Turbulent amplification of magnetic field and diffusive shock acceleration of cosmic rays

A. R. Bell

Blackett Laboratory, Imperial College, London SW7 2AZ

Accepted 2004 June 1. Received 2004 April 30

ABSTRACT

The diffusive shock acceleration of cosmic rays by supernova remnants depends upon the generation of magnetic fluctuations by cosmic rays upstream of the shock. Strongly driven, non-resonant, nearly purely growing modes grow more rapidly than the resonant Alfvén waves usually considered. Non-linear simulation shows that the magnetic field can be amplified from its seed value by orders of magnitude. The consequences for the maximum attainable cosmic ray energy in supernova remnants are explored.

Key words: acceleration of particles – magnetic fields – plasmas – shock waves – turbulence – cosmic rays.

1 INTRODUCTION

There is clear evidence that electrons, and probably protons, are accelerated to energies of at least $10^{14}$ eV in supernova remnants (SNR) (Koyama et al. 1995; Allen et al. 1997; Tanimori et al. 1998; Aharonian 1999; Naito et al. 1999; Aharonian et al. 2001; Berezhko, Ksenofontov & Völk 2003; Vink & Laming 2003). Diffusive shock acceleration (DSA) is the most likely acceleration process (Axford, Leer & Skadron 1977; Krymsky 1977; Bell 1978; Blandford & Ostriker 1978; reviewed by Drury 1983), and detailed DSA/SNR models fit well to many aspects of the observational data (e.g. Berezhko, Ksenofontov & Völk 2002). DSA naturally gives a power law with a suitable spectral index, and its high efficiency allows SNR to match the energy requirements for the production of cosmic rays (CR) in our galaxy. The observed CR energy spectrum is a power law with constant index up to the ‘knee’ at $10^{15}$ eV above which the spectrum steepens. This invites the natural conclusion that DSA at the outer shocks of SNR is responsible for galactic CR acceleration to this energy. However, the standard theory of DSA has to be stretched to its limits to reach this energy and the observational support is not unambiguous (Lagage & Cesarsky 1983a,b; Buckley et al. 1998; Kirk & Dendy 2001). Additionally, the connection of the spectrum below the knee to the steeper power law above the knee invites speculation that acceleration by SNR does not end at the knee, but continues in some form to energies of $10^{17}$–$10^{18}$ eV (e.g. Axford 1994), although the CR at the higher end of this range may be heavier ions. Obviously, it is important to know the maximum energy attainable by DSA.

The basic theoretical limit to the CR energy arises from the CR Larmor radius, which must be smaller by at least $v_s/c$ than the dimensions of the acceleration region (Hillas 1984) where $v_s$ is the shock velocity. The Larmor radius of a proton with energy $10^{15}$ eV in a typical interstellar magnetic field of 3 $\mu$G is 0.36 pc ($1.1 \times 10^{16}$ m). This is sufficiently small compared with the radius of a suitable (young) SNR only if favourable, although not unreasonable, assumptions apply. For example, CR diffusion upstream of the shock must not exceed Bohm diffusion, which assumes that the CR mean free path is limited to approximately one Larmor radius ($D = r_L/c/3$). Otherwise, CR at the maximum energy escape upstream instead of being accelerated.

The limitation to the maximum CR energy due to the large Larmor radius would be ameliorated if the characteristic magnetic field were larger than supposed. Here, we further investigate the possibility that the magnetic field in the acceleration region is amplified to larger than the typical interstellar value of 3 $\mu$G by the action of the CR. There are indications that this amplification does in fact occur. Fields as large as 1 mG are indicated by radio observations (Longair 1994) in some parts of some young SNR. Recent analysis of the morphology of X-ray emission close to the outer shocks of SN1006 (Berezhko et al. 2003) and Cas A (Vink & Laming 2003) imply that strong magnetic fields are needed for rapid synchrotron loss by high-energy electrons. Furthermore, as noted by Völk & McKenzie (1981), Völk (1984) and Bell & Lucek (2001), the interaction of CR with the upstream plasma generates strong turbulence, which could wind up a large ‘frozen-in’ magnetic field, as the linear theory indicates a value of $\delta B/B$ greater than one. A non-linear theory of CR-driven turbulence is clearly needed.

