kinematic data of disc galaxies and are used to set limits on the mass-to-light ratio of the disc component. In particular we find that the cool disc component of unbarred Sc galaxies must have $(M/L)_D \leq 1.5h$ (where $h$ is Hubble’s constant in units of 100 km s$^{-1}$ Mpc$^{-1}$) compared with a total mass-to-light ratio interior to the Holmberg radius of $\sim 8h$, derived from their rotation curves. Normalizing the stellar population models of Larson & Tinsley (1978) and Tinsley (1981) to the results for Sc galaxies and data in the solar neighbourhood, we find that the later-type galaxies (Sd–Sm) may have a total H\textsc{i} mass comparable to or exceeding the mass in disc stars. The population models also predict lower masses for the disc components of early-type disc galaxies than those deduced from their rotation curves suggesting that they possess enough dark matter to stabilize their discs.
1 Introduction

Theoretical and numerical studies of highly flattened, rotationally supported stellar systems show that they are subject to large-scale bar-like instabilities (see e.g. the recent reviews by Toomre 1977, 1981, and references therein). Ostriker & Peebles (1973, hereafter OP) have suggested the criterion,

\[ t_{\text{OP}} = \frac{T}{|W|} = 0.14, \]

(here \( T \) is the total kinetic energy of rotation and \( W \) is the total gravitational potential energy) represents the maximum allowed rotational energy an axisymmetric system can possess and remain stable to the formation of a bar. OP then argue that the presumably cold axisymmetric discs of spiral galaxies are stabilized by the presence of a hot (invisible) component.

Since then, much work has been done on testing the generality of OP's criterion (e.g. Hohl 1976; Zang & Hohl 1978) and several possible counter-examples have been discussed (Zang 1976; Miller 1978; Berman & Mark 1979). In particular, Zang's study casts serious doubt on the general applicability of OP's criterion.

It is unclear, however, how general a criterion must be in order to apply to real disc galaxies. There is a remarkable uniformity to both the observed photometric and kinematic properties of disc galaxies (Freeman 1970; Rubin, Ford & Thonnard 1980). Thus our efforts will be concentrated on the stability of discs with an exponential surface density profile,

\[ \mu_D(r) = \left( \frac{\alpha^2 M_D}{2\pi} \right) \exp(-\alpha r), \]

together with a rigid halo component resulting in the rotation curve,

\[ v_D(r) = v_m \left( \frac{r^2}{r^2 + r_m^2} \right)^{1/2} \left[ 1 - \gamma \ln \left( \frac{r^2}{r_m^2 + r^2} \right) \right]^{1/2}, \]

where \( \gamma = (\frac{1}{2} \alpha r_m)^2 (\alpha M_D G/v_m^2) \). These models form a two parameter family in the dimensionless numbers \( \alpha r_m \) and \( v_m/(\alpha M_D G)^{1/2} \) and were proposed by Fall & Efstathiou (1980, hereafter FE) as relatively simple first-order approximations to the mass distribution in real disc galaxies, especially late types. The ratio of the mass in the halo within a given radius to that in the disc may be fixed by the parameter \( v_m/(\alpha M_D G)^{1/2} \). The parameter \( \alpha r_m \) effectively controls the concentration of the halo component, thus models with small \( \alpha r_m \) possess a 'bulge'. Two examples of the rotation curve of equation (3) are shown in Fig. 1.

The purpose of this paper is two-fold. First, we wish to make precise the statement commonly found in the literature, that heavy haloes are required to stabilize disc galaxies. The OP criterion is ill defined in the case of an isothermal halo, hence in Section 2 we discuss alternative measures of stability. We describe a series of numerical experiments to investigate the global stability of the FE models and we also study the effects of variations in the shape of the rotation curve and surface density profile. The second aim is to compare the results of the models with observations and ask whether it is possible that heavy haloes (or bulges) stabilize disc galaxies. A detailed discussion is given in Section 3. The main conclusions are summarized in Section 4.

2 Numerical simulations

2.1 Methods

The simulations discussed here were performed in two dimensions using a Fourier transform potential solver (Hohl & Hockney 1969) and the cloud in cell technique for the mass assign-
ment-and force interpolation (Birdsall & Fuss 1969). The time-centred leap-frog scheme was used in order to advance the particle coordinates (Buneman 1967). We have used 20,000 particles in all simulations. The active region of the Cartesian potential solver was a $128 \times 128$ mesh with the actual computation done on a mesh four times as large in order to remove the periodic images and simulate an isolated system (Hohl 1972). The halo components have been included by the addition of a fixed axisymmetric force. In all cases the halo extends out to a radius equal to one half the active mesh length. Particles which escape the mesh experience a Keplerian force of mass equal to the sum of the disc and halo masses and may re-enter the mesh later during the run. We have used 150–500 time steps per rotation period at the half-radius of the disc depending on the degree of concentration of the halo. In several cases we have reduced the time step by a factor of 2 or 4 and confirmed that our results are insensitive to the time step.

We have chosen to do two-dimensional simulations with a fixed halo potential rather than three-dimensional ones with a properly modelled population of halo stars, since even when the latter are done using a much coarser mesh they are prohibitively expensive for a study of this sort. Fortunately, recent studies (Sellwood 1980; Hohl 1978) indicate that this is likely to be a good approximation.

### 2.2 Initial Conditions

After generating the initial density distribution, particles were given random velocities from a Gaussian distribution and the equations of stellar hydrodynamics were solved to determine the angular velocity of the disc. The radial velocity dispersions were specified in terms of the minimum dispersion required to stabilize all local axisymmetric instabilities (Toomre 1964), i.e.

$$
\sigma_r(r) = \xi(r) \sigma_{\text{min}}(r),
$$

$$
\sigma_{\text{min}}(r) = 3.36 \frac{G \mu_D(r)}{\kappa(r)}.
$$

For simplicity, these procedures were performed using analytic expressions for the density, force and rotational velocity which assumed a gravitational force proportional to $1/r^2$. Since the potential solver utilizes softened gravity a further correction was applied close to the centre of the disc. Here particles were divided up amongst several radial bins and the mean radial force $F_1$ in the $i$th ring was calculated. This was then compared to the expected radial force using $1/r^2$ gravity, $F_1^{G}$. The velocity of each particle in the $i$th ring was then multiplied by $(F_i/F_i^{G})^{1/2}$.

