Collisions and Gravitational Interactions between Particles in Planetary Rings

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Particles in planetary rings orbit the central planet, and undergo collisions and gravitational interactions with other particles. As a result, their orbits become inclined and non-circular. Also, ring particles likely have rough surfaces, and an oblique impact between them leads to rotation. In the case of dilute rings where collision frequency is sufficiently smaller than the orbital frequency, particles’ orbits evolve through successive two-body collisions and/or gravitational encounters, and the evolution can be described by the formulation based on the three-body problem. We describe basic equations for such cases, and derive evolution equations for particle velocity dispersion and spin rates. We also discuss effects of rings’ self-gravity, which become dominant in dense rings. Collisions and gravitational interactions between particles result in angular momentum transfer in planetary rings. We discuss ring viscosity, which determines the rate of angular momentum transfer in rings. Viscosity in dilute rings can be expressed by particles’ velocity dispersion and is proportional to the ring surface density for a given particle size, while the dependence of the viscosity on ring surface density is stronger in dense self-gravitating rings, where angular momentum is transferred by interactions between gravitational wakes.

§1. Introduction and summary

Dynamics of systems of interacting particles in shear flow is an important subject in granular physics and in other various fields, including astrophysics. In many astrophysical particle disks where shear flow is induced by differential rotation, the effect of gravitational interactions between constituent particles is dominant over that of physical collisions; for example, mutual gravity governs the dynamical evolution in disks of stars or planetesimal disks in the early stage of planetary formation. On the other hand, inelastic collisions between particles play an important role in planetary rings, which is located close to the central planet and largely influenced by the planet’s tidal effects.

When mutual gravity between particles is neglected, equilibrium dynamical properties of ring particles are determined by balance between energy gain from the mean shear flow and energy dissipation due to inelastic collisions (e.g., Ref. 1; see reviews by Refs. 2,3)). In the case of Saturn’s rings, for example, mutual gravity between particles is also important, especially in the outer region, where tidal effects are reduced. Furthermore, in dense rings such as Saturn’s A and B rings, collective effects govern the dynamical evolution and result in the formation of the so-called gravitational wakes (Refs. 4–6); Fig. 4).

In dilute systems consisting of particles on Keplerian orbits about a central massive body, their dynamical evolution can be described by the formalism based on the three-body problem (i.e., two interacting particles and the central body).7,8) In the present paper, summarizing results of previous works, we describe basic equations...
for such cases (§2), and show the derivation of evolution equations for particle velocity dispersion (§3) and spin rates (§4). The derived evolution equations can be applied to systems of particles with an arbitrary size distribution. We show that the evolutions calculated by solving these equations agree with results of N-body simulation for dilute systems. We briefly comment on implications of these dynamical studies for observed features of planetary rings. We also discuss the importance of the disk self-gravity in dense rings, which cannot be described by the above three-body formulation. We show that ring viscosity is significantly enhanced in dense self-gravitating rings, where angular momentum is transferred by interactions between gravitational wakes (§5).

§2. Description of particle motion

We consider motion of mutually interacting particles (with masses \( m_i \) and \( m_j \)) orbiting a central massive body (with mass \( M_c \)). We use a rotating Cartesian coordinate system with origin that moves on a circular orbit with semimajor axis \( a_0 \) at the Keplerian angular velocity \( \Omega = \sqrt{GM_c/a_0^3} \), the \( x \) axis pointing radially outward, the \( y \) axis pointing in the direction of the orbital motion, and the \( z \) axis normal to the equatorial plane. The masses of particles are assumed to be much smaller than \( M_c \), and the orbital eccentricity \( e_j \) and inclination \( i_j \) of particle \( j \) are assumed to be much smaller than unity. In this case, equations for the motion of particle \( j \) moving under the gravitational influence of the central body and particle \( i \) in the above rotating coordinate system can be linearized and are given as\(^{7,8} \)

\[
\begin{align*}
\ddot{x}_j &= 2\Omega \dot{y}_j + 3\Omega^2 x_j - \frac{Gm_i}{r^3}(x_j - x_i), \\
\ddot{y}_j &= -2\Omega \dot{x}_j - \frac{Gm_i}{r^3}(y_j - y_i), \\
\ddot{z}_j &= -\Omega^2 z_j - \frac{Gm_i}{r^3}(z_j - z_i),
\end{align*}
\]  

where \( r = [(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2]^{1/2} \). On the right-hand side, the \( 2\Omega \dot{y}_j \) and \( -2\Omega \dot{x}_j \) terms represent the Coriolis force, the \( 3\Omega^2 x_j \) and \( -\Omega^2 z_j \) terms represent the tidal force (i.e., the sum of the gravitational force of the central body and the centrifugal force), and the last terms represent the mutual gravity between particles \( i \) and \( j \). Equation (2.1) is called Hill’s equation. When the mutual gravitational terms can be neglected, the solutions for Eq. (2.1) describe the Kepler motion of particle \( j \) as

\[
\begin{align*}
x_j &= b_j - e_j a_0 \cos(\Omega(t - \tau_j)), \\
y_j &= \lambda_j - \frac{3}{2}b_j \Omega t + 2e_j a_0 \sin(\Omega(t - \tau_j)), \\
z_j &= i_j a_0 \sin(\Omega(t - \omega_j)),
\end{align*}
\]  

\[
\begin{align*}
\dot{x}_j &= e_j a_0 \Omega \sin(\Omega(t - \tau_j)), \\
\dot{y}_j &= -\frac{3}{2} b_j \Omega + 2 e_j a_0 \Omega \cos(\Omega(t - \tau_j)), \\
\dot{z}_j &= i_j a_0 \Omega \cos(\Omega(t - \omega_j)),
\end{align*}
\]

where \( b_j \equiv a_j - a_0 \) with \( a_j \) being the semimajor axis of particle \( j \), \( \tau_j \) and \( \omega_j \) determine orbital phase angles, and \( \lambda_j \) is the azimuthal position of the guiding center at \( t = 0 \). Equations (2.2) and (2.3) show that the particle’s Kepler motion is described by the epicyclic motion superimposed on the guiding center motion on a circular orbit. The orbital elements \( e_j, b_j \) etc. are constant when mutual gravity can be neglected. Particles’ motion can be described in the form of Eqs. (2.2) and (2.3) also when mutual gravity cannot be neglected; in this case, the orbital elements vary with time.