A first attempt at the non-linear calculation was made by Lucek & Bell (2000). In a hybrid model they coupled a particle-in-cell (PIC) model of CR to a three-dimensional (3D) magnetohydrodynamic (MHD) model of the background plasma. The CR were initialized with a drift relative to the MHD background and it was shown that the magnetic field frozen into the background was wound up by turbulent motions to give a maximum field reaching a factor of 8 times greater than the magnitude of the initial field. However, the...
limited computational resources meant that the calculations used unrealistic parameters and could not be continued for a sufficient period of time. Bell & Lucek (2001) used an analytic model to estimate the way in which turbulent magnetic field amplification might extend the maximum CR energy, but the model was based upon hypothesized equations for the generation of magnetic field, which need substantiation (or amendment) if the results are to be convincing. Magnetic field amplification is seen as probably the most hopeful key to understanding the acceleration of galactic CR to the knee and possibly beyond the knee (Kirk & Dendy 2001; Drury, van der Swaluw & Carroll 2003).

In this paper, we show that for shocks with high Alfvén Mach number the process by which streaming CR excite MHD turbulence is different from that usually supposed and takes on a different character. The improved understanding makes possible a non-linear simulation with realistic parameters. We show that strong magnetic field amplification can be expected in young SNR.

2 LINEAR THEORY

In this section, we develop the linear theory for the generation of fluctuations in the upstream magnetic field. These fluctuations are then responsible for the scattering of CR and their confinement in the region close to the shock. It is well known that streaming CR can resonantly excite the growth of Alfvén waves with a wavelength matching the CR Larmor radius (Wentzel 1974; Skilling 1975a,b,c). We show here that, in contrast, for the strong CR drifts expected in SNR, the waves are not accurately described as Alfvén waves. The waves are nearly purely growing, and the waves grow preferentially at wavelengths that are not resonant with the CR Larmor radius. The non-linear growth of these waves will be investigated in Section 8.

In the frame in which the shock is at rest, the hydrodynamic quantities and the CR distribution can be taken to be approximately in steady state. The upstream CR have a zero drift velocity in this frame and consequently have a drift velocity equal to the shock velocity \( v_s \) in the frame in which the upstream plasma is at rest. In general, the drift velocity along magnetic field lines is \( v_{\parallel}/\cos \theta \) where \( \theta \) is the angle between the magnetic field and the shock normal, but here we restrict ourselves to parallel shocks for which \( \theta = 0 \). This CR drift is the source of the unstable growth of MHD fluctuations in the background plasma. For simplicity we assume that all CR are protons. The CR have a number density \( n_c \) and a charge density \( n_{c,e} \), and carry a current density \( j_{c} = n_{c,e}e u \) in the upstream rest frame. For quasineutrality, the background plasma carries a charge density \(-n_{c,e}e\) if the background plasma carries a current density \( j \) in its local rest frame, then \( \nabla \cdot B = \mu_0 \mu_0 (\nabla \cdot B) = j_{c} \) in the upstream rest frame. Consequently, in the upstream rest frame (moving at speed \( v_s \) relative to the shock), the momentum equation for the background plasma is

\[
\frac{du}{dt} = - PV - \frac{1}{\mu_0} \frac{B}{\nabla \cdot (B \wedge B)} - j_{c} \wedge B + n_{c,e}e u \wedge B,
\]

where \( P \) is the background plasma pressure. The plasma pressure plays no part in the linear calculation as the fluctuations are transverse in the cases we consider. The other MHD equations are unaffected by the presence of the CR:

\[
\frac{dB}{dt} = \nabla \cdot (u \wedge B);
\frac{d\rho}{dt} = -\nabla \cdot (\rho u).
\]

These equations make it clear that the MHD turbulence is driven by the \(- j_{c} \wedge B\) force exerted in reaction to the \( j_{c} \wedge B\) force exerted on CR through the current they carry.

\[\sigma = \frac{B_0}{j_0} \sigma \int_0^\infty 2\pi p_1 dp_1 \int_{-\infty}^\infty dp_2 \frac{v_{\perp} p_1}{\omega - \omega_0 - kv_1} \times \left[ v_1 \left( \frac{\partial f}{\partial p_2^2} - \frac{\partial f}{\partial v_2^2} \right) - \omega \frac{\partial f}{k \partial p_2^2} \right].\]