As discussed in the introduction, most attention was focused on the FE models (equations 2 and 3) where 15 models were run covering the parameter range $0.1 < \alpha r_m < 1.3$ and $0.7 < u_m/(\alpha M_D G)^{1/2} < 1.3$. In these models, the exponential was abruptly truncated at $a_D = 5 \alpha^{-1}$. Unfortunately, the finite cell-size used in the simulations imposes a lower bound on the value of $\alpha r_m$ which can be modelled satisfactorily. Sellwood (1981) has shown that models may be artificially stabilized if the gravitational softening is comparable to the peak of the rotation curve. Indeed, Sellwood concludes that this effect was probably responsible for the stability of the model studies by Berman & Mark (1979). With $\alpha r_m = 0.4$, there are approximately 8–10 mesh cells within the peak of the rotation curve. Thus we do not expect these models to be unduly affected by softening. We have run two models, however, with $\alpha r_m = 0.1$ in an attempt to further explore the parameter range. These models, with only 2–3 mesh cells inside the peak of the rotation curve, require more care in interpretation as the softening is likely to have influenced the growth rates of the interesting modes.
Two additional sets of models were run. These used a fixed potential given by the mass
distribution,

$$\rho (r) = \rho_0 / \{1 + r^2/a_H^2\},$$

(5)

together with either an exponential disc with $a_H = 0.2$ (E models) or a disc with surface
density,

$$\mu_D(r) = \frac{11}{2\pi a_D^2} M_D (1 - r^2/a_D^2)^{9/2}.$$  

(6)

from the family of discs studied by Hunter (see Hohl 1970); here we chose $a_H/a_D = 0.1$ (H
models).

Observations of stellar motions in our Galaxy show that the disc is cold ($\sigma_\rho < \sigma_v$), with
$Q \approx 1$ (Toomre 1964, 1974). Therefore, we model initially cold discs. In most of these
models $Q(r)$ in equation (4a) was chosen to be constant, $Q = 1.05$, but in two of the E
models we used a temperature profile which gradually increased towards the centre of the
disc, $Q(r) = 1 + \exp (-q_0 r^2)$.

2.3 MEASURES OF STABILITY

2.3.1 Global parameters

In this section we introduce a parameter $t^*$, similar to the parameter $t_{OP}$ used in the Ostriker–
Peebles criterion. This parameter was suggested by Lake & Ostriker (1979, unpublished) and we
shall follow their presentation here.

An $N$-body system in equilibrium satisfies the virial theorem in the form,

$$2T_{\text{rand}} + 2T_{\text{mean}} + W = 0,$$

(7)

where

$$T_{\text{mean}} = \frac{1}{2} \int \langle v(r)^2 \rangle \rho (r) \, d^3r,$$

(8a)

$$T_{\text{rand}} = \frac{1}{2} \int \left[ \{v - \langle v(r)\rangle\}^2 f(v, r) \right] \, d^3r \, d^3v,$$

(8b)

$$W = \frac{1}{2} \int \Phi \, dm.$$  

(8c)

Here $\Phi$ is the gravitational potential, $\rho (r)$ is the density, $f(v, r)$ is the phase space distribution
function of stars and $\langle v(r) \rangle$ is the mean circular velocity of the material located in a cylinder
at a distance $r$ from the axis of symmetry, i.e. $\langle v(r) \rangle = \int f(v, r) v_\phi \, dz \, d^3v / \int f(v, r) \, dz \, d^3v$. Here
$v_\phi$ is the circular velocity about the axis of symmetry taken as the $z$-axis. This differs from the
definitions of OP who used the local streaming velocity in place of $\langle v(r) \rangle$.

Now consider the disc as a subsystem embedded in the potential of another system (such as
a halo or another disc). The virial theorem for the subsystem may be written,

$$2T_D \rangle_{\text{mean}} + 2T_D \rangle_{\text{rand}} + (1 + f_{\text{ext}}) W_D = 0,$$

(9)

where the subscript $D$ indicates that the integrals in equations (8) are performed over the
disc only. Here $f_{\text{ext}}$ is the fraction of the force which is due to mass external to the disc,

$$1 + f_{\text{ext}} = \int \rho (r) F_{\text{tot}} \cdot r \, d^3r / \int \rho (r) F_D \cdot r \, d^3r,$$

(10)

where $F_{\text{tot}}$ and $F_D$ are the total force and the force due to the mass in the disc respectively.
As global measures of stability we will consider the parameters,

\[ t^* = \frac{T_D \text{mean}}{(1 + f_{\text{ext}})^2 |W_D|}; \quad u^* = \frac{T_D \text{rand}}{(1 + f_{\text{ext}})^2 |W_D|} \]

and from equation (9), \( t^* + u^* = 1/2(1 + f_{\text{ext}}) \). In the special case where the densities \( \rho \) of the disc and the total system are related by \( \rho_{\text{disc}}(r) = \xi \rho_{\text{tot}}(r) \), then \( \xi = 1/(1 + f_{\text{ext}}) \) and \( t^* \) is exactly the same as \( t_{\text{OP}} \).

The parameter \( t^* \) has some notable advantages over \( t_{\text{OP}} \). The stabilizing influence of any rigid component (halo) should depend only on the force it exerts on the subsystem in question. For example, a small point mass with a divergent gravitational energy or a massive spherical shell enclosing the system may greatly lower \( t_{\text{OP}} \) whilst having little or no effect on the stability of the system. The utility of the second change in calculating \( t^* \), averaging the mean velocity over cylinders rather than locally, is due to the weak \( z \)-dependence of the bar-like mode. As a specific example, imagine a thin system of three layers with a different mean circular velocity in each layer. Collapsing this system to a single thin layer or mixing the layers should not affect the stability of this system to a \( 2\theta \) mode. The stability parameter \( t^* \) (unlike \( t_{\text{OP}} \)) is also insensitive to this change. This latter difference will not be important for the two-dimensional systems we consider here, but is relevant to three-dimensional systems.

### 2.3.2 Measurement of bar strengths and lengths

To determine these quantities, we have binned the particles in radial rings which are linearly spaced and typically contain 600–1300 particles per ring in the region of interest. We then calculate the amplitude and phase of the \( m = 2 \) component in each of the various rings. As a particular measure of bar strengths we use

\[ \delta_2 = \left( \frac{\rho_m = 2}{\rho_m = 0} \right)_{\text{max}} \]

the maximum value of the ratio of the \( m = 2 \) component to the average density in the ring.

The length of the bar is defined as the region over which the phases are coherent to \( \pm 10^\circ \). This definition is matched quite well by visual estimates from pictures generated from the simulations.