Under the approximations described above, the equations of motion can be separated into the center of mass motion and the relative motion. Here, we define the relative position \( \mathbf{x} = (x, y, z) \) and the position of the center of mass \( \mathbf{X} = (X, Y, Z) \) for particles 1 and 2 as

\[
\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1, \quad \mathbf{X} = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2}.
\]

In this case, using Eq. (2.1), the equations for the center of mass motion can be written as

\[
\begin{align*}
\ddot{X} &= 2\Omega \dot{Y} + 3\Omega^2 X, \\
\ddot{Y} &= -2\Omega \dot{X}, \\
\ddot{Z} &= -\Omega^2 Z,
\end{align*}
\]

which is similar to Eq. (2.1) but without the terms for mutual gravitational interaction. This shows that the center of mass motion is unaffected by the mutual gravitational interaction and follows a Keplerian orbit with constant orbital elements. On the other hand, equations for the relative motion can be described as

\[
\begin{align*}
\ddot{x} &= 2\Omega \dot{y} + 3\Omega^2 x - \frac{G(m_1 + m_2)}{r^3} x, \\
\ddot{y} &= -2\Omega \dot{x} - \frac{G(m_1 + m_2)}{r^3} y, \\
\ddot{z} &= -\Omega^2 z - \frac{G(m_1 + m_2)}{r^3} z.
\end{align*}
\]

The above equations are similar to Eq. (2.1) and, again, the right-hand side represents the Coriolis force, the tidal force, and the mutual gravitational force. When the mutual gravity can be neglected, the solutions for Eq. (2.6) can be written as

\[
\begin{align*}
x &= b - e a_0 \cos(\Omega(t - \tau)), \\
y &= \lambda - \frac{3}{2} b \Omega t + 2 e a_0 \sin(\Omega(t - \tau)), \\
z &= i a_0 \sin(\Omega(t - \omega)).
\end{align*}
\]
Fig. 1. Left: Contours of the potential $U$ (nondimensional form of $U_J$) for the $z = 0$ plane (see Eq. (2.17)). The $U = 0$ surface, which defines the Hill sphere, is shown by the thick line. Contour lines inside the Hill sphere are not shown. Right: Potential $U$ as a function of $\tilde{x} = x/R_H$ for $y = z = 0$.

\[
\begin{align*}
\dot{x} &= ea_0 \Omega \sin(\Omega(t - \tau)), \\
\dot{y} &= -\frac{3}{2}b \Omega + 2ea_0 \Omega \cos(\Omega(t - \tau)), \\
\dot{z} &= ia_0 \Omega \cos(\Omega(t - \omega)),
\end{align*}
\]

which are similar to Eqs. (2.2) and (2.3). In the above, $b = a_2 - a_1$, $e$, $i$, $\tau$, $\omega$, and $\lambda$ are the orbital elements for the relative motion.

It can be easily shown that Eq. (2.6) hold an energy integral given by

\[
E_J = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U_J(x, y, z),
\]

with the potential $U_J$ given as

\[
U_J(x, y, z) = -\frac{3}{2} \Omega^2 x^2 + \frac{1}{2} \Omega^2 z^2 - \frac{G(m_1 + m_2)}{r} + \frac{9}{2} R_H^2 \Omega^2.
\]

The first two terms on the right-hand side of Eq. (2.10) represents the tidal potential, the third term is the mutual gravitational potential, and the last term has been added so that $U_J$ vanishes at the Lagrangian points $(x, y, z) = (\pm R_H, 0, 0)$ (Fig. 1). Here, $R_H$ is called the Hill radius, and is defined as

\[
R_H = ha_0
\]

with

\[
h = \left(\frac{m_1 + m_2}{3M_c}\right)^{1/3}.
\]

The Hill radius represents the size of the region where the mutual gravity becomes dominant over the gravity of the central body.
If we scale the distance and time by $R_H$ and $\Omega^{-1}$, respectively, Eq. (2.6) for the relative motion can be written in a nondimensional form as:

\[
\begin{aligned}
\ddot{x} &= 2\dot{y} + 3\ddot{x} - \frac{3\dddot{x}}{r^3}, \\
\ddot{y} &= -2\dddot{x} - \frac{3\dddot{y}}{r^3}, \\
\ddot{z} &= -\dddot{z} - \frac{3\dddot{z}}{r^3},
\end{aligned}
\] (2.13)

where tildes denote scaled quantities, such as $\tilde{x} \equiv x/R_H$. The unperturbed solutions to Eq. (2.13) neglecting mutual gravity can be given in a form similar to Eqs. (2.7) and (2.8) as:

\[
\begin{aligned}
\dot{x} &= \tilde{b} - \tilde{e} \cos(\tilde{t} - \tau), \\
\dot{y} &= \tilde{\lambda} - \frac{3}{2} \tilde{b} \tilde{t} + 2\tilde{e} \sin(\tilde{t} - \tau), \\
\dot{z} &= \tilde{i} \sin(\tilde{t} - \omega),
\end{aligned}
\] (2.14)

\[
\begin{aligned}
\ddot{x} &= \tilde{e} \sin(\tilde{t} - \tau), \\
\ddot{y} &= -\frac{3}{2} \tilde{b} + 2\tilde{e} \cos(\tilde{t} - \tau), \\
\ddot{z} &= \tilde{i} \cos(\tilde{t} - \omega).
\end{aligned}
\] (2.15)

In the above, $\tilde{e}, \tilde{b},$ etc. are scaled orbital elements for the relative motion, which are defined as $\tilde{e} = e/h, \tilde{b} = b/R_H,$ etc., and $\tilde{t} = t/\Omega^{-1}$. In this case, nondimensional expressions for Eqs. (2.9) and (2.10) are given as:

\[
E = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(\tilde{x}, \tilde{y}, \tilde{z})
\] (2.16)

with

\[
U(\tilde{x}, \tilde{y}, \tilde{z}) = -\frac{3}{2} \tilde{x}^2 + \frac{1}{2} \tilde{z}^2 - \frac{3}{r} + \frac{9}{2}.
\] (2.17)

