Infinitely Strong Potential of Finite Rank

Smio Tani

Physics Department, Marquetee University
Milwaukee, Wisconsin 53233, U. S. A.

(Received August 23, 1966)

The scattering off a potential of finite rank (a sum of a finite number of separable potentials) is studied thoroughly in the neighborhood of infinity of potential strength; for this purpose we introduce the "strong coupling representation" by means of a canonical transformation. The same K matrix as for the infinitely strong repulsion is derived for the infinitely strong attraction. The difference between attraction and repulsion for a large but finite magnitude of strength can be understood in terms of the effects from bound states and excited states which exist for attraction and repulsion, respectively. It is shown that this difference disappears from the K matrix at low energies as the potential strength tends to infinity. The high energy behavior of the phase shift shows a characteristic difference between the two cases for a large but finite magnitude of strength. Levinson's theorem is discussed lucidly, and it is discussed how to interpret the identical result in the K matrix at infinity for both attraction and repulsion.

§1. Introduction

A wide class of potentials can be approximated by a sum of suitably chosen separable potentials. Since a problem with a potential of finite rank (a sum of a finite number of separable potentials) can be solved exactly by an algebraic method, various useful theorems can be demonstrated by algebraic methods of similar kind. These theorems are valid for the original potential, once the convergence is established as the rank tends to infinity. For instance, the convergence and the analytical continuation of a Born series can be treated most lucidly by this method. Weinberg originally developed it in a different form, and its merit was emphatically demonstrated by Coester; the present author analyzed its mathematical aspects from the physicist's point of view and gave a rigorous proof.

Let us quote some of the known formulas in order to state concretely the purpose of the present paper. We shall deal with a potential of the N-th rank and the strength \( \lambda \). Without loss of generality, it can be set in the form

\[
(k|V|k')=\lambda \sum_{m=1}^{N} u_m(k) u_m(k')^*,
\]

in momentum space for a partial wave. The form factors \( u_m(k) (m=1, 2, \ldots) \) are

\[\text{This work is supported by the Committee on Research of the Marquette University.}\]


..., $N$) are arbitrary for the moment, provided that all matrix elements

$$M_{mn} = M_{nm}^* = \int \frac{dk}{k^2 - E} u_m^*(k) u_n(k)$$

for the Hermitian matrix $M$ are well defined; since it is sufficient to take the standing wave solution, we shall take the principal value of the integral. We introduce the initial momentum $k_0$, and the energy $E$ is set in the form

$$E = k_0^2.$$

The normalized standing wave solution at energy $E$ takes the form

$$\phi(k; k_0) = \cos \delta(k_0) \left\{ \delta(k - k_0) - \frac{\lambda}{k_0^2 - E} \sum_{m,n} u_m(k) \left[ I + \lambda M \right]^{-1}_{mn} u_n^*(k_0) \right\},$$

where the unit matrix is denoted by $I$. The cofactor of the propagator, $(k^2 - E)^{-1}$, on the right-hand side will be referred to as the $T^{(n)}$ matrix

$$(k | T^{(n)} | k_0) = -\frac{\lambda}{k_0} \sum_{m,n} u_m(k) \left[ I + \lambda M \right]^{-1}_{mn} u_n^*(k_0),$$

while the $T^{(n)}$ matrix on the energy-shell will be called the $K$ matrix

$$K(E) \equiv (k_0 | T^{(n)} | k_0) \equiv (k_0 | K | k_0);$$

the phase shift $\delta(k_0)$ ought to be derived from the formula

$$\tan \delta(k_0) = (\pi/2k_0) K(E).$$

Now, the problem which we are concerned with is the behavior of the $K$ matrix in the neighborhood of infinity of $\lambda$. When the magnitude of $\lambda$ is so large that every eigenvalue of $M$ multiplied by $\lambda$ is larger than unity in magnitude, $K(E)$ can be expanded into inverse powers of $\lambda$:

$$K(E) = -\sum_{m,n} u_m(k_0) M_{mn}^{-1} u_n^*(k_0) + \lambda^{-1} \sum_{m,n} u_m(k_0) M_{mn}^{-1} u_n^*(k_0) + \cdots.$$ 

The salient feature on the right-hand side is that the first term which remains at infinity is independent of the sign of potential; namely, the limit of strong attraction gives the same $K(E)$ as the limit of strong repulsion. In a separate paper\textsuperscript{5) it will be pointed out that a strong square well potential, off a resonance,\textsuperscript{*}) gives rise to a scattering length similar to a strong square potential barrier (namely, a hard core potential), by using (1.8) when the rank $N$ is very large. This result in reference 5) is quite relevant, presumably, to the study of scattering by a singular potential.

\textsuperscript{*}) Here a resonance is referred to as a phenomenon in which the phase shift takes a value equal to $(1/2)\pi$ (modulo $\pi$) without differentiating whether the phase shift is increasing or decreasing as a function of energy. A resonance alluded here is very broad both in energy and in potential strength.
Needless to say, the comparison of a strong attraction with a strong repulsion (of the same shape) requires a scrutiny, because the physical nature of the two cases are completely different from each other as is shown by the presence or absence of bound states in the respective cases. It is the purpose of this paper to examine the neighborhood of infinity of $\lambda$ for a potential of finite rank in view of the importance of the problem mentioned above.

We shall introduce the strong coupling representation, which enables us to visualize the difference between attraction and repulsion clearly and also to set a limit of the validity of inverse power expansion in $\lambda$ unambiguously. In this approach every form factor $u_m(k) (m=1,2,\ldots,N)$ will be treated as the wave function of a bound (unstable) state if the potential is attractive (repulsive). There is a canonical transformation which converts the usual momentum space into the strong coupling representation. An eigenfunction belonging to the continuum of spectrum of the new “unperturbed” Hamiltonian is defined uniquely by the constraint that it be orthogonal to all form factors. The “interaction Hamiltonian” has a nonvanishing matrix element between a state in the continuum and one of the discrete states. The derivation of the scattering amplitude in the strong coupling representation is exactly the same in form as the Lee model or its version with an unstable $V$ particle; actually we shall deal with a Lee model in which there are $N$ of $V$ particles. As we shall see later in detail, the orthogonality constraint is responsible for the first term on the righthand side of (1·8), while the second and other higher order terms will be derived by the Lee model calculation.

In §2 we shall discuss the properties of an eigenfunction under the orthogonality constraints as mathematical preliminary to the later sections. In §3 we shall introduce the strong coupling representation. In §4 the Lee model calculation for the strong coupling $T^{(3)}$ matrix will be carried out. In §5 we shall discuss the scattering phase shift in the strong coupling representation. The detail of the comparison between attraction and repulsion as well as the interpretation of their superficially identical limit at infinity will be given in §6.