### 2.4 Results

In Table 1 we summarize the main details of the models. Fig. 1 shows rotation curves for four models at similar values of \( f_{\text{ext}} \). The E models have rotation curves which decline slightly at large radii but are qualitatively similar to the FE models with \( \alpha r_m = 0.4 \). The H models have rotation curves which are close to solid body over much of the disc and are, therefore, more similar to the FE models with \( \alpha r_m = 1.0 \). Fig. 2 shows the evolution of \( t^* \) and \( \delta_2 \) for some of the FE and H models. The evolution for each set of models, though qualitatively similar, do differ in detail. The FE models with \( v_m/(\alpha M_D G)^{1/2} \leq 0.8 \) rapidly develop a large \( m = 2 \) component which then decays to a roughly constant amplitude (cf. also Coombes & Sanders 1981). This occurs because the development of the bar is accompanied by the growth of a strong multi-armed spiral pattern. The initial rise is due to interference of these spiral patterns which then decay leaving a strong bar embedded in a roughly axisymmetric distribution of particles. The initial spiral phase is quite weak in the H discs. The models attain a nearly constant value of \( \delta_2 \) and thus do not show any evidence for slow
Table 1. $N$-body simulations.

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<th>$t_2^*$</th>
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(b) E models

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(c) H models

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Notes to Table 1

*N*$_{rot}$ is the total number of rotation periods (measured at the mid-point of the disc) for which the model was allowed to evolve. *N*$_{esc}$ is the number of particles which had left the active mesh at the end of the run. *a*$_d$ is the radius of the disc in units of the active mesh. The two asterisked E models had warm centres as discussed in Section 2.2.

The parameters $t_1^*$ and $\delta_2$ illustrate the stabilizing effect of a rigid component but do not readily distinguish a bar from a spiral. To do this we use the bar length $L_B$ as defined in Section 2.3. Fig. 3 shows the final bar lengths $L_B$ for the FE and E models as a function of $t_1^*$. The bar lengths plotted are the averages over four estimates made during the final two revolutions and the error bars indicate one standard deviation in the mean. There is a trend in Fig. 3 for the bar length to decrease with decreasing $t_1^*$ for models with fixed $\alpha r_m$. However, there is no noticeable relationship between the final length of the bar and the initial value of $\alpha r_m$. This result is not necessarily in conflict with that of Sellwood (1981) as he identifies two distinct stages in the formation of a bar. Initially a bar forms with length
Figure 1. Rotation curves for four models with similar values of $f_{\text{ext}}$. The numbers and coding refer to Table 1. The radius $a_D$ is defined by equation (6) for the H models, whilst for the E and FE models, $a_D = 5\alpha^{-1}$, where $\alpha^{-1}$ is the scale length of the exponential disc.

Figure 2. Evolution of $t^*$ (equation 11) and the maximum value of the $m=2$ Fourier component $\delta_2$, (equation 12): (a) for the FE models with $\alpha r_m = 0.4$, (b) for the FE models with $\alpha r_m = 1.0$, and (c) for the H models.

proportional to the turnover radius of the rotation curve. The bar then grows by transferring angular momentum to cool stars in the outer parts of the disc. Thus the final bar length may be only weakly dependent on the initial shape of the rotation curve, especially in highly concentrated models such as those discussed here. In our models it is difficult to distinguish between the various stages in the growth of the bar as the initial bar length is difficult to measure because it is accompanied by a strong spiral pattern in the early stages. It is unlikely that our results are influenced by numerical errors. Dr Sellwood has kindly repeated models 1, 5 and 8 using a quite different $N$-body technique (Sellwood 1981) and finds qualitatively similar results for bar lengths and pattern speeds.
One of the E models, with $r^*_B = 0.2$, was allowed to evolve for more than 25 revolutions in order to test for a slowly growing bar mode. The model did appear to be unstable to a weak bar-like pattern but neither its length nor pattern speed could be measured reliably. An upper limit to the bar length is shown in Fig. 3.

Our attempts at stabilizing the E models using hot centres did not prove successful, thus model 20 developed at least as strong a bar as model 17. The warm models are not entirely satisfactory as they rely on the epicyclic approximation to fix the initial random velocities, even at the centre of the disc. This problem needs to be examined in greater detail using initial conditions generated from a self-consistent solution to the collisionless Boltzmann equation. Solutions for warm exponential discs have not yet been devised and the particular ones demanded are especially complicated due to the softened gravity.

The main result from this study is summarized in Fig. 4. This shows the initial and final values of $r^*_B$ for each of the FE models of Table 1(b). The parameter $r^*$ does not in itself determine whether an FE model will be stable to bar formation as there is a clear dependence on $\alpha r_m$ (cf. Fig. 3). However, Fig. 4 shows that a cold FE model will be stable against the growth of bar-like modes if,

$$v_m/(\alpha M_D G)^{1/2} \gtrsim 1.1, \quad 0.1 \leq \alpha r_m \leq 1.3.$$  \hfill (13)

The results for the other models are summarized in Fig. 5 which also show lines of constant $Q$ (equation 4a) thus giving an indication of the degree of heating in the disc during its evolution. The lines of constant $Q$ are strictly applicable only to the initial states but yield a reasonably good estimate (accurate to $\approx 15$ per cent) of the final value of $Q$ in the central parts of the final state. Typically the discs evolve until $Q = 1.5-2$ in their inner regions ($\alpha r < 3$) and $Q \approx 3-4$, in their outer parts. This is true even if the discs are stable to the formation of bars. The implications of this result will be discussed in Section 3.3 with reference to our own Galaxy. For two models (5 and 8) we have examined the form of the random velocity distribution function. The models begin with Gaussian distributions for both $\sigma_r$ and $\sigma_\theta$. After several rotations considerable heating occurs but the distributions remain approximately Gaussian.
Figure 4. The filled circles show the initial value of $t^*$ for each model and the open triangles show the final values. The numbers refer to Table 1(a). The lines of constant $v_\text{M}/(\alpha M_\text{D} G)^{1/2}$ are computed for a cold disc ($Q = 0$).

Figure 5. Summary of the results from the E and H models. The symbols are as follows: (●) initial values of $t^*$ and $f_{\text{ext}}$ for the H modes, (○) initial values for the E models with $Q = 1.05$, (△) initial values for the E models with warm centres. The triangles show the final values of $t^*$ and $f_{\text{ext}}$. The dashed lines show lines of constant $Q$ for the H discs whilst the solid lines are appropriate for the E models.

