Breaking stress of neutron star crust

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ABSTRACT

The breaking stress (the maximum of the stress–strain curve) of neutron star crust is important for neutron star physics including pulsar glitches, emission of gravitational waves from static mountains and flares from star quakes. We perform many molecular dynamic simulations of the breaking stress at different coupling parameters (inverse temperatures) and strain rates. We describe our results with the Zhurkov model of strength. We apply this model to estimate the breaking stress for time-scales \(~\sim\)1 s–1 yr, which are most important for applications, but much longer than can be directly simulated. At these time-scales, the breaking stress depends strongly on the temperature. For coupling parameter \(\Gamma \lesssim 200\) matter breaks at very small stress, if it is applied for a few years. This viscoelastic creep can limit the lifetime of mountains on neutron stars. We also suggest an alternative model of time-scale-independent breaking stress, which can be used to estimate an upper limit on the breaking stress.

Key words: equation of state – stars: interiors – stars: neutron.

1 INTRODUCTION

Ions in the neutron star crust can form a Coulomb crystal which determines the crust’s elastic properties. For example, the breaking stress \(\sigma_b\) is the maximum stress, as a function of strain, that the crust can support. If larger stress is applied to matter it will not remain in a static configuration. Neutron star crust is under high pressure. As a result, voids or fractures do not occur (Horowitz & Kadau 2009) and this simplifies how the crust breaks. The crust will break if the local stress is larger then the breaking stress at least one point in the crust. In this Letter, we mostly refer to breaking stress (force per unit area). The breaking strain (corresponding value of the fractional deformation) can be estimated from the linear stress–strain relation, where the slope is given by the well-known elastic constant or shear modulus (Strohmayer et al. 1991; Horowitz & Hughto 2008).

There are a lot of models which associate breaking of neutron star crust with observational phenomena. First of all, crust breaking causes pulsar glitches in the starquake model of Ruderman (1969, 1991) and in some recent models (for brief review see Chamel & Haensel 2008). Secondly, some models of magnetar giant flares involve crust breaking (Thompson & Duncan 2001). Here, the crust may need to be very strong to control the release of large magnetic energies responsible for extremely energetic gamma-ray bursts. Indeed, our results, see below, predict that the crust is very strong. Finally, the breaking stress limits the maximum size of ‘mountains’ on neutron stars. Mountains, on rapidly rotating stars, can efficiently emit gravitational waves (Ushomirsky, Cutler & Bildsten 2000; Haskell 2007) that could be detected by present large-scale interferometers (Abbott et al. 2007, 2008). Gravitational wave emission could be especially important for low-mass neutron stars that can have large deformations (Horowitz 2010). Moreover the balance of angular momentum gained from accretion, and radiated in gravitational waves, can control the spin period of some accreting stars (Watts et al. 2008).

For most terrestrial materials, the durability (time before breaking) \(\tau\) at a given stress is known to depend on the temperature and, of course, the applied stress (e.g. Zhurkov 1957; Regel\textsuperscript{c}, Slutsker & Tomashhevskii 1972; Slutsker 2004, 2005; Slutsker et al. 2007). In other words, the breaking stress is not just a constant, defined by the matter parameters (density, temperature and composition), but depends on the duration of the stress – the matter can break at lower stress, if one waits long enough. In fact, the dependence of \(\sigma_b(\tau)\) on \(\tau\) is logarithmic and we will refer to \(\tau\) as the time-scale of the process. The aim of the present Letter is to determine how \(\sigma_b\) depends on \(\tau\) and on temperature \(T\) for neutron star crust. We suggest a simple expression [see equation (6) parametrized by equation (8) below] for the durability of neutron star crust material at a given stress. This expression gives the lifetime of elastic deformations (e.g. mountains), and one can easily extract the breaking stress for the time-scale of the process of interest (e.g. a few years for pulsar glitches).

We also note the possible application of our results to the physics of dusty plasmas. Such a plasma can form a Yukawa crystal, just like the ions in neutron star crust. Its breaking stress can be measured in the laboratory as a threshold stress which is needed to obtain flow. The presence of such a threshold has been recently experimentally demonstrated by Gavrikov et al. (2010).

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Until recently, breaking stress studies (Smoluchowski 1970; Ruderman 1991) have been based on analogies with terrestrial materials instead of accurate calculations, and the uncertainties were large. Recently, large-scale molecular dynamics (MD) simulations of the breaking stress were performed (Horowitz & Kadau 2009). The breaking strain was found to be large \( \sim 0.1 \) and only slightly affected by impurities, defects and polycrystalline structure (grain boundaries). In the present Letter, we extend these results with extensive MD simulations of breaking stress at different temperatures and strain rates. We use these results to extract parameters of the Zhurkov model of strength (see Section 2.2 and Zhurkov 1957; Regel’ et al. 1972 for details) and apply this model to estimate the breaking stress for time-scales of 1 s–1 yr, which are of most interest for applications, but much longer than possible for direct MD simulations.