§2. Eigenfunctions under orthogonality constraints

In order that the calculations in the strong coupling representation may become neat, we shall limit the form factors more strictly than madly by the conditions (1·2) that the integrals exist. We assume that they are members of some orthonormal set

$$\int d\kappa u_m(k) \ast u_n(k) = \delta_{mn}. \quad (2·1)$$
We assume also that the kinetic energy is represented by a diagonal matrix in this orthonormal set

\[ \int dk k^2 u_m^*(k) u_m(k) = \Omega_m \delta_{mn}. \]  

(2.2)

The assumption (2.2) can be made without loss of generality, if a Hermian matrix \( \mathbf{W} \) defined by

\[ \int dk k^2 u_m^*(k) u_m(k) = W_{mn} = W_{nm}^*, \]  

(2.3)

exists. This is because there must be, then, a unitary matrix \( \mathbf{U} \) which diagonalizes \( \mathbf{W} \)

\[ \mathbf{U}^{-1} \mathbf{W} \mathbf{U} = \mathbf{Q}, \]  

(2.4)

where \( \mathbf{Q} \) is the diagonal matrix whose \( n \)-th diagonal element is \( Q_n \). And, if we replace \( u_m \) by \( u_m' \) according to

\[ u_m(k) \to u_m'(k) = \sum_{n=1}^{N} U_{mn} u_n(k), \]  

(2.5)

the two conditions (2.1)–(2.2) concerning \( u_m'(k) \) can be satisfied simultaneously; this proves that (2.1)–(2.2) can be assumed without loss of generality once (2.3) is assumed. Finally, we shall assume, along with the existence of the integrals (2.3), that the integrands vanish at infinity

\[ \lim_{k \to \infty} k^2 u_m^*(k) u_m(k) = 0, \]  

for all \( n \) and \( m; k \to \infty \);  

(2.6)

this excludes the possibility of convergence due to “accelerated oscillation (with a constant amplitude)”, such as is exemplified by \( \sin k^a \) with \( a > 1 \), as \( k \) approaches infinity.

The projection operator onto the form factor \( u_m(k) \) will be denoted by \( A_m \)

\[ \langle k | A_m | k' \rangle = u_m(k) \cdot u_m(k')^*. \]  

(2.7)

The kinetic energy operator will be denoted by \( \mathbf{W} \)

\[ \langle k | \mathbf{W} | k' \rangle = k^2 \delta(k - k'). \]  

(2.8)

By using the projection operators \( A_m (m = 1, 2, \ldots, N) \), we shall make the kinetic energy orthogonalized to every form factors

\[ \mathbf{W}_1 = (I - \sum_{m=1}^{N} A_m) \mathbf{W} (I - \sum_{m=1}^{N} A_m^*). \]  

(2.9)

Hereafter, this operator will be called the orthogonalized kinetic energy. Its eigenfunction which belongs to energy \( E \) will be denoted by \( \Psi(k; k_0) \).
\[ \int dk'(k' | W_1 | k') \cdot \psi(k'; k_0) = E \cdot \psi(k; k_0). \] \tag{2\cdot10} \\

By taking the inner product with any one of the form factors \( u_n(k) \), it can be seen readily that \( \psi \) must satisfy the orthogonality constraint

\[ \int dk u_n(k)^* \cdot \psi(k; k_0) = 0. \] \tag{2\cdot11} \\

(When \( E \) vanishes the argument is not valid, but we can ignore this exceptional case, because we can establish the completeness of \( \psi \), (2\cdot18), without giving a special treatment to the lower limit of the spectrum.) By taking into account of the orthogonality (2\cdot11), the left-hand side of (2\cdot10) can be simplified, to be read as

\[ k^2 \psi(k; k_0) - \sum_{n=1}^{N} u_n(k) \cdot \int dk' k' u_n(k')^* \cdot \psi(k'; k_0) = E \cdot \psi(k; k_0). \] \tag{2\cdot12} \\

A normalized solution of (2\cdot12) should take the form

\[ \psi(k; k_0) = \cos \omega(k_0) \left\{ \delta(k - k_0) + \frac{1}{k^2 - E} \sum_{n=1}^{N} u_n(k) \cdot t_n(k_0) \right\}. \] \tag{2\cdot13} \\

By a straightforward calculation it will be found that

\[ t_n(k_0) = -\sum_{n=1}^{N} \frac{M_n}{k_0^2} \cdot u_n(k_0)^*. \] \tag{2\cdot14} \\

If we take the limit of (1\cdot4) as \( \lambda \) tends to infinity formally, we shall obtain the same expression inside the curly bracket as in (2\cdot13). That is, the solution (1\cdot4) at infinity of \( \lambda \), (which gives rise to the first term on the right-hand side of (1\cdot8)), is exactly the same as an eigenfunction \( \psi \), (2\cdot10), of the Hermitian operator \( W_1 \), (2\cdot9). The phase shift introduced here will be called the orthogonality phase shift; it will be derived from

\[ \tan \omega(k_0) = -\frac{1}{2\pi} \sum_{n,m=1}^{N} u_n(k_0) \cdot M_n^{-1} u_m(k_0)^*. \] \tag{2\cdot15} \\

When we use the formula for products of principal values,

\[ \frac{1}{k_0^2 - k^2} \cdot \frac{1}{k_0^2 - k'^2} = \frac{1}{k^2 - k'^2} \left\{ \frac{1}{k_0^2 - k^2} - \frac{1}{k_0^2 - k'^2} \right\} \\
+ \pi^2 \cdot \delta(k_0^2 - k^2) \cdot \delta(k_0^2 - k'^2), \] \tag{2\cdot16} \\

it is rather straightforward to verify the orthonormal relation among eigenfunctions which belong to different energies

\[ \int dk \psi(k; k_1)^* \cdot \psi(k; k_2) = \delta(k_1 - k_2). \] \tag{2\cdot17}
On the other hand, the completeness relation holds in the space orthogonal to all form factors
\[
\int dk \varphi(k; k_0) \cdot \varphi(k'; k_0)^* = \delta(k-k') - \sum_{m=1}^{N} u_m(k) \cdot u_m(k')^*,
\]
(2.18)
although, its proof needs a lengthy calculations. Since the same calculation will be necessary again in §4, the rest of this section will be devoted to the proof of (2.18).