These additional models also illustrate that our conclusions are not sensitive to small deviations from the initial conditions of the FE models. The exponential models are stable for $t^*_1 < 0.18$, similar to the results for the FE models with $\alpha r_m = 0.4$ and the Hunter models are stable for $t^*_1 < 0.2$, similar to the results for the FE models with $\alpha r_m = 1.0$ (cf. Fig. 1).
Figure 6. (a) Evolution of surface density profile for three models that are stable to bar formation. The solid lines are for model 12 and the dashed lines are for model 24. The initial profiles are shown to the left and the final profiles, after 10 revolutions, are shown on the right. The open circles show the evolution of a K4hajks O-model with a uniform halo and \( t^* = 0.25 \) and \( \rho_{\text{ext}} = 0.74 \). The final density profile shown for this model is that observed after five rotations. (b) As for (a) except for three models that form bars. The solid lines are for model 10, the dashed lines are for model 22 and the open circles are for a K4hajks O-model with \( t^* = 0.28 \) and \( \rho_{\text{ext}} = 0.30 \).

It is interesting to note that the density profiles of the Hunter models show a substantial evolution. Fig. 6(a) shows the evolution in the surface density for models 12 and 24; both are stable to bar formation. After 10 revolutions, the exponential model maintains an exponential density profile apart from a small increase in surface density close to the centre. The density profile of the Hunter model rapidly develops a form which closely matches that
of the exponential, Hohl (1970) and Zang & Hohl (1978) have noticed that the surface density profiles of bar unstable models acquire a form which may be approximated by the sum of two exponentials. Examples of the evolution of bar unstable exponential and Hunter models are shown in Fig. 6(b). The evolution in surface density is much more dramatic when the models are bar unstable and the final density profiles are not exponential. Even though the discs shown in Fig. 6(a) are stable to bar formation a significant redistribution of angular momentum can occur due to torques from higher order instabilities. This may be deduced from the change in $M(h)$, the mass in the disc with angular momentum per unit mass less than $h$.

In model 12, $dM(h)/dh$ shows evolution only for small values of $h$, whilst model 24 shows a more severe evolution. It is doubtful whether $M(h)$ can be considered an invariant (e.g. Mestel 1963; FE 1980) and therefore an indicator of the primordial angular momentum distribution even in models with a substantial halo component such as in models 12 and 24.

\[ \text{Figure 7. Low order normal modes for the Kalnajs } \Omega \text{-model with a uniform halo. The lines delineate regions of instability.} \]
We have included in Fig. 6 the evolution of the density profiles for two of Kalnajs’ (1972) $\Omega$ models. Here the initial conditions were generated directly from the distribution function given by Kalnajs and we included the force due to a uniform rigid spherical halo. The models thus have solid body rotation curves. Each model was run for five rotations using a $64 \times 64$ active mesh. The model shown in Fig. 6(a) has $t^* = 0.25$ and $J^*_{\text{ext}} = 0.74$. Apart from some diffusion of particles in the outer parts of the disc, there is little evolution of the density profile. It is straightforward from Kalnajs’ paper to assess the stability of the $\Omega$-models including a uniform halo component (Fig. 7). In this case we find that the model is unstable to the $4-4$ mode and most modes of higher order. The model shown in Fig. 6(b) has $r^*_m = 0.28$ and $J^*_{\text{ext}} = 0.30$ and develops a bar, as would be expected from the results shown in Fig. 7. This model does show some evolution in surface density but the final profile differs significantly from an exponential.

It has been suggested (e.g. Gerhard 1981) that the exponential profile may generally result due to global instabilities. Our models show that this does not occur in general. It is especially interesting that the Kalnajs’ discs show little evolution in the surface density profile. The instabilities of the Kalnajs’ discs are termed ‘edge modes’ by Toomre & Kalnajs (Toomre 1981) and do not entail any wave transport. From the results shown in Fig. 6 it appears that shear is required to excite wave transporting modes and shift material over a range in radius (cf. Lynden-Bell & Kalnajs 1972).

3 Comparison with observations

3.1 Comparison with Sc galaxies

If a disc galaxy may be adequately approximated by one of the FE models, the parameters $r_m$, $v_m$ and $\alpha^{-1}$ may be estimated from the rotation curve and photometry. The morphology of the galaxy, i.e. whether or not it possesses a bar, may then be used together with the stability criterion of equation (13) to set a limit on the mass-to-light ratio of the cool disc component. Unfortunately, there are only a handful of galaxies for which both detailed rotation curves and surface photometry exist. The available data are not ideal but will be discussed in Section 3.2.

Our models are most likely to be applicable to galaxies with weak bulges such as Sc’s and later types as these galaxies usually have rotation curves for which $\alpha r_m$ lies within the range for which the criterion of equation (13) has been tested.

Recently, Rubin et al. (1980) have obtained extended rotation curves for 21 Sc galaxies. Detailed surface photometry does not yet exist for this sample, but estimates of the disc scale length may be obtained using Freeman’s (1970) law of constant central surface brightness, together with either the total luminosity or by using the radius ($R_{25}$) corresponding to a surface brightness of 25 B mag arcsec$^{-2}$. (Throughout this paper, luminosities refer to the B-band.) Thus,

\[ \alpha^{-1} = R_{25}/3.09, \]

\[ \alpha^{-1} = 1.05 (L_D/10^9 L_\odot)^{1/2} \text{kpc}. \]

In applying equation (14b) we assume $L_D/L_T = 0.9$ for Sc galaxies corresponding to the mean bulge-to-disc ratio adopted by FE. Nine galaxies were rejected from the Rubin et al. sample either because they have rotation curves which are poorly represented by FE models, or because $\alpha r_m < 0.1$. The parameters for the remaining twelve galaxies are listed in Table 2(a). The table gives an estimate of the parameter $v_m/(\alpha L_D G)^{1/2}$ which is only weakly dependent on total luminosity consistent with the relation $L_T \propto v_m^4$ (cf. Aaronsen, Huchra
Table 2(a). Properties of Sc galaxies.