The plots of $U$ are given in Fig. 1. The $U = 0$ surface defines the so-called Hill sphere, which is actually lemon-shaped. Since $U < 0$ inside the Hill sphere, only particles with positive $E$ can enter the sphere, and they cannot escape out of the sphere if their energy is reduced to negative values by some dissipative processes such as inelastic collision, or gas drag if the particles are embedded in the nebular gas, as in the case of planetesimals in the early stage of planet formation.\(^{11}\)

When both collisions and gravitational encounters between particles are important, as in the case of planetary rings, the ratio of the Hill radius to the sum of the physical radii of colliding particles is an important parameter describing the relative importance of gravitational encounters and collisions. The ratio $r_h$ can be written as:

\[
r_h \equiv \frac{R_H}{(R_1 + R_2)} = \left(\frac{4\pi}{9}\right)^{1/3} M_p^{1/3} a_0 M_c^{-1/3} \left(1 + \delta\right)^{1/3} \left(1 + \delta^1/3\right)^{1/3},
\] (2.18)

\(^{11}\) We omit tildes for $\tau$ and $\omega$ for simplicity.
Fig. 2. Examples of orbits in the case of $r_p = 10^{-3}$ (left panel) and 1 (right panel). In both cases, Eq. (2.13) for the relative motion was numerically solved for initially circular, coplanar orbits ($\tilde{e} = \tilde{\iota} = 0$). The dashed line represents the Hill sphere. In the case of the left panel, no direct collisions are detected, and the orbital changes are caused by the mutual gravitational interaction alone. The solid circle in the right panel shows the physical size of the colliding particles. In the case shown here, four orbits lead to direct collision, and the orbital changes are calculated assuming that particles are perfectly elastic, smooth spheres.

where $R_j$ ($j = 1, 2$) is the radius of particle $j$, $\rho_p$ is the internal density of particles, and $\delta = m_2/m_1$ is the mass ratio of the colliding particles. The inverse of $r_h$, i.e.,

$$r_p = (R_1 + R_2)/R_H,$$

is also often used in the literature.\textsuperscript{5,10,11,13} In the case of encounters between planetesimals orbiting the Sun at $1 \sim 5$AU, $r_h \simeq 200 - 10^3$ ($r_p \simeq 1 \times 10^{-3} - 5 \times 10^{-3}$), while $r_h$ (and $r_p$) is on the order of unity in the case of planetary rings.\textsuperscript{11} It should be noted that the masses of particles or the distance from the central body do not appear explicitly in the nondimensional Eq. (2.13), but the orbital behavior depends on these quantities through $r_h$ or $r_p$, when direct collisions are taken into account. Therefore, results of orbital integration obtained by solving Eq. (2.13) can be applied to various combinations of masses and $a_0$, as long as the dependence of the results on $r_h$ or $r_p$ is taken into account appropriately. Figure 2 shows examples of orbits for $r_p = 10^{-3}$ ($r_h = 10^3$) and 1 ($r_h = 1$).

When direct collision is taken into account in solving the equation of motion for the relative motion of particles, the velocity change due to collision can be calculated, for example, based on the hard-sphere model, as follows. Suppose that two particles collide with each other with relative velocity of their centers in the rotating coordinate system $\mathbf{v}$, and that their relative position at impact is given by $\mathbf{r}$. We express the normal and the tangential component of $\mathbf{v}$ to the tangent plane as $\mathbf{v}_n$ and $\mathbf{v}_t$, respectively. In this case, the normal and the tangential components of the relative velocity of the two contacting points (i.e., the relative velocity of the surfaces of the two particles at the point of contact) at the time of impact can be written as\textsuperscript{14–17}

$$\mathbf{v}_n = \mathbf{v} - (\mathbf{v} \cdot \lambda)\lambda,$$
\[ u_t = v_t + \Omega \times r + \lambda \times (R_1 \omega_1 + R_2 \omega_2), \]
\[ = v_t + \lambda \times (R_1 \omega_{R1} + R_2 \omega_{R2}), \]  
(2.20)

with

\[ \omega_{Rj} \equiv \omega_j - \Omega, \]  
(2.21)

where \( \lambda \equiv r/|r| \) is the unit vector pointing from the center of particle 1 to that of particle 2; \( \Omega \equiv (0, 0, \Omega) \) is the Keplerian angular velocity vector of the particles; \( \omega_j \) is the spin rate vector of particle \( j \), which needs to be taken into account when particle surface friction is included; \( \omega_{Rj} \) is the spin rate vector measured in the rotating coordinate system; and we used the relation \( r = (R_1 + R_2) \lambda \) at impact. In the expression for \( u_t \) in Eq. (2.20), \( v_t \) is the relative tangential velocity between the centers of the colliding particles, as mentioned above, and the rest represents the relative tangential velocity due to spins of the particles (expressed in terms of spin rates measured on the inertial frame in the first line, and in terms of those measured on the rotating coordinate system in the second line). We introduce the normal and the tangential restitution coefficients, \( \varepsilon_n \) and \( \varepsilon_t \), where \( 0 \leq \varepsilon_n \leq 1 \) and \( -1 \leq \varepsilon_t \leq 1 \) (perfectly smooth spheres have \( \varepsilon_t = 1 \)). In this case, the normal and the tangential components of the relative velocity of the two contacting points after impact are given as

\[ u'_n = v'_n = -\varepsilon_n v_n, \]
\[ u'_t = \varepsilon_t v_t. \]  
(2.22)

From Eq. (2.22) and the conservation of linear and angular momenta, the changes of the normal and the tangential components of the relative velocity of the centers of the two particles are given as

\[ \Delta v_n = -(1 + \varepsilon_n) v_n, \]
\[ \Delta v_t = -\frac{K}{K + 1} \left( 1 - \varepsilon_t \right) \left[ v_t + \lambda \times (R_p \omega_R) \right], \]  
(2.23)

where \( R_p \equiv R_1 + R_2 \) and

\[ \omega_R \equiv (R_1 \omega_{R1} + R_2 \omega_{R2})/(R_1 + R_2). \]  
(2.24)

In the above, \( K \) is the coefficient of the moment of inertia, e.g., \( I_1 = K m_1 R_1^2 \), and \( K = 2/5 \) for a homogeneous sphere.