2 FORMALISM

The neutron star outer crust consists of ions and electrons. The inner crust also contains free neutrons (see, e.g., Haensel, Potekhin & Yakovlev 2007; Chamel & Haensel 2008). The electrons are degenerate and form a slightly polarizing background which screens the Coulomb interaction between ions. We describe electron screening by a Thomas–Fermi screening length

\[
\lambda_e = \frac{\sqrt{\pi}}{2} \left( \frac{\hbar c}{e^2} \right)^{1/2} \left( 3\pi^2 n_e \right)^{-1/3},
\]

(1)

where \( n_e \) is the electron number density, \( e \) is elementary charge and \( c \) is the speed of light. For simplicity, we assume the electrons are ultrarelativistic.

In this Letter, we discuss a one component plasma, where all ions have the same charge number \( Z \). The ions interact via a Yukawa potential

\[
\phi(r) = \frac{Z^2 e^2}{r} \exp(-r/\lambda_e),
\]

(2)

where \( r \) is the inter-ion distance. The state of the ion system can be characterized by the classical coupling parameter

\[
\Gamma = \frac{Z^2 e^2}{a T}.
\]

(3)

Here \( a = [13/(4\pi n_i)]^{1/3} \) is the ion sphere radius and \( T \) is the temperature in energy units. The ion number density is \( n_i = n_e/Z \), because the system is neutral.

In the ultrarelativistic limit, the ratio of the electron screening length \( \lambda_e \) to the ion sphere radius \( a \) depends only on \( Z \): \( \lambda_e/a \approx 5.41/Z^{1/3} \). We use \( Z \approx 29.4 \) in all of our calculations. This is the mean charge of the ions in the crust composition used by Horowitz, Berry & Brown (2007) and Horowitz & Kadau (2009). It corresponds to \( \lambda_e/a \approx 1.75 \). For such screening length, the crystal is thermodynamically stable at \( \Gamma > \Gamma_m \approx 176.1 \pm 0.7 \) (Horowitz et al. 2007), which is close to the melting point \( \Gamma_m = 175 \) expected for the one component plasma in the absence of electron screening (Potekhin & Chabrier 2000). We suppose that the breaking stress depends only slightly on screening length and our results can be used for matter of any composition.

A typical dynamical frequency is the ion plasma frequency \( \omega_p \),

\[
\omega_p = \sqrt{\frac{4\pi Z^2 e^2 n_i}{m_i}},
\]

(4)

where \( m_i = Am_n \) is the ion mass. The ratio of plasma temperature \( T_p = \hbar \omega_p \) to the temperature \( T \) characterizes the importance of quantum effects on ion motion. In our classical MD simulations we neglect quantum effects (assuming \( \hbar \to 0 \)), but they can be important at the conditions of realistic neutron star crust and we discuss them in Section 3.2.

2.1 Molecular dynamics simulations

To calculate the breaking stress we developed a parallel version of the YUKAWAMD code (Horowitz & Kadau 2009) where the ion system is strained by deforming periodic boundaries. We use the velocity Verlet algorithm (Verlet 1967) with a time-step \( \sim 0.1/\omega_p \).

The simulations are performed as follows: the original system is taken to be a perfect body-centred cubic crystal contained in a cubic box, that is aligned with the lattice planes. After thermalization, we start a shearing deformation of the periodic box boundaries according to \( x \to x + \epsilon y/2, y \to y + \epsilon x/2 \) and \( z \to z/ (1-\epsilon^2/4) \). This deformation conserve volume of the simulation box. Here the strain \( \epsilon = vt \) increases with time at constant strain velocity \( v \). Since the deformation is adiabatic, we define stress as \( \sigma = \delta E/\delta \epsilon \) (Landau & Lifshitz 1986), where \( \delta E \) is internal energy at unit volume. The derivative is calculated numerically as \( \sigma(\epsilon) = [\delta E(\epsilon) - \delta E(\epsilon - \Delta \epsilon)]/\Delta \epsilon \) with \( \Delta \epsilon = 0.002 \). The breaking stress \( \sigma_b \) is the maximum stress obtained during the shear simulation. The corresponding strain we refer to as the breaking strain \( \epsilon_b \).