<table>
<thead>
<tr>
<th>NGC/IC</th>
<th>D (Mpc)</th>
<th>L_T (10^9 L☉)</th>
<th>v_m (km s⁻¹)</th>
<th>R_{25} (kpc)</th>
<th>R_L (kpc)</th>
<th>r_m (kpc)</th>
<th>v_m/(αL_D G)^{1/3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1035</td>
<td>12.4</td>
<td>3.0</td>
<td>125</td>
<td>3.8</td>
<td>5.3</td>
<td>1.1</td>
<td>1.4</td>
</tr>
<tr>
<td>4062</td>
<td>7.6</td>
<td>2.5</td>
<td>165</td>
<td>4.5</td>
<td>4.9</td>
<td>1.5</td>
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<td>2742</td>
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<td>6.5</td>
<td>170</td>
<td>6.3</td>
<td>7.8</td>
<td>1.3</td>
<td>1.6</td>
</tr>
<tr>
<td>1087</td>
<td>15.4</td>
<td>13.1</td>
<td>125</td>
<td>7.8</td>
<td>11.1</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>4682</td>
<td>21.7</td>
<td>8.7</td>
<td>170</td>
<td>8.8</td>
<td>9.1</td>
<td>1.6</td>
<td>1.6</td>
</tr>
<tr>
<td>3672</td>
<td>16.6</td>
<td>16.1</td>
<td>190</td>
<td>9.4</td>
<td>12.4</td>
<td>2.2</td>
<td>1.4</td>
</tr>
<tr>
<td>1421</td>
<td>20.0</td>
<td>28.7</td>
<td>170</td>
<td>9.8</td>
<td>16.5</td>
<td>1.3</td>
<td>1.1</td>
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<tr>
<td>2715</td>
<td>15.0</td>
<td>12.1</td>
<td>130</td>
<td>10.2</td>
<td>10.7</td>
<td>2.1</td>
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<td>IC 467</td>
<td>22.3</td>
<td>13.7</td>
<td>120</td>
<td>10.8</td>
<td>11.4</td>
<td>2.2</td>
<td>1.0</td>
</tr>
<tr>
<td>2998</td>
<td>47.8</td>
<td>51.8</td>
<td>195</td>
<td>20.0</td>
<td>22.2</td>
<td>2.0</td>
<td>1.1</td>
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<tr>
<td>753</td>
<td>51.0</td>
<td>39.3</td>
<td>205</td>
<td>22.4</td>
<td>19.3</td>
<td>3.0</td>
<td>1.4</td>
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<tr>
<td>801</td>
<td>59.5</td>
<td>83.5</td>
<td>220</td>
<td>26.9</td>
<td>28.1</td>
<td>1.5</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Notes to Table 2(a)

D – distance determined from the recession velocity assuming $H_0 = 100$ km s⁻¹ Mpc⁻¹. $L_T$ – total luminosity using apparent $B_T$ mag corrected for galactic extinction and inclination from Burstein et al. (1981). $R_{25}$ – corrected radius out to a surface brightness of 25 mag arcsec⁻² taken from Burstein et al. $R_L = 3.08 (L_T/10^9 L☉)^{1/3}$ (cf. equations 14). $\alpha^{-1}$ – disc scale length estimated using Freeman’s law, we take $3.09 \alpha^{-1} = \frac{1}{2} (R_{25} + R_L)$. $v_m$ and $r_m$ were deduced from the rotation curves of Rubin et al. (1980).

& Mould 1979; Burstein et al. 1981). All of these galaxies are normal Sc’s except for IC 467 which is weakly barred. Hence applying equation (13), the global stability of these galaxies may be understood if they possess mass-to-light ratios which satisfy,

$$M_D/L_D < (1.5 \pm 0.2) h.$$  \hspace{1cm} (15)

(Here and throughout this section, $h$ is Hubble’s constant in units of 100 km s⁻¹ Mpc⁻¹.) Since barred Sc’s are quite common (e.g. Freeman (1976) quotes the ratios of SA:SAB:SB to be 39:37:26) and the dispersion about the relation $L_T \propto v_m^4$ appears to be quite small (Aaronson et al. 1979) we suggest that equation (15) is more nearly an equality than an upper limit. It is interesting to note that as the rotation curves are fairly flat out to $R_{25}$, the ratio of total mass to that in the cool disc implied by equation (15) is quite large.

The mass-to-light ratio deduced in equation (15) is quite small and confirms an earlier conclusion reached by FE. In Section 3.3 we shall enquire whether such a low mass-to-light ratio is consistent with what is known about stellar populations in disc galaxies and in particular we shall compare with the population models of Larson & Tinsley (1978).

3.2 Galaxies with Detailed Photometry and Extended Rotation Curves

The main problem in applying our results to early-type disc galaxies is the separation of the light due to the bulge from that due to the disc. The recent study of Boroson (1981) shows that there is a large scatter in the relation between $L_D/L_T$ and Hubble type, and this is likely to lead to large errors if mean relations are used to derive scale lengths and disc luminosities. It would be preferable to use detailed photometry when available. A survey of the literature resulted in a total of nine galaxies with both extended rotation curves and photometry and we summarize their properties in Table 2(b). The disc scale lengths were obtained from the references listed in the notes to Table 2(b). In cases where the disc-to-bulge ratio was not calculated by a detailed decomposition of the luminosity profile we have used the relation,

$$L_D = 0.91 \alpha^{-2} 10^{0.41(21.65-B_{0,c})} \times 10^9 L☉,$$ \hspace{1cm} (16)

where $\alpha^{-1}$ is the disc scale length in kpc and $B_{0,c}$ is the central intensity in B mag arcsec⁻².
Table 2(b). Properties of galaxies with extended rotation curves and surface photometry.

<table>
<thead>
<tr>
<th>NGC</th>
<th>Type</th>
<th>$D$ (Mpc)</th>
<th>$L_D/L_T$</th>
<th>$L_T$ ($10^9 L_\odot$)</th>
<th>$v_m$ (km s$^{-1}$)</th>
<th>$\alpha^{-1}$ (kpc)</th>
<th>$r_m$ (kpc)</th>
<th>$(B-V)_T^2$</th>
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</thead>
<tbody>
<tr>
<td>224</td>
<td>SAb</td>
<td>0.69</td>
<td>0.76</td>
<td>20.0</td>
<td>230</td>
<td>4.1–4.5</td>
<td>0.63</td>
<td>0.74</td>
</tr>
<tr>
<td>253</td>
<td>SABc</td>
<td>2.50</td>
<td>0.85</td>
<td>26.5</td>
<td>205</td>
<td>2.3</td>
<td>1.6</td>
<td>0.69</td>
</tr>
<tr>
<td>598</td>
<td>SAc</td>
<td>0.72</td>
<td>0.89</td>
<td>3.3</td>
<td>100</td>
<td>1.6</td>
<td>1.60</td>
<td>0.44</td>
</tr>
<tr>
<td>2841</td>
<td>SAb</td>
<td>6.0</td>
<td>0.76</td>
<td>7.0</td>
<td>270</td>
<td>2.0</td>
<td>0.75</td>
<td>0.73</td>
</tr>
<tr>
<td>4244</td>
<td>SAc</td>
<td>3.4</td>
<td>1.00</td>
<td>2.0</td>
<td>110</td>
<td>1.8</td>
<td>1.22</td>
<td>0.24</td>
</tr>
<tr>
<td>5457</td>
<td>SABc</td>
<td>4.0</td>
<td>0.80</td>
<td>13.3</td>
<td>210</td>
<td>2.9</td>
<td>0.52</td>
<td>0.40</td>
</tr>
<tr>
<td>5907</td>
<td>SAc</td>
<td>8.6</td>
<td>1.00</td>
<td>9.0</td>
<td>235</td>
<td>4.5</td>
<td>0.42</td>
<td>0.56</td>
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<tr>
<td>7331</td>
<td>SAbc</td>
<td>10.5</td>
<td>0.47</td>
<td>21.3</td>
<td>230</td>
<td>5.3</td>
<td>0.30</td>
<td>0.64</td>
</tr>
<tr>
<td>SMc</td>
<td>Sbm</td>
<td>0.065</td>
<td>1.00</td>
<td>0.67</td>
<td>37</td>
<td>0.82</td>
<td>1.3–1.8</td>
<td>0.36</td>
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</table>