We shall substitute \( \varphi \) given by (2.13)–(2.14) on the left-hand side; the integral can be rewritten as follows on using (2.16):
\[
\int dk_0 \cos^2 \omega(k_0) \left\{ \delta(k-k_0) \delta(k'-k_0) 
- \delta(k-k_0) \frac{1}{k'^2 - k_0^2} \sum_{m,n} u_m(k_0) M_{mn}^{-1} u_n(k')^* 
- \delta(k'-k_0) \frac{1}{k^2 - k_0^2} \sum_{m,n} u_m(k) M_{mn}^{-1} u_n(k_0)^* 
+ \frac{1}{k^2 - k_0^2} - \frac{1}{k'^2 - k_0^2} \sum_{m,n} u_m(k) M_{mn}^{-1} u_n(k_0)^* u_n(k_0) M_{n'n}^{-1} u_n(k')^* \right\}
= \delta(k-k') + \frac{1}{k^2 - k_0^2} \sum_{m,n} u_m(k) \left\{ N_{mn}(k^2) - N_{mn}(k'^2) \right\}
+ \cos^2 \omega(k) M(k^2)^{-1} - \cos^2 \omega(k') M(k'^2)^{-1} \right\} u_n(k')^*,
\]
(2.19)
where the argument of \( M_{mn}^{-1} \) is \( k_0^2 \) unless stated otherwise explicitly. The matrix element \( N_{mn}(k^2) \) is defined by the integral
\[
N(k^2)_{mn} = \int dk_0 \cos^2 \omega(k_0) \frac{1}{k^2 - k_0^2} \sum_{m,n} M(k_0^2)^{-1} u_n(k_0)^* u_n(k_0) M(k_0^2)^{-1}.
\]
(2.20)
As a preparation for carrying out the integral (2.20), we shall generalize the real function \( M_{mn} \) of a real variable \( E \) into an analytic function of a complex variable \( \varepsilon \) in the complex plane cut along the positive real axis; thus, in place of (1.2) we set
\[
f_{mn}(\varepsilon) = \int dk \frac{u_m^*(k) u_n(k)}{k^2 - \varepsilon}.
\]
(2.21)
The real part of the boundary value as \( \varepsilon \) tends to real positive \( E \) from above is \( M_{mn} \), (1.2),
\[
\lim_{\varepsilon \to 0} \Re[f_{mn}(E+i\varepsilon)] = M_{mn},
\]
(2.22)
while its imaginary part, to be denoted by \( I(E)_{mn} \), is given by
The determinant of the complex matrix \( f_{m*} \) will be denoted by \( \Delta \)

\[
\Delta(z) = \det |f_{m*}(z)|. \tag{2.24}
\]

The real and imaginary parts of the boundary value of \( \Delta \) as \( z \) tends to real positive \( E \) from above will be denoted by \( U(E) \) and \( V(E) \), respectively,

\[
\lim_{\varepsilon \to 0} \Delta(E + i\varepsilon) = U(E) + iV(E). \tag{2.25}
\]

We must note that the imaginary part \( I_{m*} \) is a product of separate factors, \((\pi/2k_0)^{1/2}u_m(k_0)^*\) concerning \( m \) and \((\pi/2k_0)^{1/2}u_n(k_0)\) concerning \( n \); this fact is extremely helpful in reducing the determinants into a simpler form. Thus it follows that the real part \( U(E) \) in (2.25) is simply the determinant of \( M_{m*} \), (2.22),

\[
U(E) = \det |M(E)_{m*}|. \tag{2.26}
\]

We shall denote the \((m,n)\)-cofactor in the determinant \( U \) by \( U'_{(m,n)} \). *)

Because of the separability of \( I_{m*} \), the imaginary part \( V(E) \) in (2.25) is given by

\[
V(E) = \sum_{m,n} U(E)'_{(m,n)} \cdot I(E)_{m*}. \tag{2.27}
\]

Obviously the matrix element of the inverse \( M^{-1} \) is given by

\[
M(E)_{m*}^{-1} = U(E)'_{(m,n)} [U(E)]^{-1}. \tag{2.28}
\]

The right-hand side of (2.15) can be rewritten as

\[
\tan \omega(k_0) = -\sum_{m,n} M(E)_{m*}^{-1} I(E)_{m*} = -V(E) \cdot [U(E)]^{-1}. \tag{2.29}
\]

The above equation means that the orthogonality phase shift \( \omega(k_0) \) is the imaginary part of the boundary value of an analytic function in the cut plane, namely

\[
\omega(k_0) = -\lim_{\varepsilon \to 0} \Im \left[ \log \Delta(E + i\varepsilon) \right]. \tag{2.30}
\]

Accordingly, we shall set

\[
\cos \omega(k_0) = -\frac{U(E)}{\sqrt{U(E)^2 + V(E)^2}}. \tag{2.31}
\]

*) A co-factor means the relevant minor of the determinant multiplied by a suitable signature; the expansion of the determinant is given by

\[
U = \sum_{n=1}^{N} U'_{(n,m)} M_{nm}, \quad \text{for any } n.
\]
where the sign is chosen in such a way as the behavior of \( \omega(k_0) \) as a function of momentum \( k_0 \) is in conformity with Levinson’s theorem and the absolute definition of phase shift which will be discussed in §6.

We shall establish then that the integrand of (2·20) is the imaginary part of the boundary value of the analytic function \( f(z) \). If we use (2·23), (2·28), and (2·31), it can be rewritten as

\[
\cos^2 \omega(k_0) \sum_{p,q} M(k_0)^{-1} u_p(k_0)^* u_q(k_0) M(k_0)^{-1} = (2k_0/\pi) [U(E)^2 + V(E)^2]^{-1} \sum_{p,q} U(E)(p,m) \cdot I(E)_{pq} \cdot U(E)(q,n).
\]

(2·32)

We shall use the formula

\[
U_{(p,m)} \cdot U_{(q,n)} = U_{(p,m)} \cdot U_{(q,n)} + U_{(p,m)} U''_{(q,n)}.
\]

(2·33)

where \( U''_{(p,m)} \) denotes the \((n, q)\) cofactor of \( U_{(p,m)} \). Because of the sign involved in the definition of a cofactor, we shall find

\[
U''_{(p,m;n,q)} = - U''_{(n,m;p,q)}.
\]

(2·34)

On using (2·33) and (2·34) on the right-hand side of (2·32), we shall find that it can be transformed into

\[
(2k_0/\pi) [U^2 + V^2]^{-1} \left\{ \sum_{p,q} U_{(p,q)} \cdot I_{pq} \cdot U_{(q,n)} - \sum_{p,q} U''_{(n,m;p,q)} \cdot I_{pq} \cdot U \right\},
\]

which is equal to

\[
(2k_0/\pi) [U^2 + V^2]^{-1} \left\{ V \cdot U_{(n,m)} - U \cdot \sum_{p,q} U''_{(n,m;p,q)} \cdot I_{pq} \right\}
\]

(2·35)

because of (2·27). Obviously we have

\[
\lim_{\varepsilon \to 0} \text{Re} [\Delta(E + i\varepsilon)^{-1}] = U \cdot [U^2 + V^2]^{-1},
\]

\[
\lim_{\varepsilon \to 0} \text{Im} [\Delta(E + i\varepsilon)^{-1}] = -V[U^2 + V^2]^{-1}.
\]

(2·25·a)