To decrease the simulation time, we prepare a well-thermalized system at strain \( \epsilon = 0.05 \) and use it as an initial configuration for deformations with different \( v \). For the longest runs (with lowest \( v \)) we start from a configuration at larger strain. The initial parameters of this configuration are extracted from other simulations with larger \( v \). We also use a cut-off radius \( r_{\text{cut}} \) to omit interactions at \( r > r_{\text{cut}} \) and increase computation speed. Most of the simulations are done for 9826 ions in a periodic box and \( r_{\text{cut}} = 13 \lambda_e \). This large \( r_{\text{cut}} \) value is needed to accurately reproduce the shear modulus (see Horowitz & Hughto 2008). We discuss these simulation parameters in more detail in Section 3.

2.2 Zhurkov model of strength

The Zhurkov model (see e.g. Zhurkov 1957; Regel’ et al. 1972; Slutsker 2005) is based on the kinetic conception of strength. The main assumption is the following. The breaking event occurs due to thermal fluctuations. To achieve breaking, a fluctuation should have energy \( U \). This energy threshold is reduced by the product of the stress \( \sigma \) times an activation volume \( V \). The probability of a breaking fluctuation is \( \propto \exp[-(U - \sigma V)/T] \). Following this model, the matter will break at stress \( \sigma_b \), if it is applied over the time interval \( \tau \)

\[
\tau = \tau_0 \exp \left( \frac{U - \sigma_b V}{T} \right).
\]

(5)

where \( \tau_0 \) is a typical time-scale for ion motion, that we take to be \( \tau_0 = \omega_p^{-1} \). Defining dimensionless threshold energy \( \bar{U} = Ua/(Z^2 e^2) \) and stress \( \bar{\sigma} = \sigma/(n_i Z^2 e^2/a) \), equation (5) can be rewritten in the form

\[
\tau = \frac{1}{\omega_p} \exp \left( \bar{U} \Gamma - \bar{\sigma} \bar{N} \bar{\Gamma} \right).
\]

(6)

where \( \bar{N} = V n_i \) is the number of ions in the activation volume. We use \( \bar{U} \) and \( \bar{\Gamma} \) as fit parameters to reproduce MD data. Equation (6) is written for static stress, but our MD simulations are dynamical. The strain is linearly increasing with time, this produces a stress that also
increases, almost linearly, with time \( \sigma \approx \mu \epsilon = \epsilon \mu \). Of course, this linear dependence of \( \sigma \) on \( t \) is not correct after the breaking event. Here \( \mu \approx \sigma_o/\epsilon_b \) is the corresponding elastic constant. Equation (6) can be easily generalized for such deformation (see e.g. Slutsker & Aidarov 1983). The strain velocity \( v \) and breaking stress are related by the equation

\[
\frac{v}{\omega_p} = \frac{\epsilon_b}{N \Gamma \sigma_b} \exp(-\bar{U} \Gamma + \bar{\sigma}_b N \Gamma).
\] (7)

The stress is lower than \( \sigma_e \) during most of the deformation time. As a result, the time before breaking \( \tau = \epsilon_b/v \) is larger than \( \tau \) given by equation (6) by a factor of \( N \Gamma \bar{\sigma}_b \).

3 RESULTS

Our MD simulation results for the breaking stress of a system of 9826 ions are shown by symbols in Fig. 1. The data are fitted by equation (7) with parameters \( \bar{U} \) and \( \bar{N}(\Gamma) \),

\[
\bar{U} = 0.366, \quad \bar{N} = \frac{500}{\Gamma - 149} + 18.5. \quad (8)
\]

This fit is shown as lines in Fig. 1. The \( \epsilon_b \) in equation (7) was taken from spline interpolation of MD data over \( v/\omega_p \). In fact, \( \bar{\sigma}_b \) depends on \( \epsilon_b \), only logarithmical and one can set \( \epsilon_b = \epsilon_b(\Gamma) \) (and neglect the dependence on \( v/\omega_p \)) in equation (7) to reproduce the plot. There is good agreement between the fit and MD results, except for \( v/\omega_p \gtrsim 2 \times 10^{-4} \). We suppose that here the deformation velocity is too rapid and the breaking stress increases because there is not enough time for the system to break and relax the lattice. We exclude these points from the fit.

If equations (6) and (8) are formally applied for \( \Gamma \) below the melting value \( \Gamma_m \), we find that the crystal has a non-vanishing durability at fixed stress until \( \Gamma \gtrsim 150 \). This is in qualitative agreement with the results of Daligault (2006), who found Coulomb crystals to be metastable for \( \Gamma \gtrsim 150 \).