The CR drift velocity is of the order of \( v_{\perp} \), which is much smaller than \( c \), so we can approximate the zeroth-order distribution to

\[f(p, z, t) = f_0(p, z, t) + f_1(p, z, t)p_\parallel/p\]

using the first two terms of the Chapman–Enskog expansion in the small parameter \( v_{\perp}/c\) for a diffusive drift. Making the reasonable

\[\frac{\partial^2 B_{\parallel}}{\partial t^2} = \frac{\nabla^2 B_{\parallel}}{\partial z^2} + \frac{j_1}{\rho v_{\parallel}} B_{\perp} \wedge B_{\parallel} + \frac{B^2_{\parallel}}{\rho} \frac{\partial j_1}{\partial v_{\parallel}} \wedge B_{\parallel} = 0.\]
approximation, which can be checked at the end of the calculation, that \( \omega \) is small compared with both the CR gyrofrequency \( \omega_c \) and \( kc, \sigma \) can be written as

\[
\sigma = \frac{1}{\gamma_i} \int_0^\infty \frac{4\pi}{3} e^{p^2} \left( \nu f_1 - \frac{\omega}{k} \rho \frac{\partial f_0}{\partial p} \right) \rho(\omega_k/kv) dp, \tag{6}
\]

where

\[
\rho(\lambda) = \frac{3}{4} \lambda(1 - \lambda^2) \left[ \ln \left( \frac{1 + \lambda}{1 - \lambda} \right) + i\pi \right] + \frac{3}{2} \lambda^2.
\]

\( \lambda = 1/k\gamma_c \) is the wavelength normalized to \( 2\pi \) times the CR Larmor radius. In the case of spatial resonance between the wavelength and the Larmor radius, \( \lambda \sim 1 \). The real and imaginary parts of \( \rho \) are essentially the same as \( I_1 \) and \( I_1 \) in the calculation by Achterberg (1983) except that Achterberg’s \( x^2 \) is our ‘1/\( \lambda^2 \).’ Substituting this expression for \( \rho \) into the expression for \( \sigma \) gives the dispersion relation

\[
\omega^2 - k^2 v_s^2 = \frac{k^2 B_1^2}{\rho} \left( 1 - \frac{\omega}{kv_s} \right) = 0.
\]

In most cases \( \omega k v_s \) can be neglected as the phase speed \( |\omega|/k \) is small compared with the shock velocity or the growth rate is small compared with \( kv_s \).

### 3 LINEAR THEORY FOR AN UPSTREAM CR DISTRIBUTION

Upstream of a strong shock the CR distribution function can be approximated by a power-law distribution \( f_0 \propto p^{-\beta} \) between a lower limit \( p_1 \) and an upper limit \( p_2 \) in momentum.

\[
f_0(p) = \begin{cases} \Phi p^{-\beta} & \text{if } p_1 < p < p_2 \\ 0 & \text{otherwise} \end{cases}
\]

In reality, if the shock is not highly supersonic, or if the shock itself is strongly modified by the CR pressure, the index of the power law deviates from 4, but the simple theory ofDSA for a strong shock gives 4 and the following analysis is simpler if this assumption is made. The spatial distribution of CR with momentum \( p \) decreases upstream of the shock with a scale length \( L(p) \) proportional to the CR scattering mean free path, which is proportional to the CR momentum in the case of Bohm diffusion (mean free path equal to the Larmor radius), giving \( L(p) = pc/3eBv_s \). Hence at a distance \( z \) upstream of the shock, the minimum CR momentum \( p_1 \) is determined by \( L(p_1) \sim z \), giving \( p_1 \sim 3v_s/cBz/c \).

Provided \( p_2 \gg p_1 \), the actual value of \( p_2 \) does not matter in what follows and can be set to \( \infty \), except that it appears logarithmically in the expression for the local CR energy density. The CR energy density at the shock is approximately \( U_{\Theta} = 4\pi \Phi \Phi_c \ln (p_2/m_p c) \). If the CR drift at velocity \( v_{\perp} \), relative to the upstream plasma, then \( f_1 = (3v_{\perp}/c)f_0, \) and \( j_1 = 4\pi e v_{\perp} \Phi / p_1 = (v_{\perp}/c)e(U_{\Theta}/p_1)/\ln(p_2/m_p c) \). Substitution into equation (6) gives

\[
\sigma = \frac{\omega}{3k v_s} \rho(1/k\gamma_c)
\]

\[
+ \left( 1 - \frac{4\omega}{3k v_s} \right) k\gamma_c \int_0^{1/k\gamma_c} \rho(\lambda) d\lambda.
\]

For \( k\gamma_c \ll 1 \), the Alfvén term is negligible and the dominant mode has a frequency with a large imaginary component. Taking as typical parameters, \( U_{\Theta}/\rho v_s^2 = 0.2 \) (equivalent to 10 per cent energy transfer to CR),

\[
\int_0^{1/k\gamma_c} \rho(\lambda) d\lambda = \frac{3}{8} \lambda^2(1 - 1) - \frac{3}{16} \lambda^2(1 - 1)^2 \ln \left[ 1 + \lambda \right] - \lambda^2(1 - 1)^2 \int_1^{1/k\gamma_c} \rho(\lambda) d\lambda.
\]

\( \lambda_c = 1/k\gamma_c \) is an important parameter because it represents the ratio between the wavelength and the Larmor radius of CR at the lower momentum cut-off \( p_1 \).