If we denote the \((n, m)\) cofactor of the complex determinant \( \Delta \) by \( \Delta_{(n,m)} \), we shall find, because of the separability of \( I_{pq} \),

\[
\lim_{\varepsilon \to 0} \text{Re} [\Delta(E + i\varepsilon)'_{(n,m)}] = U'_{(n,m)},
\]

\[
\lim_{\varepsilon \to 0} \text{Im} [\Delta(E + i\varepsilon)'_{(n,m)}] = \sum_{p,q} U''_{(n,m;p,q)} \cdot I_{pq}.
\]

(2·36)

Thus, except for the factor \(- (2k_0/\pi)\), the integrand of (2·20) is equal to the imaginary part of the boundary value of \( f(z) \) as \( z \) tends to real positive \( E \) from above. Therefore we can write down (2·20) in the form
Infinitely Strong Potential of Finite Rank

\[ N(k^2)_{\text{ns}} = -\frac{1}{\pi} \int_0^\infty dE \frac{1}{E-k^2} \text{Im} \left[ f(E+i0)_{\text{ns}} \right]. \]  
(2.37)

When we recall the two properties (2.1) and (2.2) of the form factors, we can easily show that the asymptotic form of \( f(z)_{\text{ns}} \) for \( z \) of very large magnitude is

\[ f(z)_{\text{ns}} = -\delta_{\text{ns}} (z^{-1} + \Omega_n z^{-2}) + O(|z|^{-3}), \]

as \( |z| \to \infty \),
(2.38)

by starting from its definition (2.21); the order of magnitude of the remainder is smaller than \( |z|^{-3} \). Thus, if we define

\[ F(z)_{\text{ns}} = f(z)_{\text{ns}} + (z-\Omega_n) \delta_{\text{ns}} = \left[ \Delta(z)'_{(n,n)}/\Delta(z) \right] + (z-\Omega_n) \delta_{\text{ns}} \]
(2.39)

for every \( m \) and \( n \), we shall derive an analytic function of \( z \) in the cut plane, which vanishes at infinity, and yet gives rise to the same imaginary part of its boundary value as \( f(z)_{\text{ns}} \) (on the real axis). Thus the integral in (2.37) reduces to a typical dispersion theoretic integral, which can be carried out straightforwardly;

\[ N(k^2)_{\text{ns}} = -\text{Re} [F(k^2+i0)_{\text{ns}}]. \]
(2.40)

Explicitly the result reads

\[ N(k^2)_{\text{ns}} = (\Omega_n-k^2) \delta_{\text{ns}} - \cos^2 \omega(k) M(k^2)_{\text{ns}}^{-1} \]
\[ - \{ V(k^2) / [U(k^2) + V(k^2)] \} \cdot \sum_{r,s} U''_{(n,n,s,s)} I(k^2)_{rs}. \]
(2.41)

However, when substituted on the right-hand side of (2.19), the last term of the above result can be dropped. This is because

\[ \sum_{m,n,r,s} U(k^2)'_{(n,m,s,s)} \cdot I(k^2)_{rs} \cdot u_m(k) \cdot u_n(k')^* \]
\[ = \sum_{m,n,r,s} (\pi/2k) U''_{(n,m,s,s)} [u_n(k')^* u_m(k)] \cdot [u_p(k)^* u_q(k)] = 0, \]
(2.42)

which follows from the fact that it is a sum, over all possible pairs of \( n \) and \( p \), of the determinants derived from \( U = \det [M(k^2)] \) by replacing its \( n \)-th and \( p \)-th rows by \( u_n(k')^* u_m(k) \) and \( u_p(k)^* u_m(k) \), respectively; since the elements in the corresponding columns are proportional to a fixed ratio \( u_n(k')^* : u_p(k)^* \), the determinant vanishes for every pair of \( n \) and \( p \). Similar arguments apply in order to drop the corresponding term from the result for \( N(k^2)_{\text{ns}} \). Thus, the sum on the right-hand side of (2.19) reduces, on using (2.41) and (2.42), to...
\[ \sum_{m,n} u_m(k) \{ N(k^2)^{m,n} + \cos^2 \phi(k) M(k^2)^{m,n} - N(k'^2)^{m,n} - \cos^2 \phi(k') M(k'^2)^{m,n} \} u_n(k')^* = (k^2 - k'^2) \sum_{m,n} u_m(k) \cdot u_n(k')^*. \]  

(2·43)

The completeness relation (2·18) follows immediately, when (2·43) is substituted into the last member of (2·19).

§3. **Strong coupling representation**

The two statements formulated by (2·1) and (2·18) guarantee that there is a complete set in momentum space which consists of:

\[ u_m(k), \quad m=1, 2, \ldots, N, \quad \text{for discrete states,} \]

(3·1·a)

and

\[ \varphi(k; k_0), \quad 0 \leq k_0 \leq \infty, \quad \text{for the continuum.} \]

(3·1·b)

Therefore the transformation of base from free waves (from momentum space) into the set defined by (3·1·a)–(3·1·b) is a canonical transformation. After this canonical transformation is performed, the wave function \( \Phi(k; k_0) \) belonging to the energy \( k_0^2 \) is set in the form

\[ \Phi(k; k_0) = \sum_{n=1}^{N} u_n(k) A_n(k_0) + \int dk \varphi(k; k_0) \chi(k; k_0). \]

(3·2)

We shall obtain the Schrödinger equation in the new representation by operating the total Hamiltonian on the wave function \( \Phi \). The new representation will be called the strong coupling representation because it is suitable to explore the neighborhood of infinity of potential strength. As for the potential energy, (1·1), it will be represented by a constant matrix \( \lambda \mathbf{I} \) for the discrete states, because of (2·1) and (2·11),

\[ \int dk' \langle k | V | k' \rangle \Phi(k'; k_0) = \sum_{n=1}^{N} \lambda u_n(k) A_n(k_0), \]

(3·3)

In dealing with the matrix elements of the kinetic energy, we shall take advantage of the equation

\[ k^2 \varphi(k; k_0) = (k^2 - k_0^2 + k_0^2) \varphi(k; k_0) \]

\[ = k^2 \varphi(k; k_0) - \cos \phi(k_0) \sum_{m,n} u_m(k) M(E)^{m,n} u_n(k_0)^*, \]

(3·4)

which follows from (2·13)–(2·14). By using (2·11) again, we shall have

\[ \int dk u_m(k)^* k^2 \varphi(k; k_0) = - \cos \phi(k_0) \sum_{n=1}^{N} M(E)^{m,n} u_n(k_0)^* \]

\[ = (m | H' | k_0); \]

(3·5)
we shall also obtain, on remembering (2·17),

$$\int dk \Phi(k; k_1) \Phi^*(k; k_2) = k_1^2 \delta(k_1 - k_2). \quad (3·6)$$

As consequences of (2·2), (3·3), (3·5), and (3·6), the Schrödinger equation in the strong coupling representation reads

$$\left(\lambda + \Omega_m - E\right) A_m(k_0) = -\int dk \langle m | H' | k \rangle \cdot \chi(k; k_0), \quad (3·7)$$

$$\left(k^2 - E\right) \chi(k; k_0) = -\sum_{m=1}^{N} \langle k | H' | m \rangle \cdot A_m(k_0). \quad (3·8)$$