3.1 Dependence on system size and cut-off radius

To study finite size effects, we have performed a simulation with 250,000 ions at \( \Gamma = 800 \). For a strain speed \( v/\omega_p = 6.25 \times 10^{-6} \), we obtain a breaking stress \( \bar{\sigma}_b = 0.0187 \). This is only 4 per cent larger than the value 0.0180 obtained for a 9826 ion simulation. We conclude that finite size effects are not very large.

To study the dependence on cut-off radius \( r_{cut} \), we perform a set of additional runs with \( r_{cut} = 10 \lambda_c \) and \( r_{cut} = 16 \lambda_c \). The value \( r_{cut} = 10 \lambda_c \) is too small (for \( \Gamma = 800 \) the breaking stresses are systematically 10 per cent larger than for \( r_{cut} = 13 \lambda_c \)), but results for \( r_{cut} = 13 \lambda_c \) and \( r_{cut} = 16 \lambda_c \) are almost the same (within a few per cent statistical accuracy). We conclude that \( r_{cut} = 13 \lambda_c \) is large enough. It is easy to understand why so large a value of \( r_{cut} \) is needed. Let us estimate the interaction energy \( U_{cut} \) of an ion with other ions at distances \( r > r_{cut} \). Assuming the ions are uniformly distributed for \( r > r_{cut} \), one obtains

\[
U_{cut} = 3 \frac{Z^2 e^2 \lambda_c (r_{cut} + \lambda_c)}{a^2} \exp(-r_{cut}/\lambda_c). \quad (9)
\]

For \( r_{cut} = 10 \lambda_c \) and \( \lambda_c = 1.75a \), we obtain \( U_{cut} \approx Z^2 e^2/220a \) which is small compared with the Coulomb energy, but for \( \Gamma \gtrsim \Gamma_m \approx 176 \) it is of the same order of magnitude as the thermal energy, which is crucial for the breaking of the crystal. For \( r_{cut} = 13 \lambda_c \) the energy \( U_{cut} \approx Z^2 e^2/3400a \), which is small compared with thermal energy up to \( \Gamma \lesssim 1600 \).

3.2 Correction for quantum effects

In our classical MD simulations we cannot properly include quantum effects, such as zero-point vibrations. But such effects can be important for applications, since the crust temperature can be less than the plasma temperature. At \( T \ll T_p \) the breaking event takes place because of sub-barrier tunnelling, but not overbarrier thermal fluctuations as in a classical crystal (see e.g. Slutsker & Aidarov 1983). To include quantum effects, we make a simple assumption that the breaking stress mainly depends on the rms displacement of the ions. A similar idea was suggested by Salgani (1970) to describe polymers breaking and the results were shown to be in a good agreement with experiments with boron samples (Slutsker & Aidarov 1983). Let us introduce a renormalized coupling parameter \( \Gamma \) in such a way that the classical ion crystal at \( \Gamma = \bar{\Gamma} \) has approximately the same rms displacement as the quantum system at a given \( \Gamma \) and \( T/T_p \).

\[
\Gamma = \bar{\Gamma} \left[ \frac{1 + \frac{1}{4} T_p^2 u_1^2}{T^2 u_2^2} \right]^{-1/2} \approx \bar{\Gamma} \left[ 1 + 0.013 \frac{T_p^2}{T^2} \right]^{-1/2}. \quad (10)
\]

Here \( u_1 \approx 2.7986 \) and \( u_2 \approx 12.973 \) are moments of the phonon spectrum (see Baiko, Potekhin & Yakovlev 2001, and the values given are for a body-centred cubic crystal). We assume that \( \Gamma = \bar{\Gamma} \) substituted in equations (6) and (8) provide qualitatively correct results not only for classical crystals (\( T \ll T_p \)) but also for quantum crystals. In Section 4, it will be shown that the quantum corrections are typically not very large.

4 DISCUSSION

The knowledge of the parameters (equation 8) of the Zhurkov model of strength (equation 6) allow us to estimate the breaking stress for time-scales \( \sim 1 \text{s} \ll 1 \text{yr} \) which are the most interesting for astrophysical applications. Direct MD simulations for such time-scales are impossible because of the very small dynamical time-scale of the
ions $\sim v_{\text{th}} \sim 10^{-20}$ s. One would need at least a few $10^{20}$ MD steps to simulate one second, but one time-step takes approximately 10 core seconds of computer time.