### 4 MODE BEHAVIOUR IN THREE WAVELENGTH REGIMES

An important dimensionless parameter determining the strength of the CR driving term is

\[
\zeta = \frac{1}{\ln(p_2/m_p c)} \frac{U_{\Theta} v_s}{\rho v_s^2 c} = \frac{|B_1 j_1 r_{\Theta}}{\rho v_s^2}.
\]

As remarked above, \( \omega/kv_s \) is usually small, reflecting the fact that the Alfvén Mach number \( M_A \) is very large in young SNR, so we ignore terms multiplied by this parameter during our examination of mode behaviour. The dispersion relation (equation 7) simplifies to

\[
\omega^2 - v_s^2 k^2 = \frac{1 - \sigma_1}{\gamma} = 0,
\]

where

\[
\sigma_1 = k\gamma_c \int_0^{1/k\gamma_c} \rho(\lambda) d\lambda.
\]

The real part of \( \sigma_1 \) and the imaginary part of \( \sigma_1 \) as functions of \( k\gamma_c \).
The real and imaginary parts of the frequency are equal in magnitude, leaving the imaginary part \( kr \) only as the square root of the CR current for fixed regime. Because the wavelength is much smaller than the Larmor radius upstream of the outer shocks of young SNR and cannot be characterized in terms of linear or non-linear Alfvén waves. In the weakly driven regime, the linear growth rate is proportional to the wave depends on the sign of \( \delta B / B \) and the tension in the magnetic field is of any significance for CR acceleration. In this intermediate regime the most significant for CR acceleration. The maximum growth rate occurs towards the high-\( k \) end of this regime, and waves throughout most of the regime have e-folding times short enough for significant amplification in the time taken for a fluid element to pass through the CR scaleheight. Except at the low-\( k \) end of this regime, \( \sigma \) is small (1 - \( \sigma \) \( \approx \) 1 in Fig. 1), so the dispersion relation for the growing mode approaches to

\[
\omega^2 - \frac{v_A^2}{\ell} k^2 = \pm \frac{v_s^2}{\ell} k \frac{k}{\ell} = 0,
\]

where the replacement of \( \pm \) by \( - \) or \( + \) determines whether or not the mode is either oscillatory or purely growing or decaying. For the purely growing mode, the growth rate varies as \( k^{1/2} \) within regime II, and the maximum growth rate \( \gamma_{\text{max}} \) occurs at the wavenumber \( k_{\text{max}} \) where

\[
k_{\text{max}} = \frac{\zeta_{\nu} v_s}{2} \frac{1}{\ell} \quad \text{and} \quad \gamma_{\text{max}} = \frac{\zeta_{\nu}}{2} \frac{v_s}{v_A} \frac{v_A}{\ell_{\nu}}.
\]

For the parameters adopted above, and for a lower limit of \( 10^{15} \) eV to the local CR spectrum (\( \ell_{\nu} = 1.1 \times 10^{16} \) m), \( k_{\text{max}} = 2 \times 10^{13} \) m and \( \gamma_{\text{max}} = 98 \) yr. The relative time-scales and their implication for CR acceleration in SNR will be discussed in Section 6.

## 5 Wave Modes in Regime II

In regime II, the small value of \( \sigma \) for \( kr_{g1} \gg 1 \) indicates that \( j_\perp \) is small. As might be expected, CR trajectories are negligibly affected by modes with a wavelength much shorter than the CR gyroradius. Consequently, the \( j_\perp \wedge B \) force is small, and the waves are driven mainly by the interaction of the zeroth-order current \( j_1 \) with the first-order perpendicular field \( B_\perp \). Because kinetic effects are small, modes in regime II, especially those with the largest growth rate, can be understood in purely MHD terms. This allays the concerns of McKenzie & Volk (1982) and Achterberg (1983) that kinetic effects such as particle trapping might become important when the growth rate is comparable with the oscillation frequency and \( \delta B / B \) approaches one, although it will be shown in Section 9 that kinetic effects may cause wave saturation when \( \delta B / B \gg 1 \).

The MHD equations describing the waves in regime II take a very simple form (i) under our assumption that the wave-vector \( \mathbf{k} \), the zeroth-order magnetic field and the zeroth-order current are all parallel, thus making the wave transverse and \( \rho \) a constant, and (ii) if the wavelength is significantly longer than that at the border with regime III, thus making irrelevant the tension in the magnetic field lines. The equations for regime II are then

\[
\frac{\partial u}{\partial t} = - j_1 \wedge B; \quad \frac{\partial B}{\partial t} = \nabla \wedge (u \wedge B).
\]

These simple equations can be used to check the results reached above for regime II and to understand the nature of the modes in this regime. Many of the results presented in this paper can be derived from these two equations alone. The more complicated derivation of the dispersion relation in Sections 2 and 3 serves mainly to establish the validity and applicability of these equations.

