THE EDDINGTON FACTOR FOR RADIATIVE TRANSFER IN SPHERICAL GEOMETRY

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SUMMARY

The present paper shows that, contrary to the statement of Hummer & Rybicki, the half-range method (Wilson & Sen) can handle the outward peaking effect of the radiation field in extensive spherical media. The source function $J(r)$ and the Eddington factor $f(r) = [K(r)/J(r)]$ have been calculated by the half-range method with $k(r) = r^{-2}$ for $0 < r \leq R$, where $R = 3$ and $R = 10$. The values for $R = 10$ are compared with those of Hummer & Rybicki and are found to be in fairly good agreement.

1. INTRODUCTION

Eddington (1926) suggested a method for solving transfer problems in a plane-parallel medium using an iterative scheme on $f(r)$, the ratio of the second to the zero-order moment of the specific intensity. This treatment has been repeated by Woolley & Stibbs (1953) and referred to by Böhm-Vitense (1963), Feautrier (1964) and Auer & Mihalas (1970).

In a recent paper, Hummer & Rybicki (1971) have developed a numerical method for the solution of the radiative transfer problem in a spherically symmetric medium. This scheme is based on iterating on an assigned space dependant Eddington factor $f(r)$ which was presupposed to take care of the 'peaking effect' suggested by Chapman (1966). The essential feature of this method consists in assuming a form for $f(r)$ which increases monotonically from $\frac{1}{3}$ to 1, and developing an iterative scheme for calculating successive values for the source function $J(r)$ and the Eddington factor $f(r)$. In doing this, they sacrificed the flux equation (which can be solved exactly) in favour of the assumed nature of $f(r)$ based on Chapman's observations. Incidentally Chapman also avoids strict adherence to the exact flux integral.

In their paper, Hummer & Rybicki seem to suggest that the half-range method (Wilson & Sen 1965) is of 'doubtful' accuracy in not meeting Chapman's suggestion. In the present paper, it is demonstrated that the half-range method does not suffer from the limitation imputed by Hummer & Rybicki. In fact $f(r)$ calculated from the moment equation, with $J(r)$ obtained from the half-range method in the second approximation, shows all the essential features suggested by the peaking effect. It is also found that the values of $J(r)$ obtained by the half-range method in this approximation compares favourably with those of Hummer & Rybicki. This agreement in the values of $J(r)$ is not surprising as the half-range method is geared to accommodate the exact boundary conditions, whereas the customary spherical harmonic method or the method of gaussian quadrature is not. Furthermore, in the half-range method the flux integral is exactly satisfied.

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2. BASIC EQUATIONS

The transfer equation for conservative scattering in a grey spherical medium may be taken as

$$\mu \frac{\partial}{\partial r} I(r, \mu) + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu) + k(r) I(r, \mu) = \frac{1}{2} k(r) \int_{-1}^{+1} I(r, \mu) d\mu \quad (2.1)$$

where $r$ is the distance measured from the centre of the spherical medium and $\mu$ is the cosine of the angle measured from the positive direction of the radius vector. $I(r, \mu)$ is the specific intensity and $k(r)$ is the volume coefficient of attenuation. The equation (2.1) is to be solved subject to the boundary conditions

$$I(R, \mu) = 0, \quad -1 \leq \mu \leq 0, \quad (2.2a)$$

where $R$ is the radius of the medium, and the convergence of intensity as

$$r \to 0. \quad (2.2b)$$

Defining the zero, first and second moments of intensity respectively as

$$J = \frac{1}{2} \int_{-1}^{+1} I d\mu; \quad H = \frac{1}{2} \int_{-1}^{+1} I \mu d\mu; \quad K = \frac{1}{2} \int_{-1}^{+1} I \mu^2 d\mu;$$

we obtain from (2.1)

$$\frac{dH}{dr} + \frac{2H}{r} = 0, \quad (2.3)$$

and

$$\frac{dK}{dr} - \frac{1}{r} \left[ J - 3K \right] = -k(r)H. \quad (2.4)$$

Introducing the Eddington factor

$$f(r) = [K/J],$$

equation (2.4) becomes

$$\frac{d}{dr} (fJ) - \frac{J}{r} (1 - 3f) = -k(r)H. \quad (2.5)$$

Also from (2.3)

$$H = H_0 r^{-2}. \quad (2.6)$$

Combining (2.5), (2.6) with $k(r) = r^{-2}$, we obtain

$$\frac{d}{dr} (fJ) - \frac{J}{r} (1 - 3f) = -\frac{H_0}{r^4}. \quad (2.7)$$

The value of $J(r)$ is evaluated by the half-range method from the equation

$$J(r) = \frac{1}{2} \left[ \int_{0}^{r} I_+(r, \mu) d\mu + \int_{-1}^{0} I_-(r, \mu) d\mu \right],$$

where

$$I_+(r, \mu) = A(r) + \sum_{l=0}^{r} (2l + 1) I_l^+(r) \mu P_l(2\mu - 1)$$
and

\[ I_\lambda (r, \mu) = A(r) + \sum_{l=0}^{1} (2l+1) I_l^{-}(r) \mu P_l(2\mu+1) \]

with

\[ A(r) = \alpha + \frac{\beta}{r^3} + \frac{\gamma}{r^2} + \frac{\delta}{r} + e^{-\kappa/r} \left[ \frac{A}{r^3} + \frac{B}{r^2} + \frac{C}{r} + D \right]. \]