The Schrödinger equation (3·7)–(3·8) reveals that the strong coupling representation can be visualized in the following fashion: There are $N$ of discrete states whose unperturbed levels are at

$$E_m = \Omega_m + \lambda, \quad m=1, 2, \ldots, N, \quad (3·9)$$

respectively. Obviously, if the potential is strongly attractive,

$$\lambda < 0, \quad \text{and} \quad |\lambda| \gg \Omega_m, \quad \text{for all} \quad m, \quad (3·10)$$

they are deep lying bound states with levels of the order of $|\lambda|$

$$E_m \approx -|\lambda|, \quad \text{for all} \quad m; \quad (3·11)$$

on the other hand, if the potential is strongly repulsive,

$$\lambda \gg \Omega_m > 0, \quad \text{for all} \quad m, \quad (3·12)$$

they are highly excited states with levels of the order of $\lambda$

$$E_m \approx \lambda, \quad \text{for all} \quad m. \quad (3·13)$$

The matrix element $\langle m | H' | k \rangle$ on the right-hand sides of (3·7)–(3·8) represents the "interaction Hamiltonian" in the strong coupling representation; its form is defined by (3·5) and independent of the potential strength $\lambda$. These equations (3·7)–(3·8) are exactly the same in form as the Lee model or its version with an unstable particle, except that here the number of $V$ particles is increased to $N$. If we solve these equations by perturbation expansion, (by developing into powers of the matrix elements of the "interaction" $H'$), we shall have, to the lowest order,

$$\chi(k; k_0) = \delta(k - k_0) + \frac{1}{k^2 - E} \sum_{m=1}^{N} \frac{\langle k | H' | m \rangle \langle m | H' | k_0 \rangle}{\lambda + \Omega_m - E} + \ldots \quad (3·14)$$

If this is valid, the first term for the $T^{(0)}$ matrix in the strong coupling repre-
sentation is of the order of $\lambda^{-1}$ when the magnitude of $\lambda$ is very large; this is shown in (3·14) by the presence of $\lambda$ in the denominator of the second term. As will be shown in more detail in the following sections, small corrections derived in the strong coupling representation account for the second and the following smaller terms in (1·8), while the first term is due to the distortion of the wave function $\psi$ as defined by (2·13)–(2·15). We shall not develop the detail of the perturbation expansion, however, because the fact that the matrix elements of the interaction are of special form given in (3·5) enables us to solve the problem exactly.

§4. $T^{(0)}$ matrix in strong coupling representation

It follows from (3·8) that, disregarding the normalization, the continuum wave function in the strong coupling representation $\chi(k; k_0)$ can be set in the form

$$\chi(k; k_0) = \delta(k - k_0) - \frac{1}{k^2 - E} \sum_{n=1}^{N} (k|H'|m) A_m(k_0). \quad (4·1)$$

Substituting (4·1) into (3·7), we shall find for $A_m(k_0)$

$$A_m(k_0) = \sum_{n=1}^{N} [(E - \lambda) I - \Omega + N]^{-1} \langle n|H'|k_0 \rangle, \quad (4·2)$$

where a matrix element of the matrix $N$ is primarily defined by

$$N(E)_{mn} = \int \frac{dk}{k^2 - E} \frac{(m|H'|k)(k|H'|n)}{k^2 - E} \quad (4·3)$$

to be used in (4·2); however, on substituting the matrix element $(m|H'|k)$ defined in (3·5) and its Hermitian conjugate into the integrand, we shall find that it is the same integral as in (2·20). The matrix $\Omega$ in (4·2) is the diagonal matrix defined by its diagonal element $\Omega_m$, (2·2). The integral (4·3) has been carried out already in §2; but we shall quote the result in a slightly different form which turns out to be more convenient later. Thus, referring to (2·41), we shall set

$$N(E)_{mn} = (\Omega_m - E) \delta_{mn} - M(E)_{mn}^{-1} + P(E)_{mn}, \quad (4·4)$$

where $P(E)_{mn}$ is defined by

$$P(E)_{mn} = V(E) \cdot [U(E)^2 + V(E)^2]^{-1} \{ V(E) \cdot M(E)_{mn}^{-1} - \sum_{p,q} U_{pq} U_{mpr} I(E)_{nr} \}. \quad (4·5)$$

From (4·2) and (4·4), we have
If we set (4.1) in the form
\[ \chi(k; k_0) = \delta(k - k_0) + \frac{1}{k^2 - E} (k | T_{i}^{(i)} | k_0), \]
then \( T_{i}^{(i)} \) is given, on using (3.5) and (4.6), by
\[ (k | T_{i}^{(i)} | k_0) = \cos \omega(k) \cos \omega(k_0) \sum_{n,m} u_n(k) [\lambda M(E) M(k^2) + M(k^2) - M(E) P(E) M(k^2)]_{m,n}^{-1} u_m(k_0)^* . \]
Then, we can derive formally a phase shift from
\[ \tan \delta_{s}(k_0) = \frac{1}{2} (\pi / k_0) (k_0 | T_{i}^{(i)} | k_0) ; \]
this phase shift will be called the strong coupling correction to the phase shift, because it is very small for a large potential strength, except in some domain of high energies, which we shall show in what follows.

Now, we shall discuss the behavior of the strong coupling phase shift \( \delta_{s}(k_0) \) at low energies when \( |\lambda| \) is very large. There is a unitary transformation \( S \) which diagonalizes the Hermitian matrix \( M \), (1.2), and an eigenvalue of \( M \) will be denoted by \( \eta_m (m = 1, 2, \cdots, N) \); thus we set
\[ S^{-1}MS = \eta . \]
We shall introduce also the transformed form factors
\[ \sigma_n(k) = \sum_{m=1}^{N} u_n(k) S_{mn}, \quad n = 1, 2, \cdots, N, \]
and their Hermitian conjugates. After some calculations, we shall find
\[ \tan \delta_{s}(k_0) = \frac{1}{2} (\pi / k_0) \cos^2 \omega(k_0) \sum_{m,n} \sigma_n(k_0) D_{mn}^{-1} \sigma_m(k_0)^\dagger , \]
where the matrix element of \( D \) is given by
\[ D(E)_{mn} = [\lambda \eta(E)_m^2 + \eta(E_n)] \delta_{mn} + \frac{1}{2} \pi \tau [k_0 (1 + t^2)]^{-1} \eta_m(k_0) \cdot \sigma_m(k_0)^\dagger , \]
in which \( \tau \) denotes the negative of the tangent of the orthogonality phase shift represented in terms of the above introduced quantities, (cf. (2.15)),
\[ \tau(E) = \frac{1}{2} (\pi / k_0) \sum_{m=1}^{N} \eta(E)_{m}^{-1} |\sigma_m(k_0)|^2 = - \tan \omega(k_0) . \]
On the other hand, it follows from (1.2), (4.10) and (4.11) that each
eigenvalue \( \eta(E) \) is given by an integral

\[
\eta(E) = \int \frac{\sigma_m(k)}{k^2 - E} \, dk.
\]