Waves in regime II can be oscillatory, purely growing or purely decaying depending on their polarization. The time dependence of the wave depends on the sign of \( k B_\perp j_\parallel \). If the current \( j_\parallel \) and the...
zeth-order magnetic field $B_1$ are both aligned in the same direction, then waves with clockwise polarization ($k < 0$) are oscillatory, and waves with clockwise polarization ($k > 0$) are either purely decaying or purely growing depending on the relative directions of $u_\perp$ and $B_\perp$. If $j_1$ and $B_1$ are antiparallel, the oscillatory wave has a clockwise polarization, and the polarization of the growing or decaying mode is anticlockwise. In regime II, a CR proton moving along the magnetic field in the same direction as $j_1$ spirals in the opposite sense to the growing wave it produces. If the wave were kinetically driven, as in regime III, the wave and the CR particles driving it would rotate in the same sense as required by the resonance condition. This is not, and need not be, the case in regime II in which the kinetic effects are unimportant and the wave is not driven resonantly. However, the opposite sense of rotation means that resonant scattering by small amplitude waves is impossible for CR drifting in the same direction as $j_1$, i.e. those streaming away from the shock. The waves have the correct polarization for resonant scattering of CR moving towards the shock, but this cannot inhibit the upstream escape of CR. Non-linear scattering by large amplitude waves is probably needed if these waves are to inhibit escape upstream of the shock.

For the typical parameters adopted above, waves throughout regime II can be expected to grow to non-linear amplitudes unless the seed perturbation is very small. Waves with high $k$ will become non-linear earlier because of their larger growth rate. It may appear that this is unhelpful as their wavelength is much smaller than the CR Larmor radius and the CR scattered only weakly by linear non-linear earlier because of their larger growth rate. It may appear that the growth of strongly driven modes is

$$v_\perp^2 v_\parallel^2 \approx \frac{1}{\ln(p_\perp/m_e c)} \frac{U_{\perp}}{\rho v_\perp^2} \frac{v_\perp}{c v_\parallel^2} > 1.$$  \hspace{0.5cm} (19)

This is a condition on the magnitude of the CR current, and can be interpreted in terms of the efficiency of CR production. In the rest frame of the shock, kinetic streaming energy enters the system at a rate $\rho v_\perp^2/2$, and the downstream CR energy flux is $U_{\perp}/v_\perp^4$ where it is assumed that the downstream velocity is $v_\perp/4$ as expected for a strong shock. The ratio of these defines an efficiency $\eta_{\text{CR}} = U_{\perp}/2\rho v_\perp^2$ for CR production, and wave generation is strongly driven if

$$\eta_{\text{CR}} > 10^{-4} \left( \frac{n_0}{\text{cm}^{-3}} \right)^{-1} \left( \frac{v_\perp}{10^7 \text{ms}^{-1}} \right)^{-3} \left( \frac{B}{3 \mu G} \right)^2$$

$$\times \left[ \frac{\ln(p_\perp/m_e c)}{14} \right].$$  \hspace{0.5cm} (20)

If CR acceleration in SNRs accounts for galactic CR production then $\eta_{\text{CR}}$ must take a value around 0.1 and the condition for strong driving is easily satisfied for SNRs in the free expansion phase.

Having established that the strongly driven regime applies to young SNRs, the next step is to find whether wave growth is sufficiently rapid. This again depends on the CR current and the efficiency with which energy is transferred to CR. In $(p_\perp/m_e c)$ is a further efficiency factor giving the fraction of the total CR energy given to CR in a momentum range $\Delta p \sim p$. It is convenient to define an overall efficiency $\eta_{\text{CR}} = \eta_{CR}/\ln(p_\perp/m_e c) = U_{\perp}/2\rho v_\perp^2 \ln(p_\perp/m_e c)$ for energy transfer into a momentum range $\Delta p \sim p$. The value of $\eta_{\text{CR}}$ will be discussed further in Section 7. As given in equation (16), the most rapidly growing modes with wavenumber $k_{\text{max}}$ grow with a growth rate $\gamma_{\text{max}}$. The inverses of these can be written as