§3. Velocity dispersion

Velocity dispersion of ring particles is an important quantity related to rings’ dynamical evolution as well as their structures.\(^2\),\(^3\) For example, the thickness of a ring is determined by the particles’ vertical velocity dispersion, and the rate of angular momentum transfer in dilute rings is related to particle velocity dispersion (§5). In the case of dilute particle disks, the velocity dispersion evolves through successive binary interactions (collisions or gravitational encounters), and the evolution can
be described by the formulation based on the three-body problem discussed in §2. Such an approximation is valid for planetesimal disks in planetary formation\(^9,18\) or rings with low optical depth.\(^{10}\) Here, we will mostly focus on the evolution of velocity dispersion in such dilute rings, and briefly discuss the case of dense rings where collective effects are important.

Here, we introduce eccentricity and inclination vectors defined by\(^7–10\)

\[
e_j \equiv (e_j \cos(\Omega \tau_j), e_j \sin(\Omega \tau_j)),
\]
\[
i_j \equiv (i_j \cos(\Omega \omega_j), i_j \sin(\Omega \omega_j)).
\] (3.1)

From Eqs. (2.2) and (2.3), we find that the position and the velocity vectors of a particle can be expressed in linear combinations of orbital elements if we use the components of \(e_j\) and \(i_j\), such as \(e_j \cos(\Omega \tau_j)\). Therefore, if we further define the eccentricity and the inclination vectors for the relative motion as (see Eqs. (2.7) and (2.8))

\[
e \equiv (e \cos(\Omega \tau), e \sin(\Omega \tau)),
\]
\[
i \equiv (i \cos(\Omega \omega), i \sin(\Omega \omega)).
\] (3.2)

from Eqs. (2.2), (2.3), (2.4), (2.7), and (2.8), we find

\[
e = e_2 - e_1,
\]
\[
i = i_2 - i_1.
\] (3.3)

Thus, \(e_j\) and \(i_j\) can be treated like velocity vectors in the Cartesian coordinate system. In the same manner, the eccentricity and inclination vectors for the center of mass motion can be defined as

\[
E = \frac{m_1 e_1 + m_2 e_2}{m_1 + m_2}, \quad I = \frac{m_1 i_1 + m_2 i_2}{m_1 + m_2}.
\] (3.4)

Since the center of mass motion is unaffected by collision or gravitational interaction between particles,

\[
\Delta E = 0, \quad \Delta I = 0,
\] (3.5)

where \(\Delta\) denotes the changes due to a collision or gravitational encounter.

Now, we consider dilute rings consisting of particles with two size-components \(m_1\) and \(m_2\), and derive an evolution equation for their velocity dispersion.\(^{10}\) From Eqs. (3.3) and (3.4),

\[
e_1 = E - m'_2 e,
\] (3.6)

where

\[
m'_j = \frac{m_j}{m_1 + m_2}.
\] (3.7)

Using Eq. (3.5), we have

\[
\Delta e_1^2 = \Delta(E - m'_2 e)^2
\]
\[
= m'_2^2 \Delta e^2 - 2m'_2 E \cdot \Delta e.
\] (3.8)
In the above, $\Delta e_1^2$ is the change in $e_1^2$ during a single encounter, and is a function of initial orbital elements of the interacting two particles. The number of particles 2 approaching the particle 1 per unit time with $b_2$ corresponding to the range $b \sim b + db$ is given by $N_{s2}(3/2)\Omega|b|db$, where $N_{s2}$ is the surface number density of particle 2, and $(3/2)\Omega|b|$ is the approaching velocity of the guiding center of particle 2 due to the Kepler shear. Integrating over $b$ and averaging over the distribution functions of eccentricities and inclinations of particles 1 and 2, the evolution of $\langle e_1^2 \rangle$ can be written as

$$\frac{d\langle e_1^2 \rangle}{dt} = N_{s2} \int f(e_1, \iota_1) f(e_2, \iota_2) \Delta e_1^2 \frac{3}{2} \Omega |b| db \frac{d\tau_1 d\omega_1}{(2\pi)^2} \frac{d\tau_2 d\omega_2}{(2\pi)^2} \langle e_1^2 \rangle \langle e_2^2 \rangle \langle e_2^2 \rangle,$$  

(3.9)

$N$-body simulation of planetesimal disks that evolve through their mutual gravitational encounters show that planetesimals’ orbital eccentricities and inclinations follow the Rayleigh distribution\(^{19}\)

$$f(e_j, \iota_j)de_j di_j = \frac{4e_j \iota_j}{\langle e_j^2 \rangle \langle \iota_j^2 \rangle} \exp \left( -\frac{e_j^2}{\langle e_j^2 \rangle} - \frac{\iota_j^2}{\langle \iota_j^2 \rangle} \right) de_j di_j,$$  

(3.10)

which is equivalent to the Gaussian distribution of random velocities (i.e., velocity components corresponding to the deviation from the coplanar, circular orbits).\(^{20}\) In the case of dilute rings, it was also shown by $N$-body simulation that the distribution can be approximated by Eq. (3.10).\(^{21}\)

Velocity dispersions of particle $j$ in the radial and vertical directions are related to the mean square eccentricities and inclinations as

$$\sigma_{r,j} = \sqrt{\frac{\langle e_j^2 \rangle}{2}} a_0 \Omega, \quad \sigma_{z,j} = \sqrt{\frac{\langle \iota_j^2 \rangle}{2}} a_0 \Omega.$$  

(3.11)