The value of the constants \( \kappa, \alpha, \beta, \gamma, \delta, A, B, C, D \) are calculated from (2.1) and the boundary conditions (2.2) which reduce to

\[ A(R) = 0 \]

\[ I_0^{-}(R) = 0 \]

\[ I_1^{-}(R) = 0. \]  \hspace{1cm} (2.8)

The numerical values are calculated as in Wilson & Sen (1965, p. 351). For \( R = 3 \) and \( R = 10 \), the values of \( J(r) \) become respectively

\[ J(r) = 4H_0 \left[ -2.2850884 + \frac{1}{r} (1.2613636) + \frac{1}{r^3} (0.2500000) \right. \]

\[ + e^{-\kappa/r} \left( 2.2418276 + \frac{1}{r} (3.0132227) + \frac{1}{r^2} (1.7713424) \right. \]

\[ + \frac{1}{r^3} (0.5518122) + \frac{1}{r^4} (0.0556705) \left] \right. \]  \hspace{1cm} (2.9)

and

\[ J(r) = 4H_0 \left[ -2.0490197 + \frac{1}{r} (1.2613636) + \frac{1}{r^3} (0.2500000) \right. \]

\[ + e^{-\kappa/r} \left( 2.0480095 + \frac{1}{r} (2.4908316) + \frac{1}{r^2} (1.3186856) \right. \]

\[ + \frac{1}{r^3} (0.3790084) + \frac{1}{r^4} (0.0368658) \left] \right. \]  \hspace{1cm} (2.10)

where

\[ \kappa = 1.8257419 \]  \hspace{1cm} (2.11)

Equations (2.9) and (2.10) have been obtained by evaluating the constants correct up to seven decimal places.

3. Computation of \( f(r) \)

The Eddington factor \( f(r) \) is obtained by numerical integration of equation (2.7) using the \( J(r) \)'s from equations (2.9) and (2.10) with the initial condition

\[ f = \frac{1}{3} \text{ at } r = 0.01. \]  \hspace{1cm} (3.1)

The calculations were performed on the IBM 1130 Computer using Euler's method with an increment \( \Delta r = 0.0005 \), and this ensures an accuracy of \( f(r) \) within a half per cent.
Table I shows the values of $J(r)$ and $f(r)$ for $R = 3$. In Table II the values of $J(r)$ and $f(r)$ are tabulated for $R = 10$ and compared with those of Hummer and Rybicki (1971, p. 11).

Table II

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<tr>
<th>$r$</th>
<th>$J(r)$</th>
<th>$f(r)$</th>
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<tbody>
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<td>0.01</td>
<td>$9.291 \times 10^5$</td>
<td>0.333</td>
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<tr>
<td>0.05</td>
<td>$8.092 \times 10^3$</td>
<td>0.336</td>
</tr>
<tr>
<td>0.10</td>
<td>$1.041 \times 10^3$</td>
<td>0.342</td>
</tr>
<tr>
<td>0.40</td>
<td>$2.042 \times 10^1$</td>
<td>0.426</td>
</tr>
<tr>
<td>0.71</td>
<td>$4.355$</td>
<td>0.512</td>
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<td>1.01</td>
<td>$1.780$</td>
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<tr>
<td>1.51</td>
<td>$6.819 \times 10^{-1}$</td>
<td>0.626</td>
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<td>0.685</td>
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<tr>
<td>2.51</td>
<td>$1.995 \times 10^{-1}$</td>
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<td>2.96</td>
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<table>
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<tr>
<th>$r$</th>
<th>$J(r)$</th>
<th>$f(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
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<td>0.333</td>
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<td>0.02</td>
<td>$1.252 \times 10^5$</td>
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<td>0.03</td>
<td>$3.720 \times 10^4$</td>
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<td>10.00</td>
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4. CONCLUSION

From Table II we note that the values of $J(r)$ obtained by the half-range method in the 2nd approximation are comparable with those obtained by Hummer & Rybicki. Also the values of $f(r)$ are in fairly good agreement with those obtained by Hummer & Rybicki. As this function $f(r)$ is supposed to reflect the 'peaking effect' of the intensity, it is reasonable to believe that the critical feature of the 'peaking effect' is not suppressed by the half-range method in any way.
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If the internal field calculation through $J(r)$ is the aim, it is clear that the half-range method in as low an approximation as the second gives a fairly reliable value for practically all ranges of $r$ without any reference to the nature of $f(r)$. Moreover, the exact flux equation need not be sacrificed in this method.

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**REFERENCES**