$$k_{\text{max}}^{-1} = \frac{\eta_{\text{log}}}{1/140} \left( \frac{v_\perp}{10^7 \text{ms}^{-1}} \right)^{-3} \left( \frac{n_0}{\text{cm}^{-3}} \right)^{-1} \left( \frac{B}{3 \mu G} \right)^{2}$$

$$\times \left( \frac{\epsilon_1}{10^{15} \text{eV}} \right) \left( \frac{B_1}{3 \mu G} \right) 2 \times 10^{13} \text{m},$$  \hspace{0.5cm} (21)

$$\gamma_{\text{max}}^{-1} = \frac{\eta_{\text{log}}}{1/140} \left( \frac{v_\perp}{10^7 \text{ms}^{-1}} \right)^{-3} \left( \frac{n_0}{\text{cm}^{-3}} \right)^{-1/2} \left( \frac{B}{3 \mu G} \right)^{2}$$

$$\times \left( \frac{\epsilon_1}{10^{15} \text{eV}} \right) 98 \text{yr},$$  \hspace{0.5cm} (22)

where $\epsilon_1 = c p_1$ is the local lower energy cut-off to the CR energy distribution. For waves with other wavenumbers in regime II, the growth time is

$$\gamma_{\text{max}}^{-1} = \frac{k}{k_{\text{max}}} \left( 2 - \frac{k}{k_{\text{max}}} \right)^{-1/2}.$$  \hspace{0.5cm} (23)

For the magnetic field to grow to $\delta B/B \sim 1$, the growth time must be less than the SNR expansion time. The free expansion phase at velocity $v_\parallel$ lasts for a time $T_{\text{FE}}$ until the SNR has swept up approximately its own mass $M$ of ISM and expanded to a radius $R_{\text{FE}}$

$$R_{\text{FE}} = \left( \frac{M}{M_o} \right)^{1/3} \left( \frac{n_0}{\text{cm}^{-3}} \right)^{-1/3} 6.7 \times 10^{16} \text{m}$$  \hspace{0.5cm} (24)

$$T_{\text{FE}} = \left( \frac{M}{M_o} \right)^{1/3} \left( \frac{n_0}{\text{cm}^{-3}} \right)^{-1/3} \left( \frac{v_\perp}{10^7 \text{ms}^{-1}} \right)^{-1} 210 \text{yr}$$  \hspace{0.5cm} (25)

giving mode amplification by \( \exp(N_{\text{BE}}) \) where

\[
N_{\text{BE}} = \gamma_{\max} \tau_{\text{BE}} = 2.1 \left( \frac{\eta_{\log}}{1/140} \right) \left( \frac{v_s}{10^5 \text{ m s}^{-1}} \right)^2 \left( \frac{n_0}{\text{cm}^{-3}} \right)^{1/6} \times \left( \frac{E_1}{10^{15} \text{ eV}} \right)^{-1} \left( \frac{M}{M_0} \right)^{1/3}.
\]  

(26)

If we take \( N_{\text{BE}} > 2 \), i.e. two e-foldings at the maximum growth rate during the free expansion time of an SNR, as the condition for substantial wave growth, and hence a necessary condition for CR acceleration, then the maximum attainable CR energy is limited to

\[
\epsilon_{\max} = \left( \frac{\eta_{\log}}{1/140} \right) \left( \frac{v_s}{10^5 \text{ m s}^{-1}} \right)^2 \left( \frac{n_0}{\text{cm}^{-3}} \right)^{1/6} \times \left( \frac{M}{M_0} \right)^{1/3} \times 10^{15} \text{ eV},
\]  

(27)

Although the numerical value of the energy limit given by equation (27) is similar to the LC limit, the two are in fact not directly connected. This is apparent from their different dependences. In equation (27), \( \epsilon_{\max} \) is independent of the magnetic field strength, whereas the LC maximum energy is proportional to the magnetic field. The LC limit is independent of the CR acceleration efficiency, whereas our limit is proportional to \( \eta_{\log} \).

The strong dependence on the shock velocity \( v_s \) implies that the maximum CR energy falls rapidly once the SNR is well into the Sedov phase. Acceleration to \( 10^{15} \text{ eV} \) can take place only during the free expansion phase unless the efficiency \( \eta_{\log} \) is much greater than 1/140. Indeed, SNR such as SN1906 with shock velocities of around 3–4000 km s\(^{-1}\) may be past the stage of evolution at which they produce CR at \( 10^{15} \text{ eV} \).