Substituting Eqs. (3.8) and (3.10) into (3.9) and employing suitable variable transformation, the integration over the orbital elements that are related to the center of mass motion can be carried out to obtain\(^{10}\)

$$\frac{d\langle e_1^2 \rangle}{dt} = N_{s2} \int f(e, \iota) \left[ m_2' \langle e^2 \rangle^2 + m_2 \frac{m_1' (\langle e_2^2 \rangle - \langle e_1^2 \rangle)}{\langle e_1^2 \rangle + \langle e_2^2 \rangle} \langle -2e \Delta e \rangle \right]$$

$$\times \frac{3}{2} |b| \Omega db \frac{d\tau d\omega}{(2\pi)^2} dedi$$

$$\equiv N_{s2} h^4 a_0^2 \Omega m_2' \left( m_2' \langle PVs \rangle + \frac{m_1' (\langle e_2^2 \rangle - \langle e_1^2 \rangle)}{\langle e_1^2 \rangle + \langle e_2^2 \rangle} \langle P_{DF} \rangle \right).$$  

(3.12)

In the above,

$$\Delta e_\parallel = \cos(\Omega \tau) \cdot \Delta(e \cos(\Omega \tau)) + \sin(\Omega \tau) \cdot \Delta(e \sin(\Omega \tau))$$  

(3.13)

is the component of $\Delta e$ that is parallel to $e$, and

$$\langle PVs \rangle = \int f(\tilde{e}, \tilde{\iota}) \Delta \tilde{e}^2 \frac{3}{2} |\tilde{b}| db \frac{d\tau d\omega}{(2\pi)^2} ded\tilde{\iota},$$

$$\langle P_{DF} \rangle = \int f(\tilde{e}, \tilde{\iota}) \left( -2 \tilde{e} \Delta \tilde{e}_\parallel \right) \frac{3}{2} \tilde{b} |\tilde{b}| db \frac{d\tau d\omega}{(2\pi)^2} ded\tilde{\iota},$$  

(3.14)
are the viscous stirring and dynamical friction rates of eccentricities.\(^9\),\(^10\),\(^18\),\(^22\) Further taking account of the interaction with particles of the same size, we obtain\(^10\)

\[
\frac{d\langle e^2 \rangle}{dt} = \sum_{j=1,2} N_{sj} h_{1j}^4 a_0^2 \Omega m'_j \times \left( m'_j \langle P_{VS} \rangle + \frac{m'_j \langle e^2_j \rangle - m'_1 \langle e^2_1 \rangle}{\langle e^2_1 \rangle + \langle e^2_j \rangle} \langle P_{DF} \rangle \right),
\]

(3.15)

where \(m'_j = m_j/(m_1 + m_j)\), \(h_{1j} = [(m_1 + m_j)/(3M_e)]^{1/3}\), and \(\langle P_{VS} \rangle\) and \(\langle P_{DF} \rangle\) are the viscous stirring and dynamical friction rates for interactions between \(m_1\) and \(m_j\). Similarly, we obtain the corresponding equation for orbital inclinations as

\[
\frac{d\langle i^2 \rangle}{dt} = \sum_{j=1,2} N_{sj} h_{1j}^4 a_0^2 \Omega m'_j \times \left( m'_j \langle Q_{VS} \rangle + \frac{m'_j \langle i^2_j \rangle - m'_1 \langle i^2_1 \rangle}{\langle i^2_1 \rangle + \langle i^2_j \rangle} \langle Q_{DF} \rangle \right),
\]

(3.16)

where \(\langle Q_{VS} \rangle\) and \(\langle Q_{DF} \rangle\) are the viscous stirring and dynamical friction rates for inclinations. Note that \(\langle P_{VS} \rangle\) etc. are functions of scaled orbital elements. Therefore, these rates can be obtained by numerically or analytically solving Eq. (2.13) and expressed as a function of \(\langle \tilde{e}^2 \rangle\) and \(\langle \tilde{i}^2 \rangle\).

In a general case where particles have a size distribution, the evolution equations can be written as\(^10\)

\[
\frac{d\langle e^2_{m_1} \rangle}{dt} = a_0^2 \Omega \int n_s(m) m' h_{m_1,m}^4 \times \left( m' \langle P_{VS} \rangle_{m_1,m} + \frac{m' \langle e^2 \rangle_{m_1,m} - m' \langle e^2 \rangle_{m_1,m}}{\langle e^2 \rangle_{m_1,m} + \langle e^2 \rangle_{m_1,m}} \langle P_{DF} \rangle_{m_1,m} \right) dm,
\]

\[
\frac{d\langle i^2_{m_1} \rangle}{dt} = a_0^2 \Omega \int n_s(m) m' h_{m_1,m}^4 \times \left( m' \langle Q_{VS} \rangle_{m_1,m} + \frac{m' \langle i^2 \rangle_{m_1,m} - m' \langle i^2 \rangle_{m_1,m}}{\langle i^2 \rangle_{m_1,m} + \langle i^2 \rangle_{m_1,m}} \langle Q_{DF} \rangle_{m_1,m} \right) dm,
\]

(3.17)

where \(m' = m/(m_1 + m)\), and \(n_s(m) dm\) is the surface number density of particles within the mass range of \(m\) to \(m + dm\).

Using the above stirring and dynamical friction rates obtained by three-body orbital integration and solving the evolution equation, we can calculate the evolution of the velocity dispersion of particles in dilute systems and examine their steady state. Figure 3 shows examples of calculation of the evolution of velocity dispersion for planetesimal disks and dilute planetary rings. In both cases, results are compared with \(N\)-body simulation, and we confirm excellent agreement between the two results. We can see that large particles tend to have small velocity dispersion owing to the effect of dynamical friction. \(N\)-body simulations of planetary rings with particle size distribution also show a similar tendency; large particles are concentrated near
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Fig. 3. Evolution of r.m.s. eccentricities and inclinations for two size-component systems obtained by two different methods. The solid lines represent results of N-body simulation, while the dashed lines were obtained by solving the evolution equation for eccentricities and inclinations (Eqs. (3.15) and (3.16)). (a) Evolution due to gravitational encounters between planetesimals at 1AU from the Sun. 800 smaller planetesimals ($m = 1 \times 10^{24}$ g) and 200 larger planetesimals ($m = 4 \times 10^{24}$ g) are initially distributed with the total surface density of 10 g cm$^{-2}$. L and S stand for the values for the larger and smaller size components, respectively. Redrawn from Ref. 18. (b) Case of dilute planetary rings with parameters corresponding to $a_0 = 1.1 \times 10^5$ km from Saturn. Radii of the large and the small particles are 1 m and 50 cm, respectively, and their internal density, restitution coefficient in the normal direction, and optical depth are 0.9 g cm$^{-3}$, 0.5, and 0.005, respectively. Redrawn from Ref. 21.

the midplane with small vertical velocity dispersions, while small particles have a large vertical scale height.\textsuperscript{23)--25)} Such vertical heterogeneity needs to be taken into account in developing models to explain various observations of Saturn’s rings.