(4.15)

Since the integrand is positive-definite, an eigenvalue is positive at zero energy; the same must be true over a certain domain at low energies, because all eigenvalues are continuous functions of the energy \( E \). Consequently, the first term, \( \lambda \eta^2 \), on the right-hand side of (4.13) dominates in the limit as \( \lambda \) approaches infinity. That is, the right-hand side of (4.12) allows an expansion into inverse powers of \( \lambda \). The result in the first order is given by

\[
\tan \delta_s(k_0) = \left[ 2\lambda k_0 (1 + \epsilon^2) \right]^{-1} \pi \sum_{m=1}^{N} \eta(E)^2 \cdot |\sigma_m(k_0)|^2,
\]

(4.16)

which agrees with the corresponding term derived from the perturbation calculation (3.14).

The high energy behavior of the strong coupling phase shift \( \delta_s(k_0) \) can be discussed by using similar formulas, but the conclusions change completely. It follows from (2.38) that at high energies we have

\[
M(E)_{nn} = -\delta_{nn} [E^{-1} + E^{-2} \Omega_n] + O(E^{-3}).
\]

(4.17)

This means that the matrix \( M \) is already diagonal (without a unitary transformation \( S \) in (4.10)) in the approximation in which we retain the first two terms in the asymptotic expansion. Thus, by a substitution according to

\[
\eta(E)_{nn} \rightarrow -[E^{-1} + E^{-2} \Omega_n], \quad \sigma_m(k_0) \rightarrow u_m(k_0) \quad \text{and} \quad \sigma_m^*(k_0) \rightarrow u_m(k_0)^* \]

(4.18)

for the respective quantities in (4.12) and (4.13), we can derive the corresponding formulas at high energies. We shall be concerned with \(^\ast\)

\[
D_{mn}^{\Omega} = \eta(E)^2 \left\{ (\lambda + \Omega_m - E) \delta_{nn} - (\pi/2)^2 u_m(k_0) \cdot u_m(k_0)^* \sum_{m=1}^{N} |u_p(k_0)|^2 \right\}
\]

(4.19)

where the last term on the right-hand side comes from \( P \), but it vanishes in the high energy limit because of (2.6). Consequently the asymptotic form of (4.12) is

\[
\tan \delta_s(k_0) = \frac{1}{2} \pi \cos^2 \omega(k_0) \sum_{m=1}^{N} \frac{E [k_0 |u_m(k_0)|^2]}{\lambda + \Omega_m - E}.
\]

(4.20)

This result clearly shows that when the potential is repulsive (\( \lambda \) is positive) there are \( N \) resonances at the unperturbed levels (3.9) of unstable states. When the potential is attractive (\( \lambda \) is negative), the strong coupling

\(^\ast\) It is important to factor out \( \eta_{nn} \), because we are interested in the root of \( D_{nn} \) in (4.20).
correction remains small because of (2·6). In both cases we must keep in
mind that the expansion into inverse powers of λ fails on the right-hand
side of (4·20) in the extreme high energy region where
\[ E \gg |\lambda|, \quad (4·21) \]
and therefore we have to expand the result into powers of (\( \lambda/E \)). The
last point is important when we want to establish the validity of the first
Born approximation of the total phase shift in the domain (4·21), which
will be discussed in §6.

§5. Addition theorem for phase shifts

In order to derive the scattering phase shifts, we shall take the
Fourier transform of the wave function (3·2) and derive its asymptotic
form at large distance in configuration space. For simplicity we shall
assume that we are dealing with the \( S \) wave; a similar proof should be
available for other partial waves. The contribution of the discrete states
should be small outside the range of the potential. Therefore, discarding
the first term, we shall start from the integral
\[ \tilde{\Phi}(r; k_0) = \int dk_1 \sin kr \cdot \varphi(k; k_1) \cdot \chi(k_1; k_0). \quad (5·1) \]
The transformed wave function will be denoted by the tilde above the
same notation in momentum space. If we restrict ourselves to the asymptotic
form, the following formula for a principal value integral is useful\(^9\)
\[ \int_0^\infty dk \frac{1}{k^2 - k_1^2} \sin kr \cdot f(k) = \frac{\pi}{2k_1} \cos(k_1 r) \cdot f(k_1). \quad (5·2) \]
On taking into account of (2·13)–(2·15) also, the asymptotic form of (5·1)
will be given by
\[ \tilde{\Phi}(r; k_0) = \int_0^\infty dk_1 \sin [k_1 r + \omega(k_1)] \cdot \chi(k_1; k_0). \quad (5·3) \]
We substitute (4·7) for the form of \( \chi \), apply the formula (5·2) again, and
take into account of (4·10) to find
\[ \tilde{\Phi}(r; k_0) = [\cos \delta_0(k_0)]^{-1} \sin [k_0 r + \omega(k_0) + \delta(k_0)]. \quad (5·4) \]
The last result shows that the total scattering phase shift is defined by the
sum of the orthogonality phase shift and the strong coupling correction
\[ \delta(k_0) = \omega(k_0) + \delta_0(k_0). \quad (5·5) \]
The same kind of addition theorem for phase shifts as has led to (5·5)
can be applied to establish, concerning the orthogonality phase shift, that
\[ \omega(0) - \omega(\infty) = N\pi. \quad (5.6) \]

In the next section, we shall discuss Levinson’s theorem on the base of (5.6), and here we are concerned only with its proof. Suppose, we consider the orthogonality constraint to only the first \( m \) form factors. Then, instead of (2.9), we deal with

\[
W^{(m)}_1 = \left( I - \sum_{i=1}^{m} A_i \right) W \left( I - \sum_{i=1}^{m} A_i \right), \quad (5.7)
\]

where by definition we shall set

\[
W^{(m)}_1 = W. \quad (5.8)
\]

An eigenfunction \( W^{(m)}_1 \) which belongs to energy \( E \) will be denoted by \( \Psi^{(m)} \)

\[
\int dk' (k | W^{(m)}_1 | k') \Psi^{(m)}(k'; k_0) = EW^{(m)}(k; k_0). \quad (5.9)
\]

By applying the same method as discussed in §2, the completeness in the subspace orthogonal to the first \( m \) form factors

\[
\int dk'' \Psi^{(m)}(k; k'') \cdot \Psi^{(m)}(k'; k'')^* = \left( k | I - \sum_{j=1}^{m} A_j | k' \right), \quad (5.10)
\]

can be established. Now, we shall add the orthogonality to the \((m+1)\)-st form factor. Since \( u_{m+1} \) is orthogonal to all the form factors taken previously, we can expand it in terms of \( \Psi^{(m)} \) without changing its normalization; namely, if we set

\[
\bar{u}_{m+1}(k) = \int dk' u_{m+1}(k') \Psi^{(m)}(k'; k)^*, \quad (5.11)
\]

we shall have

\[
\int dk |\bar{u}_{m+1}(k)|^2 = 1. \quad (5.12)
\]