Our estimates of the long time breaking stresses are shown in Fig. 2 for $^{56}$Fe matter at a density $10^9$ g cm$^{-3}$. For each of five temperatures ($T = 1 \times 10^6$, $2 \times 10^6$, $5 \times 10^7$, $1 \times 10^8$ and $2 \times 10^8$ K) the figure contains two lines: the solid line corresponds to the Zhurkov model for classical crystals (equations 6 and 8) and the dotted line includes corrections for quantum effects. The dotted lines are corrected for quantum effects in accordance with Section 3.2.

One can see a strong dependence of the breaking stress on the temperature. For the highest temperature $T = 2 \times 10^8$ K, the estimated breaking stress almost vanishes on the time-scale of a few years. This can limit the lifetime of mountains, supported by elasticity, on neutron stars with warm crust. For lower temperatures, the breaking stress is significantly larger and the $T$-dependence becomes weaker – the breaking stresses at $T = 10^9$ and $10^6$ K differ by less than a factor of two. Finally, for $T \lesssim 10^8$ K, the breaking stress is almost independent of temperature. The time-scale dependence is strong for high temperatures $T \sim 2 \times 10^8$ K and almost vanishes for low temperatures $T \lesssim 2 \times 10^6$ K.

In our model (Section 3.2), corrections for quantum effects are not very large (compare solid and dotted lines in Fig. 2). The reason is simple. At large temperatures the ions are almost classical, and quantum corrections are small. To describe low-temperature limit, let us estimate breaking stress $\sigma_b \approx U/N + \Delta \sigma_0$, where correction $\Delta \sigma_0 \sim -\ln(\tau_0)/\langle \Gamma N \rangle$ is small at low temperature. In our model, the quantum corrections are described by decreasing of the effective coupling parameter $\Gamma$, and affect only the correction term $\Delta \sigma_0$.

We should note that our estimates for 1 s–1 yr time-scales are based on the extrapolation of the MD data of more than 10 orders of magnitude in time. However our data only span 3 orders of magnitude in time ($10^{-7} \lesssim v/\omega_p \lesssim 10^{-4}$). So the extrapolation results should be taken with caution, especially for low coupling $\Gamma \lesssim 250$, where the dependence of the breaking stress on time is most significant. For large enough $\Gamma \gtrsim 800$, the breaking stress extrapolated to time-scales of a few years is only slightly lower than at time-scales achieved in our MD simulations (compare Fig. 1 and the right-hand axis of Fig. 2). We expect that the extrapolation for such strong coupling is more reliable. However, we cannot exclude that there are instabilities of the deformed crystal which have large enough time-scales to be insignificant in our MD simulations but could reduce the breaking stress for few second time-scales. In any case, the breaking stress at long time-scales cannot exceed the breaking stress $\sigma_b^{\text{max}}$ obtained in our longest simulations. The corresponding values can be fitted as

$$\sigma_b^{\text{max}} = \left(0.0195 - \frac{1.27}{\Gamma - 71}\right) n_s Z^2 e^2. \quad \text{(11)}$$

This fit provides an upper limit for the long time breaking stress.

5 CONCLUSIONS

We have performed extensive MD studies of the breaking stress of neutron star crust. Our results are in good agreement with the Zhurkov model of strength, and we have determined the corresponding parameters, equations (6) and (8). We apply this parametrization to estimate the breaking stress for very long time-scales $\sim 1$ s–1 yr, which are too large for direct MD studies. We demonstrate that for a coupling parameter $\Gamma \lesssim 200$ neutron star crust matter can break under very small stress, if it is applied for a few years. This result is based on an extrapolation of over 10 orders of magnitude in time-scale and should be treated with caution. We construct a qualitative model to include the influence of quantum corrections on breaking stress (Section 3.2). We show that these corrections are small (solid and dotted lines in Fig. 2). We also present a fit for the upper limit on the breaking stress for long times, equation (11), which is based on the breaking stresses for the slowest deformations directly obtained in our MD simulations. Our results can be used to estimate the lifetime of mountains on neutron stars and to obtain the breaking stress for the relevant time-scales of other astrophysical phenomena associated with crust breaking.

In this Letter, we concentrated on shear deformations of a perfect body-centred cubic crystal along the lattice planes. Realistic neutron star crust can contain impurities, grain boundaries and the deformation may not be aligned with the lattice. These effects were found to be unimportant by Horowitz & Kadau (2009), but more detailed studies are needed. In this Letter, we also considered only one ratio of screening length to ion sphere radius $\lambda_\infty/\alpha = 1.75$, but this ratio depends on the composition and can affect the breaking stress. In addition, breaking stress for the tension deformations can be important. We plan to study these effects in the near future.

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