In dilute systems, their equilibrium velocity dispersions can be examined either by direct N-body simulation\textsuperscript{5),21)} or by solving the above evolution equation with the use of the stirring and dynamical friction rates obtained by three-body orbital integration.\textsuperscript{10)} For example, in the case of dilute rings consisting of equal-sized particles, the velocity dispersion is found to be on the order of $R\Omega$ when $r_h \ll 1$ and inelastic collisions dominate particles’ velocity evolution, while it is on the order of particles’ mutual escape velocity $v_{esc} = \sqrt{2Gm/R}$ when $r_h \approx 1$ and gravitational encounters become important. The above method based on the evolution equation can be easily applied to systems consisting of particles with a wide range of sizes.\textsuperscript{18)} On the other hand, in the case of dense self-gravitating rings, collective effects become important, and the velocity evolution deviates from the above results based on the three-body formalism.\textsuperscript{5),21)} In such dense rings, irregular structures called gravitational wakes form due to rings’ self-gravity (Refs. 4)–6); Fig. 4). The velocity dispersion in such dense rings is adjusted so that the Toomre parameter $Q \equiv \sigma_r \Omega/(3.36G\Sigma)$ (\Sigma is the ring’s surface mass density) is about 2 − 3.\textsuperscript{5)} When such irregular structures form, self-gravity also plays an important role in angular momentum transfer within the disk (§5).

When $r_h$ is sufficiently smaller than unity, as in the case shown in Fig. 4, the effect of the finite size of particles prevents gravitational accretion of particles. Therefore,
Fig. 4. Snapshots of particle spatial distribution obtained by local \( N \)-body simulation of self-gravitating planetary rings with periodic boundary conditions. Each particle is a smooth sphere with radius of 1 m, and the restitution coefficient in the normal directions is 0.5. The size of the simulation cell shown here is about 340 m, and the simulations were performed for parameters corresponding to \( a_0 = 1 \times 10^5 \) km from Saturn if particle internal density is 0.9 g cm\(^{-3}\) (i.e. \( r_h = 0.82 \)). The four panels show results for different values of dynamical optical depth \( \tau \equiv N_s \pi R^2 \), where \( N_s \) is the particle surface number density. The value of \( \tau \) for each panel is (a) 0.1, (b) 0.2, (c) 0.5, and (d) 1, respectively. Redrawn from Ref. 31).

Fig. 5. Examples of particle orbits leading to collision with another particle (smooth spheres with \( \varepsilon_n = 0.5 \)), for two different values of \( r_p \); the difference in \( r_p \) corresponds to, for example, changing the distance from the central body for a given internal density of the particles (see Eqs. (2.18) and (2.19)). In the case of \( r_p = 0.75 \), the orbit results in escape after the first collision, while the orbit in the case of \( r_p = 0.6 \) leads to accretion (i.e., \( E \) becomes negative) after the second impact. Redrawn from Ref. 11).
gravitational wakes are not permanent structures, but are forming and dissolving in a timescale of about one orbital period.\(^4\)-\(^6\) However, when \(r_h\) is larger than unity (or equivalently \(r_p < 1\)), gravitational accretion of particles becomes possible, if sufficient energy is dissipated at collision (Fig. 5).\(^4\),\(^5\),\(^11\) Observations of small moons near the outer edge of Saturn’s A ring by the Cassini spacecraft suggest that such gravitational accretion of particles took place in the region.\(^26\)

§4. Particle spins

In the studies of the origin of the rotation of terrestrial planets, mean and mean square angular momenta that a planet acquires by accretion of planetesimals are evaluated both analytically and numerically, assuming that planetesimals are accreted by the planet whenever they collide (see a review by Ref. 27)). These studies show that the systematic component of rotation that a planet obtains from a disk of small planetesimals is too small to explain the current rotation of Mars or the total angular momentum of the Earth-Moon system, and that the stochastic component imparted by large impactors needs to be taken into account.

Ring particles likely have rough surfaces, and an oblique impact between them leads to rotation. In ring-satellite systems where the tidal force of the central planet is important, inelastic rebound of particles needs to be taken into account. Rotation rates of a moonlet caused by collisions of ring particles was studied by analytic calculation and three-body orbital integration.\(^16\),\(^17\) These studies show that the systematic component of rotation arising from a number of small impacts leads to the prograde equilibrium rotation rate comparable to the synchronous rotation rate (Fig. 6), while the stochastic component is dominant when the mass of impacting particles is comparable to the moonlet’s mass. In the study of rotation rates of ring particles, energy exchange between rotation and random orbital motion needs to be taken into account. \(N\)-body simulations of rings with spinning, equal-sized particles showed that the rotation rates are on the order of the orbital angular velocity \(\Omega\), while small particles were found to spin much faster when size distribution is taken into account.\(^15\),\(^24\),\(^25\),\(^28\)

As an alternative approach, we can derive and solve an evolution equation for the particle rotational energy, which is similar to the evolution equation for the velocity dispersion we discussed in §3.\(^29\),\(^30\) As we can see in Eqs. (2.20) and (2.23), in the equation of motion in the rotating coordinate system, the effect of particle spins always appears in the form of \(R_i \omega_i R_i\). In terms of \(\omega_{Ri}\), we define the “random rotational energy” of a particle in the rotating coordinate system as

\[
E_{\text{rot},i} \equiv \frac{1}{2} k m_i R_i^2 \omega_{Ri}^2.
\]