The orthogonalized kinetic energy will be revised to

\[
W^{(m+1)}_1 = (I - A_{m+1}) W^{(m)}_1 (I - A_{m+1}), \quad (5.13)
\]

while its eigenfunction \( \Phi^{(m+1)} \) will be expanded into \( \Psi^{(m)} \) in the form

\[
\Phi^{(m+1)}(k; k_0) = \int dk' \Psi^{(m)}(k; k') \cdot \Phi^{(m+1)}(k'; k_0). \quad (5.14)
\]

The Schrödinger equation for \( \Phi^{(m+1)} \) reads

\[
(k^2 - k_0^2) \cdot \Phi^{(m+1)}(k; k_0) = \bar{u}_{m+1}(k) \int dk' k'^2 \bar{u}_{m+1}(k')^* \cdot \Phi^{(m+1)}(k'; k_0); \quad (5.15)
\]
this is obtained from (5·9) by replacing \( m+1 \) by \( m \) and by using (5·11)–(5·14), wherein its structure is the simplest case, \( N=1 \), of the equation discussed in (2·12)–(2·15). Its normalized solution can be set in the form

\[
\varphi^{(m+1)}(k; k_0) = \cos \omega^{(m+1)}(k_0) \left\{ \theta(k-k_0) + \frac{1}{k^2-k_0^2} (k | R^{(m+1)} | k_0) \right\}. \tag{5·16}
\]

The wave function \( \varphi^{(m+1)} \) represents the distortion (of \( \Psi^{(m+1)} \)) in reference to \( \Psi^{(m)} \); the phase shift associated with it, \( \omega^{(m+1)} \), will be called the \((m+1)\)st component of the orthogonality phase shift. It will be derived from

\[
\tan \omega^{(m+1)}(k_0) = (\pi/2k_0)(k_0 | R^{(m+1)} | k_0). \tag{5·17}
\]

As has been discussed some time ago,\(^ 10 \) we can establish the following formula (which is crucial in the discussion of Levinson’s theorem), if there is only a single orthogonality constraint,

\[
\omega^{(m+1)}(0) - \omega^{(m+1)}(\infty) = \pi. \tag{5·18}
\]

Since the above arguments apply in general for any \( m \), the corresponding results are valid for any \( m=0,1,\cdots,N \). The wave function \( \Psi^{(m+1)} \) as defined by (5·14) is given by the convolution of two wave functions, \( \Psi^{(m)} \) and \( \varphi^{(m+1)} \), each of which produce its own phase shift; thus the arguments which led from (5·1) to (5·5) can be repeated to show that the total phase shift involved in \( \Psi^{(m+1)} \) is the sum of the phase shifts involved in \( \Psi^{(m)} \) and \( \varphi^{(m+1)} \). Consequently, the orthogonality phase shift \( \omega(k_0) \) defined in §2 is the sum of all components

\[
\omega(k_0) = \sum_{n=1}^{N} \omega^{(n)}(k_0). \tag{5·19}
\]

Then, (5·6) is a consequence of (5·18) and (5·19).

### §6. Comparison of strong repulsion with strong attraction

In addition to what has been discussed in the later half of §4, now we shall show that the strong coupling correction to the phase shift \( \delta_s(k_0) \) is in general small for a very strong potential, except in the resonance region discussed in (4·20). We can easily show that

\[
\tan \delta_s(k_0) = (\pi/2k_0) \sum_{n=1}^{N} |\sigma_s(k_0)|^2 [\gamma(E)^n(1+\lambda\gamma(E)^n)]^{-1} \times \left\{ 1 + (\pi/2k_0)^2 \sum_{n,r=1}^{N} |\sigma_s(k_0)\cdot\sigma_s(k_0)|^2 [\gamma(E)^n(1+\lambda^{-1}\gamma(E)^n)]^{-1} \right\}^{-1}, \tag{6·1}
\]

where \( \sigma_s(k_0) \) and \( \gamma(E)^n \) have been defined in (4·10)–(4·11). Equation
(6·1) follows readily from (5·5), on using also (1·5)-(1·7) and (4·14); the straightforward calculation based on (4·12)-(4·13) is more tedious but leads to the same result, as it should. If none of the eigenvalues $\eta(E)_m$ is smaller in magnitude than a fixed number $\varepsilon$

$$|\eta(E)_m| \geq \varepsilon, \text{ for all } m,$$  \hspace{1cm} (6·2)\\
and if the magnitude of the potential strength is so large that we can set

$$|\lambda| \gg \varepsilon^{-1},$$  \hspace{1cm} (6·3)\\
the right-hand side of (6·1) can be expanded into inverse powers of $\lambda$; under the same condition we can set a bound to the right-hand side of (6·1) so that, if the strong coupling correction $\delta_s(k_0)$ is continued smoothly as a function of energy all the way under the assumption (6·2), we can set

$$|\delta_s(k_0) - \delta_s(0)| \ll \frac{1}{2} \pi;$$  \hspace{1cm} (6·4)\\
because (6·1) remains bounded everywhere, while (6·2) holds at zero energy as discussed right after (4·15). The detail of the behavior of $\eta(E)_m$ as a function of energy $E$ depends on the particular form of the form factors $u_m(k)$. However, we never encounter a degeneracy among the $\eta(E)_m$ in such examples as a square well or an exponential potential. In view of this, we shall assume, now, that only one of the eigenvalues, say $\eta(E)_1$, is very small, while (6·2) still holds for the rest of eigenvalues:

$$|\eta(E)_1| \ll \varepsilon,$$
$$|\eta(E)_m| \geq \varepsilon, \text{ } m = 2, 3, \ldots, N.$$  \hspace{1cm} (6·5)\\
In particular, when $\eta(E)_1$ vanishes we have

$$\tan \delta_s(k_0) = (2k_0) [\lambda \pi |\sigma_s(k_0)|^2]^{-1},$$  \hspace{1cm} (6·6)\\
where the righthand side is in general (unless $\sigma_s(k_0)$ vanishes) still of the order of $\lambda^{-1}$. This argument makes it plausible that (6·4) is always true when $|\lambda|$ is sufficiently large so that $\varepsilon$ is small except at high energies where all eigenvalues violate (6·2). The high energy behavior of the strong coupling phase shift has been discussed in §4, and it has been established that $\delta_s(k_0)$ is small and hence (6·4) is valid, if we exclude the resonances (4·20) for strong repulsion.