The separation between the free expansion and Sedov phases is not distinct as the SN ejects material at a range of velocities. The shock velocity can decrease substantially well before the SNR has swept up a mass of ambient material equal to the total mass ejected by the SN. SN1987A expanded at an average velocity of \( 3.5 \times 10^5 \text{ m s}^{-1} \) into a medium of density 30–100 cm\(^{-3}\) for the first 4 yr before slowing considerably (Hasinger, Aschenbach & Trümpfer 1996; Chevalier 1997; Gaensler et al. 1997). Another well-studied example, SN1993J, slowed from an initial expansion velocity of \( 1.7 \times 10^5 \text{ m s}^{-1} \) in a medium of density \( \sim 10^6 \text{ cm}^{-3} \) to a velocity after 9 yr of \( 0.9 \times 10^5 \text{ m s}^{-1} \) in a medium of density \( \sim 10^4 \text{ cm}^{-3} \), during which time it interacted with an ambient mass of \( \sim 0.3 \text{ M}_\odot \). (Bartel et al. 2002; Bietenholz et al. 2003). Inserting the parameters for these SN into equations (22) or (27) implies the possibility of wave growth excited by CR with energies beyond \( 10^{15} \text{ eV} \). Because of its strong dependence on velocity and its weak dependence on ejected mass, equation (27) directs us to very young SN expanding into a dense circumstellar medium as the source of very high energy CR. Equation (27) does not supersede the condition of Lagage & Cesarsky (1983a,b). Both conditions must be satisfied. However, the LC condition will be eased by the generation of magnetic fields much greater than 3 \( \mu \text{G} \) as indicated here and supported by observation (Fransson & Björnsson 1998). It will be shown in Section 8 that magnetic field continues to grow non-linearly well past the initial seed value, in which case the dominant limit on the maximum CR energy may often be equation (27).

7 THE VALUE OF \( \eta_{\log} \)

Equations (22) and (27) highlight the importance of the CR acceleration efficiency \( \eta_{\log} \). \( \eta_{\log} \) determines the CR current \( j_1 \), which drives wave growth. The assumed characteristic value of \( \eta_{\log} = 1/140 \) is constructed from two components. We assume firstly that 10 per cent of the SN energy is given to CR, and secondly that the energy is spread evenly in logarithmic terms over six orders of magnitude in energy (ln \( 10^6 \) = 14) as expected for an \( f_0(p) \propto p^{-4} \) power law. The energy requirements for galactic CR production stipulate that the overall efficiency cannot be less than 10 per cent, and there are many indications that it might be as high as 50 per cent (Eichler 1979; Völk, Drury & McKenzies 1984; Malkov & Drury 2001; Ellison, Decourchelle & Ballet 2004), which would increase \( \eta_{\log} \) from 1/140 to 1/28, and increase the maximum CR energy by a factor of 5. The further factor of 1/14 resulting from the energy spread might also be unduly conservative. Non-linear calculations of the feedback of CR on to the shock structure indicate a concave spectrum, which is flatter than \( p^{-4} \) at the high energy end, thereby putting more of the available energy into CR close to the knee (Bell 1987; Falle & Giddings 1987). Instead of 1/14, the more appropriate factor might be around 1/5 if the distribution function were \( f_0(p) \propto p^{-3.5} \) depending on the shape of the spectral turnover at high CR energy. This would increase \( \eta_{\log} \) and the maximum CR energy by 1/5. Putting together both these optimistic assumptions gives \( \eta_{\log} = 1/10 \), thereby raising the maximum CR energy by a factor of 14. Conversely, one might take the pessimistic view that the CR distribution function is steeper than we assume. The data might allow \( f_0(p) \propto p^{-4.1} \), and this would reduce the energy spread factor from 1/14 to around 1/200, thereby reducing the maximum CR energy by 14/200 unless the overall efficiency were greater than 10 per cent. On balance, \( \eta_{\log} = 1/140 \) seems reasonably conservative, but this number is uncertain and there is corresponding uncertainty in our conclusions on the maximum attainable CR energy as determined by the necessity of wave growth.

8 NON-LINEAR SIMULATION

Lucek & Bell (2000) used a hybrid (MHD/particle) code (mhd3d) to model the self-consistent CR/MHD interaction. mhd3d solved the three-dimensional ideal MHD equations coupled to a particle-in-cell model of the CR. We use the same code, also used as a purely MHD simulation by Lucek & Bell (1997), except that the particle model can be dispensed with as kinetic effects are unimportant. This makes the simulation very much faster and allows more realistic parameters and longer simulation times. The code solves the usual ideal MHD equations in dimensionless form with the addition of a force \( -j_1 \times B \) where \( j_1 \) is fixed, uniform and constant. \( j_1 \) is in the \( z \)-direction. The energy equation is adiabatic apart from viscous heating, which allows for dissipation in shocks. These equations represent the linear and non-linear growth of waves in regime II. They model the cutoff in wave growth due to tension in magnetic field lines at the transition from regime II to III, but underestimate the already small wave growth in regime III because kinetic effects are neglected. Because CR kinetic effects are not included, the turnover in growth rate at the boundary between regimes I and II is not modelled, but the largest growth rate occurs close to regime III and far from regime I (although see Section 9 for the importance for saturation of kinetic effects).