The change of spin angular velocity of particle \(i (i = 1, 2)\) due to a collision between particles 1 and 2 can be written as\(^29\)

\[
\Delta \omega_{Ri} = \frac{\mu (1 - \varepsilon_i)}{(K + 1) m_i R_i} \lambda \times [v + \lambda \times (R_p \omega_R)],
\]
where \( \mu = \frac{m_1 m_2}{(m_1 + m_2)} \) is the reduced mass. From Eqs. (2.24) and (4.2), we have

\[
R_p \Delta \omega_R = R_1 \Delta \omega_{R1} + R_1 \Delta \omega_{R2} = \frac{1 - \varepsilon_t}{\kappa + 1} \lambda \times [v_t + \lambda \times (R_p \omega_R)].
\]

(4.3)

Using Eq. (4.3), Eq. (4.2) can be rewritten as

\[
\Delta \omega_{Ri} = \frac{\mu}{m_i R_i} R_p \Delta \omega_R.
\]

(4.4)

Here, we define the spin velocity vector of particle \( i \) as

\[
s_i \equiv R_i \omega_R.
\]

(4.5)

Furthermore, we define \( s \) and \( S \) as

\[
s \equiv s_1 + s_2, \quad S \equiv \frac{m_2 s_2 - m_1 s_1}{m_1 + m_2}.
\]

(4.6)

Then, from Eq. (4.4) to (4.6), we find that \( S \) is conserved during impact, i.e.,

\[
\Delta S = 0.
\]

(4.7)
On the other hand, using Eq. (4.6), \( s_1 \) and \( s_2 \) can be written as

\[
\begin{align*}
  s_1 &= m'_1 s - S, \\
  s_2 &= m'_1 s + S,
\end{align*}
\]

(4.8)

where \( m'_i = m_i / (m_1 + m_2) \). From Eqs. (4.7) and (4.8), the change of \( s_1^2 \) due to a single collision with particle 2 can be written as

\[
\Delta s_1^2 = m'^2_2 \Delta s^2 - 2m'_2 S \cdot \Delta s.
\]

(4.9)

Then, the change rate of the mean random rotational energy of component 1 particles due to collisions with component 2 particles can be written as

\[
\frac{d\langle E_{\text{rot},1} \rangle}{dt} = \frac{1}{2} \mathcal{K} N s_2 m_1 \int f(e_1, i_1) f(e_2, i_2) f_s(s_1) f_s(s_2)
\]

\[
\times \Delta s_1^2 \frac{3}{2} f |b| db d\tau_1 d\varpi_1 d\tau_2 d\varpi_2 d\omega_1 d\omega_2 ds_1 ds_2,
\]

(4.10)

which is similar to Eq. (3.9) for the evolution of mean square eccentricities. In the above, \( f_s \) is the distribution function of spin velocity vectors; we assume that each component of \( s_j = (s_{jx}, s_{yy}, s_{zz}) \) (\( j = 1, 2 \)) follows the Gaussian distribution given as

\[
 f_s(s_j) ds_j = \frac{1}{(2\pi)^{3/2} (s_{jx}^2)^{1/2} (s_{yy}^2)^{1/2} (s_{zz}^2)^{1/2}} \exp \left( -\frac{s_{jx}^2}{2 s_{jx}^2} - \frac{s_{yy}^2}{2 s_{yy}^2} - \frac{s_{zz}^2}{2 s_{zz}^2} \right) ds_j.
\]

(4.11)

Furthermore, we assume that the orientation of spin axes is randomly distributed, i.e.,

\[
\langle s_{jx}^2 \rangle = \langle s_{yy}^2 \rangle = \langle s_{zz}^2 \rangle = \frac{1}{3} \langle s_j^2 \rangle.
\]

(4.12)

\( N \)-body simulations show that the above assumptions are reasonable ones.\(^{24,28}\)

Substituting Eqs. (4.9) and (4.11) into Eq. (4.10) and employing suitable variable transformation, the integration over the orbital elements that are related to the center of mass motion can be carried out to obtain\(^{29}\)

\[
\frac{d\langle E_{\text{rot},1} \rangle}{dt} = \frac{1}{2} \mathcal{K} N s_2 m_1 \int f(e, i) f_s(s)
\]

\[
\times \left( m'_2 R^2_1 \Delta \omega^2_R - 2 m'_2 m'_1 \langle s_1^2 \rangle \langle s_2^2 \rangle \Delta \omega_R \right)
\]

\[
\times \frac{3}{2} |b| db d\tau d\varpi db d\omega ds
\]

\[
\equiv \frac{1}{2} \mathcal{K} N s_2 h^4 m_1 m'_2 a_0^4 \Omega^4 \left( m'_2 \langle S_{CS} \rangle + \frac{m'_2 \langle s_2^2 \rangle - m'_1 \langle s_1^2 \rangle}{\langle s_1^2 \rangle + \langle s_2^2 \rangle} \langle S_{RF} \rangle \right),
\]

(4.13)

where \( \Delta \omega_R \) is the component of \( \Delta \omega \) that is parallel to \( \omega_R \), and \( \langle S_{CS} \rangle \) and \( \langle S_{RF} \rangle \) are the non-dimensional rates of evolution (collisional stirring and rotational friction

rates) defined in terms of the relative orbital elements ($\tilde{e}$ and $\tilde{i}$) and the relative spin rate ($\tilde{\omega}_R \equiv \omega_R/\Omega$) as \(^{29}\)

\[
\langle S_{CS} \rangle = \int f(\tilde{e}, \tilde{i}) f_s(\tilde{s}) r_p^2 \Delta \omega_R^2 \frac{3}{2} \tilde{b} d\tilde{b} \frac{d\tau d\omega}{(2\pi)^2} d\tilde{e} d\tilde{i} d\tilde{s},
\]

\[
\langle S_{RF} \rangle = \int f(\tilde{e}, \tilde{i}) f_s(\tilde{s}) (-2r_p^2 \tilde{\omega}_R \Delta \tilde{\omega}_R) \frac{3}{2} \tilde{b} d\tilde{b} \frac{d\tau d\omega}{(2\pi)^2} d\tilde{e} d\tilde{i} d\tilde{s}. \quad (4.14)
\]

Taking further into account interactions among particles of the same size, we obtain

\[
d\langle E_{\text{rot},1} \rangle dt = \frac{1}{2} K m_1 a_0^4 \Omega^3 \sum_{j=1,2} N_{s_j} h_{1_j}^4 m_j' \times \left( m_j' \langle S_{CS} \rangle + \frac{m_j' \langle s_j^2 \rangle - m_1' \langle s_1^2 \rangle}{\langle s_1^2 \rangle + \langle s_j^2 \rangle} \langle S_{RF} \rangle \right), \quad (4.15)
\]

where $\langle S_{CS} \rangle$ and $\langle S_{RF} \rangle$ are functions of the relative orbital elements and spin rates between $m_1$ and $m_j$.