Hereafter we shall differentiate between the attraction and the repulsion. The phase shifts for a strong attraction [repulsion] will be designated by the superscript $(a)$ [$\text{(r)}$]. Since the tangent of the strong coupling phase shift for attraction remains small at all energies, (6·4) is valid, or we may
Infinitely Strong Potential of Finite Rank

\[ |\delta_s^\infty(k_0) - \delta_s^\infty(0)| < \frac{1}{2}\pi. \]  \hfill (6.7)

As for the high energy limit, we may assume
\[ \delta_s^\infty(0) - \delta_s^\infty(\infty) = 0 \]  \hfill (6.8)

in analogy with Levinson's theorem for a weak potential. On the other hand, for a strong repulsion, we may assume
\[ \delta_s^\infty(0) - \delta_s^\infty(\infty) = -N\pi, \]  \hfill (6.9)

because there are \( N \) resonances \((4.20)\) and otherwise the tangent of \( \delta_s^\infty \) is small in general. As for the orthogonality phase shifts, according to \((5.6)\), we have
\[ \omega_s^\infty(0) - \omega_s^\infty(\infty) = N\pi, \]  \hfill (6.10)
\[ \omega_s^\infty(0) - \omega_s^\infty(\infty) = 0. \]  \hfill (6.11)

Therefore, Levinson's theorem for the total phase shift follows from \((6.8)-(6.11)\): it is represented in each case by the equations
\[ \delta_s^\infty(0) - \delta_s^\infty(\infty) = N\pi, \]  \hfill (6.12)
\[ \delta_s^\infty(0) - \delta_s^\infty(\infty) = 0, \]  \hfill (6.13)

which are in conformity with the number of bound states in the respective cases.

In the derivation of the phase shift from its tangent, it suffers indefiniteness by a multiple of \( \pi \). However, we can use the Born series for the phase shift within radius of convergence to remove this kind of ambiguity; outside the radius of convergence we can use its analytic continuation; this is the so-called absolute definition of the phase shift.\(^{11}\)

When we investigate the extreme high energy region defined by \((4.21)\), we shall find from \((1.5)-(1.7)\) and \((4.17)\) that the first Born approximation for the total phase shift is valid
\[ \delta(k_0) \approx (\pi\lambda/2k_0) \sum_{m=1}^{N} |u_m(k_0)|^2. \]  \hfill (6.14)

In view of \((2.6), (4.17)\) and \((4.21)\), we can easily verify that the Born series quickly converges and it gives vanishing high energy limits for the total phase shift
\[ \delta_s^\infty(\infty) = 0, \]  \hfill (6.15)
\[ \delta_s^\infty(\infty) = 0. \]  \hfill (6.16)
The absolute definition of the strong coupling correction \( \delta_s \) is available at low energies when the potential strength \( \lambda \) is very large. The convergence of the inverse power series in \( \lambda \), shown by using (4.15)–(4.16), implies the convergence of the power series in the matrix elements of the “interaction” \( \langle k|H'|m \rangle \); after some calculations we can actually verify the convergence of the Born series in \( \langle k|H'|m \rangle \). Since the form factors vanish at zero energy in such a way as

\[
\lim_{k \to 0} [k^{-1}u_{\omega}(k)u_{\omega}(k)^*] = 0, \tag{6.17}
\]

it follows from (4.16) that the absolute definition of the strong coupling phase shift vanishes

\[
\delta_s^{(\omega)}(0) = 0, \tag{6.18}
\]
\[
\delta_s^{(a)}(0) = 0. \tag{6.19}
\]

It follows from (6.8) and (6.9) that

\[
\delta_s^{(\omega)}(\infty) = 0, \tag{6.20}
\]
\[
\delta_s^{(a)}(\infty) = N\pi. \tag{6.21}
\]

In order to satisfy (6.15) and (6.16), we have to set

\[
\omega_s^{(\omega)}(\infty) = 0, \tag{6.22}
\]
\[
\omega_s^{(a)}(\infty) = -N\pi; \tag{6.23}
\]

hence, according to (6.10)–(6.11), we set

\[
\omega_s^{(\omega)}(0) = N\pi, \tag{6.24}
\]
\[
\omega_s^{(a)}(0) = 0. \tag{6.25}
\]

Therefore, we must assign different values to the orthogonality phase shift, depending on the sign of the potential, in order to get accordance with the absolute definition. In fact, they must differ from each other by \( N\pi \)

\[
\omega_s^{(\omega)}(k_0) - \omega_s^{(a)}(k_0) = N\pi \tag{6.26}
\]

and thus satisfy the same equation (2.15) for their tangent. If we take the limit as \( \lambda \) tends to infinity for a fixed energy, the high energy resonances (4.20) which characteristically differentiate the strong repulsion from strong attraction will be shifted to an infinitely highr energy, and their effects disappear from the scattering at finite energies. The phase shift at infinity of potential strength is accounted for by the orthogonality phase shift alone.
The difference in terms of the absolute definition (6·26) has no experimentally observable effect on the scattering length, for instance. Thus, as far as the results of the scattering experiments are concerned, strong attraction and strong repulsion appear very close to each other at low energies.

§7. Conclusions

We have analyzed the neighborhood of infinity of potential strength for a potential of finite rank, \( N \). Obviously there are \( N \) bound states when the potential is strongly attractive, while there is none when it is strongly repulsive, which indicates that the two cases are of completely different physical nature. This difference is represented by the different values of the absolutely defined orthogonality phase shift, (6·26). However, since they differ exactly by \( N \pi \) everywhere, there is no observable effect to discriminate the two. Thus the results of scattering experiments are superficially similar in the neighborhood of infinity, without regard to the sign of potential; mathematically, this can be well formulated by the orthogonality constraints which define the \( K \) matrix uniquely at infinity. The physical origin of the constraints is different between the two cases: for attraction, there are \( N \) deep lying bound states whose wave functions are almost identical with form factors with the same subscript, and in addition the wave function of a scattering state must be orthogonal to all the bound states; on the other hand, for repulsion, a strong repulsion prohibits a wave function (when transformed into configuration space) from having a substantial amount of amplitude inside anyone of the form factors, which will be formulated by the constraints. Thus on excluding the high energy resonances which differentiate the strong repulsion from the strong attraction, their results appear to be quite similar to each other.

References

2) F. Coester, Phys. Rev. 133 (1964), B1516.
5) S. Tani, to be published; cf. also, K. W. McVoy, L. Heller, and M. Bolsterli, to be published.
8) The same kind of separation into a discrete state and continuum was actually used by Professor Tomonaga in his formulation of the intermediate coupling theory of meson scattering; Z. Maki, M. Sato and S. Tomonaga, Prog. Theor. Phys. 9 (1953), 607.
9) This formula appears to be the same as the Hilbert transform; the deviation from the
Hilbert transform vanishes at an infinite distance; it has been used first by W. Kohn, Phys. Rev. 84 (1951), 495.


11) S. Rosendorff and S. Tani, Phys. Rev. 131 (1963), 396; further references are quoted there.