Notes to Table 2(b)

Source of photometry: 224, 598 and SMc, Freeman (1970). 2841 and 7331, Boroson (1981). 4244 and 5907, Van der Kruit & Searle (1981). 224, de Vaucouleurs (1958). 253, Pence (1980). 5457, Schweizer (1976) and Okamura et al. (1976). $L_D/L_T$ was estimated from equation (16) for NGC 598 and 5457. The colours are from the Second Reference Catalogue of Bright Galaxies (de Vaucouleurs et al. 1976) with the exception of NGC 253 for which $(B-V)_T^2$ was taken from Pence (1980). Pence notes that the internal reddening is likely to have been underestimated in this galaxy thus the value quoted may be too large. Sources of rotation curves: 224, 2841, 4244, 5907, 5457, 7331, Bosma (1978). 253, Pence (1980). 598, Rogstad et al. (1976). SMc, Hindman (1967). 224, Rubin & Ford (1970). 7331, Rubin et al. (1965). $\alpha r_m$ is not quoted for NGC 224 because the rotation curve is double peaked, the inner maximum is at $\approx 0.8$ kpc. Uncertain numbers are indicated by a colon. Uncertainties in $\alpha r_m$ are introduced by beam smearing. The rotation curve of NGC 5457 is a poor fit to equation (3), thus $v_m$ is uncertain. The asymmetry of the rotation curve of the SMc and the poorly determined inclination angle (see de Vaucouleurs & Freeman 1972) lead to uncertainties in $v_m$.

This sample is far from ideal. Uncertainties in $\alpha^{-1}$ arise from high inclination angles (NGC 4244, 5907) and some of the 21-cm rotation curves suffer from lack of resolution due to beam smearing. A second major worry is that the rotation curves in the central regions of some spirals may not be good indicators of the mass distribution. Consider, for example, the case of NGC 7331, Whitmore, Kirshner & Schechter (1979) have measured a central velocity dispersion for the bulge and obtain a value $\sigma = 176 \pm 18$ km s$^{-1}$, whilst Boroson (1981) derives an effective radius of 1.6 kpc. Thus we would expect the rotation velocity to reach a peak value of $\approx 250$ km s$^{-1}$ at a radius of $\approx 1/3 R_e$, i.e. $\alpha r_m = 0.1$, but such an early peak is not seen in Rubin et al. (1965) optical rotation curve. Similar conclusions have been reached concerning other early-type disc galaxies such as NGC 4594 (Faber & Gallagher 1979). Also many early-type disc galaxies would be expected to show a doubly peaked rotation curve, such as is observed in M31 and in our Galaxy. As central velocity dispersions and effective radii for the bulge component are not available for most of the galaxies in our sample, it is difficult to assess the reliability of individual values for $\alpha r_m$ deduced from the rotation curve data. It is likely that the problem is severe for most early-type disc galaxies.

These points deserve attention, because although we have verified the criterion of equation (13) over a wide range in $\alpha r_m$, there is evidence to indicate that an extrapolation of the result to $\alpha r_m < 0.1$ should be viewed with some caution. As Fig. 4 illustrates, the global parameter $t^*$ depends only weakly on $\alpha r_m$ and is thus insensitive to an increase in the value of $\Omega - \kappa/2$ in the central regions of the disc. In this case, the avoidance of an inner Lindblad resonance is possible only by a rapidly rotating mode which, if ending at corotation, would be quite short. One might conjecture that such a disc might be stabilized by a small central bulge (Sellwood 1981). Zang’s (1976) models also support the idea that global instabilities avoid inner Lindblad resonances (see also the discussion by Toomre 1981). However, the
numerical experiments reported here, and in particular those of Sellwood (1981), illustrate the importance of the non-linear growth of the bar. The final bar may have a pattern speed that is substantially smaller than the maximum in $\Omega - \kappa/2$ (this is indeed the case for model 1) and the final length may be only weakly dependent on the initial shape of the rotation curve (cf. Fig. 3 and Section 2.4). From the seven galaxies of type Sb and earlier for which Boroson derives parameters for both disc and bulge we find $1/3 \alpha R_e = 0.12 \pm 0.03$. Thus early-type disc galaxies have concentrations that are comparable to those of the models with $\alpha r_m = 0.1$ for which equation (13) has been found to apply. This would imply that even early-type disc galaxies would require a substantial halo component if they are to be stable to bar formation.

It is worth mentioning another possibility which may confuse the correspondence of our models with early-type galaxies. The bulges of spirals are observed to rotate rapidly (Kormendy & Illingworth 1981) and many may rotate rapidly enough to satisfy the inequality $r^* > 0.14$, which does appear to delineate the stability of self-gravitating, rotationally supported, spheroidal systems (Hohl & Zang 1979; Miller & Smith 1979; Villumsen 1981). Indeed, the bulge of M31 appears to be triaxial (Stark 1977). The presence of a bar-like mode in the bulge may have a substantial effect on the disc component.

### 3.3 Comparison with Stellar Population Models

We summarize the results on mass-to-light ratios in Fig. 8. Here we have included all galaxies from Table 2(a) together with additional galaxies with $\alpha r_m > 0.1$ from Table 1 of FE for

![Figure 8](attachment:mass-to-light-ratios.png)

**Figure 8.** Mass-to-light ratios for the disc components of galaxies required if $v_m/(\alpha M_D G)^{1/2} = 1.1$. The filled box shows the average result for 12 Sc galaxies discussed in Section 3.1. This point is placed at the mean colour for Sc's (cf. Table 3). The open box shows the local $M/L$ in our own Galaxy deduced from the Oort limit (Section 3.3). The downward pointing triangles are the results for SA galaxies, circles are for SAB's and upward pointing triangles for SB's. Filled symbols show the results for galaxies with detailed photometry and rotation curves listed in Table 2(b) whilst open symbols show results for galaxies from Table 1 of FE for which scale lengths and disc luminosities were obtained using mean relations. Two of these galaxies did not have $(B-V)_F$ listed in the RC2 and have been placed at the mean colour corresponding to their morphological type (Table 3). These points are enclosed in brackets. The crosses show $M_{H1}/L_D$. 