If particles have a continuous size distribution, Eq. (4.15) can be written as (cf. Eq. (3.17))

\[
d\langle E_{\text{rot},1} \rangle dt = \frac{1}{2} K m_1 a_0^4 \Omega^3 \int n_s(m) m' h_{1_{m1,m}}^4 m' \langle S_{CS} \rangle_{m1,m} + \frac{m' \langle s_{m1}^2 \rangle - m_1' \langle s_{m1}^2 \rangle}{\langle s_{m1}^2 \rangle + \langle s_{m1}^2 \rangle} \langle S_{RF} \rangle_{m1,m} \right) dm. \quad (4.16)
\]

Figure 7 shows an example of numerical results obtained by solving the above evolution equation for the rotational energy for a ring consisting of two size-components. We confirm excellent agreement with N-body simulation.

Although effects of particle spins on rings’ dynamical evolution are shown to be rather limited,\(^5\),\(^24\),\(^31\) spin rates of ring particles are important in relation to observations of rings’ thermal emission. The primary heat source for Saturn’s rings is the sunlight. Also, thermal radiation and reflected sunlight from Saturn contribute to the heating of the inner rings, and mutual heating between nearby particles are important for dense rings. The response of ring particles to such heating depends on their physical and dynamical properties, including their spin rates. Particles with spin period much longer than their thermal relaxation time can be regarded as slow rotators, which radiate their thermal emission mainly from the face illuminated by the sunlight, while fast rotators with random spin orientations radiate over their whole surface area. Comparison between observations of rings’ thermal emission with model calculations provides constraints on physical properties as well as spin states and structure of the rings.\(^32\),\(^33\) For example, the observed temperature decrease with increasing solar phase angle suggests a population of slowly spinning particles in Saturn’s rings. Furthermore, the derived thermal inertia values suggest that slowly spinning large particles more easily retain surface regolith layers than rapidly spinning small particles.\(^33\) More detailed thermal modeling incorporating
particle spatial and spin distributions obtained from dynamical simulations would provide better constraints on particle properties.

§5. Viscosity

As a result of mutual collisions and gravitational interactions between particles, angular momentum transfer takes place in planetary rings. Angular momentum transfer determines the time scale of radial spreading, and is described in terms of viscosity.

In dilute rings where the mean collision time is much longer than the orbital period, viscosity is determined by the extent of radial excursion of particles ($\sim \sigma/\Omega$; $\sigma$ is the particle velocity dispersion) and the mean collision frequency ($\sim \tau \Omega$) as

$$\nu \sim (\sigma/\Omega)^2 \times \tau \Omega,$$

or

$$\nu \sim \sigma^2 \tau/\Omega,$$

(5.1)

where $\tau$ is the ring optical depth. As we mentioned in §3, $\sigma \sim R\Omega$ when $r_h \ll 1$ and inelastic collision determines particle velocity evolution, while $\sigma \sim v_{esc}$ when $r_h \simeq 1$ and gravitational encounters are important. In the case of dense rings with no or weak self-gravity, mean free path of particles becomes on the order of the particle radius $R$ and the momentum transfer by a collision over the distance $R$ is important. In this case, the viscosity can be given as

$$\nu \sim R^2 \tau \Omega.$$  

(5.2)

The validity of the above expression was confirmed by $N$-body simulations. On the other hand, $N$-body simulations of self-gravitating collisional particle disks show that viscosity is strongly enhanced when particle disks become gravitationally unstable and gravitational wakes are formed (Fig. 8). In this case,

![Fig. 8. Normalized viscosity ($\nu/(R^2 \Omega)$) as a function of ring optical depth $\tau$. The solid line shows results of $N$-body simulation for $r_h = 0.82$, corresponding to $10^5$ km from Saturn if particle internal density is $0.9 \text{ g cm}^{-3}$. The dashed line represents the linear $\tau$-dependence relevant to the case in the limit of low optical depth. Redrawn from 31).](https://academic.oup.com/ptps/article-abstract/doi/10.1143/PTPS.195.29/1864195/Collisions-and-Gravitational-Interactions-between)
the typical length scale of the wakes is the critical wavelength related to axisymmetric gravitational instability (∼ \( G \Sigma / \Omega^2 \), where \( \Sigma \) is the disk surface mass density), and the viscosity in such a self-gravitating disk can be estimated as \( \nu \sim G^2 \Sigma^2 / \Omega^3.36\)

In fact, N-body simulations of dense collisional rings with self-gravitating particles show that the viscosity can be expressed in a similar form but with a correction factor, as

\[
\nu = C(r_h)G^2 \Sigma^2 / \Omega^3,
\]

(5.3)

and \( C(r_h) \propto r_h^3 \) has been suggested from results of N-body simulations.3),12),31) This correction factor represents the effect of finite size of particles in planetary rings. When \( r_h \simeq 1 \), the strong self-gravity of the disk results in the formation of enhanced wake structures and efficient angular momentum transfer. On the other hand, when \( r_h \) is significantly smaller than unity, the effect of finite size of particles prevents particle clumping, thus the self-gravity effect on ring viscosity is negligible. N-body simulations show that the value of the viscosity also depends on the elastic properties of particles, and the viscosity varies by a factor of 2 – 3 when particle restitution coefficients are changed.3),12),31) Although the above semi-analytic expressions for ring viscosity have been derived empirically by comparing with numerical simulations, detailed theoretical modeling of viscosity in self-gravitating collisional disks would be useful for a better understanding of the dynamics in dense planetary rings.

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