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which scale lengths were estimated by using the mean relations of FE. Also plotted is a theoretical relation between $M_\bullet/L$ and $B-V$ from Fig. 2 of Tinsley (1981, see also Larson & Tinsley 1978); here $M_\bullet$ refers to the mass in stars and does not include interstellar gas. The model uses an initial mass function (IMF) which closely approximates that of the solar neighbourhood and includes only known stars of mass above $0.1 M_\odot$. The agreement between the model and the mass-to-light ratio deduced in equation (15) is quite good. If the IMF were weighted towards low-mass stars the stability of Sc galaxies would be difficult to understand. It is interesting to note that the agreement would be much worse had we taken $H_0 = 50$ km s$^{-1}$ Mpc$^{-1}$ ($h = \frac{1}{2}$).

It is well known that a correlation exists between the fraction of mass in H I and morphological type (e.g. Shostak 1978 and references therein). Table 3 summarizes the results from the recent H I survey of Fisher & Tully (1981). Tinsley’s models predict a very steep decline in $M_\bullet/L_T$ for galaxies bluer than $B-V \approx 0.6$, thus with the models normalized as in Fig. 8, $M_{H/1} > M_\bullet$ for galaxies of type later than about Sd. If this were true, then we would not expect the criterion of equation (13) to apply to the bar instability in such gas rich systems. The total masses in Table 3 imply $M_D/L_D \leq 1.6$ for stability for types Sc and later, whilst the mean H I mass in Sdm system is high enough to yield $M_{H/1}/L_D \approx 0.7$. The bar instability in very late-type systems may be due to a global instability in the gas component (e.g. Bardeen 1975) or the appearance of a bar may be related to large-scale regions of enhanced star formation as in the stochastic models of Gerola & Seiden (1978). This latter picture may be favoured as Sd–Sm galaxies are usually of much lower luminosity than early-type spirals, which in the stochastic picture generally results in a more ragged and irregular type galaxy. Also, the bars in such systems are generally displaced from the centre of the disc (Freeman 1975) which is readily explained by the stochastic star formation models (Feitzinger et al. 1981).

An alternative hypothesis would be to argue that $M_\bullet/L_D$ is much higher in late-type galaxies than predicted by Tinsley’s models. This may result if (1) there exist large metallicity gradients across the disc, (2) the very blue colours are due to recent bursts of star formation, (3) there are radial variations in the IMF which result in a large fraction of mass residing in

<table>
<thead>
<tr>
<th>Table 3. H I content and total masses.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td>Sa</td>
</tr>
<tr>
<td>Sab</td>
</tr>
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<td>Sb</td>
</tr>
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<td>Sc</td>
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<td>Scd</td>
</tr>
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<td>Sdm</td>
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<td>Sm</td>
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Notes to Table 3
Here we have included all galaxies in the Fisher–Tully catalogue (1981) which have morphological classifications, $v_D > 500$ km s$^{-1}$ and inclinations $> 35^\circ$. Column 2 lists the number of galaxies of each morphological type. Total masses were calculated using the formula $M_T = (W_T/2.2 \sin l)^2 (R_{HI}/G)$ where $W_T$ is the H I profile width measured by Fisher and Tully, $l$ is the inclination angle, and the coefficient derives from a comparison of 26 galaxies with extended rotation curves. $R_{HI} = 4.2 (L_D/10^n)^{1/2}$ kpc. The Holmberg radii deduced in this way are compared to the measured Holmberg radii in column 5 and give an indication of the consistency of using mean relations to derive photometric parameters such as scale lengths. Column 6 lists the mean corrected colours as a function of morphological type from Table 4b of de Vaucouleurs (1977).
low mass stars ($\leq 0.1 M_\odot$). Tinsley gives various arguments against (1) and (2) but there is little direct observational evidence that might contradict (3). Tinsley argues against all the above hypotheses on the grounds that if $M_*/L_D$ is increased much above the model predictions, late-type galaxies would be unstable to bar-like modes. However, it is important to note that very late-type galaxies do appear to be predominantly barred; for example, from table 2 of Freeman (1975) we find that out of 104 spirals of types Sd–Sm, 67 are classified as SB, 21 as SAB and 16 as SA (the classifications are by de Vaucouleurs 1963) whilst for early-type spirals the fractions are more nearly equally divided (see also Wyse 1981). It is clear that it is difficult to rule out the above hypotheses using arguments based on bar instabilities. A detailed study (such as surface photometry and extended rotation curves) of late-type galaxies would help resolve this issue.

It is also interesting to note that the scaling of Tinsley’s models in Fig. 8 is in fairly good agreement with the dynamical mass-to-light ratio in the plane of our Galaxy deduced from the Oort limit (Oort 1965). Recent investigations are reviewed by Faber & Gallagher (1979) and by Peebles (1980) and the “best” numbers appear to be $\rho_{\text{dyn}} = 0.15 M_\odot \text{pc}^{-3}$, $L_B \sim 0.055 L_\odot \text{pc}^{-3}$, $0.08 \leq \rho_*/\rho_{\text{ISM}} \leq 0.12 M_\odot \text{pc}^{-3}$, $B-V \approx 0.62$. Thus the Oort limit yields $M/L \approx 2.7$ (this is marked as the open square in Fig. 8), whilst the mass-to-light ratio of known stars and gas is $1.5 \leq M/L \leq 2.2$. Peebles also points out that it is difficult to account for the flat rotation curve of our galaxy without a massive halo component for if we suppose that the flat rotation curve is due solely to the disc component and that $M/L$ is independent of position off the plane of the galaxy, a local value of $M/L$ of $\approx 14$ is required, which is far in excess of that deduced from the Oort limit.

We have mentioned in Section 2.4 that the $N$-body models usually heat up so that $Q \sim 1.5–2$ over the range $ar \leq 3$. Following Toomre (1964, 1974) it is of interest to examine whether this is consistent with observational data for the solar neighbourhood. We take the following data from the review by Peebles (1980):

$$v_0 = 220 \pm 20 \text{ km s}^{-1},$$

$$R_0 = 8.5 \pm 1.6 \text{ kpc},$$

$$i_B(R_0) = 17 \pm 1 L_\odot \text{pc}^{-2},$$

where $v_0$ is the local rotational velocity and we assume that the rotation curve is flat. $R_0$ is the distance of the Sun from the centre of the Galaxy and $i_B(R_0)$ is the local surface brightness in the B-band. From equation (4) we find

$$\sigma_r(R_0) = (6.7 \pm 2.2) Q(R_0) (M/L) \text{ km s}^{-1}. \hspace{1cm} (18)$$

The numerical coefficient in equation (4a) depends on both the finite thickness of the disc and on the fraction of the disc mass in gas. Toomre (1974) suggests that these effects decrease the predicted $\sigma_r(R_0)$ in equation (18) by about 15 per cent. Now, if the disc has an exponential surface brightness profile which obeys Freeman’s law, equation (17c) implies $\alpha R_0 = 2.1 \pm 0.5$ (taking the central surface brightness of the disc to be $B_{0,c} = 21.65 \pm 0.5$ mag arcsec$^{-2}$). Thus, if $\sigma_r \approx 40 \text{ km s}^{-1}$ (Toomre 1974 and references therein), we find $Q(R_0) \sim 1.5–2$ in equation (18) if $2.6 \leq M/L \leq 7$, consistent with the value of 2.7 for the material in the galactic plane deduced from the Oort limit. A value of $Q \sim 2$ is suggested if our Galaxy is to satisfy the stability criterion of equation (13). Combining equations (2) and (4) we obtain,

$$\frac{v_m}{(\alpha M_D G)^{1/2}} = 0.62 \left[ \frac{Q v_m}{\sigma_r} \alpha R_0 \exp (-\alpha R_0) \right]^{1/2}. \hspace{1cm} (19)$$
Fortunately, the rhs of equation (19) is insensitive to the precise value of \( \alpha R \). Thus,

\[
\frac{v_m}{(\alpha M_D G)^{1/2}} \approx (0.3 \pm 0.04) \left( \frac{v_m Q}{\sigma_r} \right)^{1/2} \approx 0.7 Q^{1/2}
\]

(20)

If \( v_m/(\alpha M_D G)^{1/2} \geq 1.1 \), then \( Q \geq 2 \). These arguments are sensitive to the precise value of \( \sigma_r \), but it appears that the presently available data cannot exclude \( Q \approx 2 \).

In Section 3.2 we mentioned several problems in the interpretation of our results for early-type disc galaxies. Nevertheless, the mass-to-light ratios for the disc component deduced if \( v_m/(\alpha M_D G)^{1/2} = 1.1 \), are consistent with the mass-to-light ratios from Tinsley’s models. These are much smaller than the total mass-to-light ratio within the Holmberg radius (cf. Tables 1 and 3), thus it seems likely that even early-type disc galaxies possess dark haloes sufficient to yield \( v_m/(\alpha M_D G)^{1/2} \geq 1.1 \).

4 Conclusions

We have run a series of numerical models which we have argued are reasonable first approximations to the photometric and kinematic properties of spiral galaxies. The models are characterized by two dimensionless numbers, \( \alpha \sigma_m \) which measures the concentration of the halo component and \( v_m/(\alpha M_D G)^{1/2} \) which essentially determines the mass of the halo relative to that in the disc. The models are stable to axisymmetric instabilities (\( Q = 1.05 \), equation 4) but are unstable to bar formation unless they possess a hot component such that \( v_m/(\alpha M_D G)^{1/2} \geq 1.1 \) and this criterion has been found to apply over the range \( 0.1 < \alpha \sigma_m < 1.3 \). Results from exponential models with modified halo components and the Hunter \( N = 5 \) indicate that our results are insensitive to changes in the shape of the rotation curve or the adopted surface density. The stability properties also appear to be insensitive to a substantial increase in the random motions in the central parts of the disc.

By comparing the results on stability with observations of the rotation curves of Sc galaxies we find that the cool disc component must have a low mass-to-light ratio, \( M_D/L_D \approx 1.5 \). We have argued that this is likely to be a near equality and find that it is in good agreement with the stellar population models of Tinsley (1981) based on the local IMF and which include only stars of mass \( > 0.1 M_\odot \). Since the mean \( M/L \) within the Holmberg radius derived from the rotation curve is \( \approx 8 \), the dark material is the dominant contributor to the total mass.

If the population models apply to real disc galaxies, then the mass in \( H_1 \) is an extremely large fraction of (and may exceed) the total mass in disc stars in galaxies of types Sd–Sm. The bar instability in such systems would then be difficult to understand in terms of the global instabilities discussed here. This conclusion depends on the assumption that there are no extreme radial changes in the IMF, but if most of the mass in these systems were in the cool disc component we would expect them to be strongly barred. It is interesting that a large fraction (\( \approx 85 \) per cent) of Sd–Sm galaxies are classified as barred or intermediate by de Vaucouleurs (1963).

The population models yield a mass-to-light ratio consistent with that deduced from the Oort limit in our own Galaxy. They also predict much lower masses for early-type disc galaxies than those deduced from their rotation curves. Thus these galaxies may already possess enough dark matter to stabilize their discs.

The Hunter models develop a surface density profile which is approximately exponential over a wide range of radii, if the models are stable to bar formation. Models which are highly unstable develop a density which is poorly approximated by a single exponential. These results may provide an important clue as to the origin of exponential discs.
Disc galaxies

Our work suggests several interesting lines of future research. On the theoretical side, it would be advantageous to simulate disc galaxies with varying temperature profiles constructed from a distribution function rather than using the epicyclic approximation and the equations of stellar hydrodynamics. It is also important that discs with small $r_m$ be modelled with greater resolution. Our results also indicate that gas is likely to be a large fraction of the total disc mass in late-type galaxies. Even for Sc's our results imply $M_{H_1}/M_D \approx 0.26$. The mass in molecular hydrogen in external galaxies is quite uncertain and could constitute a substantial fraction of the total mass in gas. It is possible that a gas component could exert a braking influence on the growth of global instabilities and an estimate of this effect would be of considerable interest.

On the observational side, the comparisons of our models with spiral galaxies has been hampered by the lack of coordinated photometric and kinematic studies. As the situation improves, the conclusions discussed here are likely to become much more precise.

Acknowledgments

Thanks are due to Professor J. Silk for encouraging the authors' collaboration, to Professors C. McKee and M. Roberts for providing computer time and to Drs J. Sellwood, M. Fall, J. Binney and Professor J. Ostriker for many valuable comments. Drs F. Hohl and T. Zang generously provided materials which aided these calculations and their interpretation. G.E and G.L. thank the Space Sciences Laboratory, Berkeley, for financial support during the early stages of this work. G.E. thanks the SRC for support during the later stages. J.N. acknowledges support from NSF grant AST 79-15244 and thanks the Astrophysics Group at LBL for the use of its facilities.

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