The simulation is initialized with density \( \rho = 1 \), \( B_z = 1 \), pressure \( P_u = 1 \), velocity \( u = 0 \), and CR current \( j_1 = 4\pi r/5 \). With these parameters, \( k_{\max} = 2\pi/5 \) and \( \gamma_{\max} = 2\pi/5 \). The grid-spacing is \( \Delta x = \Delta y = \Delta z = 0.5 \) and the boundary conditions are periodic with 128 cells in each direction. A small random seed perturbation is added to the magnetic field with the energy in each mode proportional to \( k^{-3} \) for all modes with wavelengths more than a few grid-points.
This dependence places equal magnetic energy densities in each decade in $|k|$. Modes with $k > k_{\text{max}}$ play an insignificant role as the boundary between regimes II and III is at $k = 2k_{\text{max}}$ and the linear growth rate is zero in regime III in the MHD approximation.

The initial root mean square (rms) perturbation to each component $(B_x, B_y, B_z)$ of the magnetic field perpendicular to $B_\parallel$ is 0.057. If we assume the same typical parameters adopted above, $v_A = 1$, $v_i = 1520$, $c = 45$ 500 and $r_{gl} = \zeta v_i^2/v_A^2 = 1100$ in dimensionless units.

Fig. 3 plots the growth of magnetic field, where $B_\perp$ is the mean value of $(|B_x|^2 + |B_y|^2)/2$. At very early times ($t \sim 1$), wave growth is slow because the initial perturbation does not consist only of growing eigenmodes. Fig. 3 also plots the evolution of the mean magnetic, thermal and kinetic energy densities. The plasma is initialized at rest, so the growth of the instability begins with the $j \wedge B$ force causing motion perpendicular to the CR current. Once the plasma is set into motion, $B_\perp$ begins to increase because the magnetic field is frozen into the plasma. Once the growing eigenmodes have emerged, the overall growth rate of $B_\perp$ (between $t = 2$ and 5) is close to one, which is a little less than the maximum growth rate $\gamma_{\text{max}} = 1.26$ as modes around $k = k_{\text{max}}$ also grow. As expected for modes close in $k$ to the fastest growing mode, the kinetic energy density is approximately equal to the perturbed magnetic energy density. At approximately $t = 5$, the amplitude becomes non-linear, $\delta B/B \sim 1$, but $B_\perp$ continues to grow exponentially until its rms value is approximately twice the zeroth-order field, whereas the growth continues but at a decreased rate. Continued growth is not surprising as the $j \wedge B$ driving force continues to inject energy into the turbulence. A number of non-linear effects are important, particularly when the spatial displacement of fluid elements reaches $\sim 1/k$. The wave modes were seeded with $k_0$ in all directions, so the transverse coherence length of the modes is $\sim 1/k$. This means that fluid elements set in motion in the $(x, y)$ plane collide with fluid elements moving in other directions in the same plane. This causes fluid element motions to be changed and deflected causing motion in the $z$-direction, which in turn begins to amplify $B_\perp$. Fig. 3 shows that $U_{\text{LC}}$ exceeds $U_{\text{thermal}}$ after the instability becomes non-linear, so weak shocks develop within the plasma and shock viscosity cause $U_{\text{thermal}}$ to increase. For $8 < t < 18$, the thermal, kinetic and magnetic field energies grow more or less in equipartition. At later times the magnetic energy is the smallest of the three and the thermal energy is significantly larger. Although non-linear effects reduce the growth rate, the field continues to grow until $t \approx 18$. The rms field reaches 30 times its initial value, and the maximum field reaches 100 times the initial value.

Fig. 4 plots slices in $z$ of the magnitude of the magnetic field at various times. In particular, during the early stages of non-linear growth, the magnetic field is concentrated in small areas separated by large areas of relatively low field. As the calculation progresses, the characteristic scalelength increases as expected from the $k^{1/2}$ dependence of the growth rate for $k < k_{\text{max}}$. Longer-wavelength modes grow more slowly, but catch up with the short-wavelength modes as the non-linear growth at short-wavelength falls below the exponential linear growth.

9 NON-LINEAR SATURATION

Although the magnetic fluctuations grow well beyond $\delta B/B \sim 1$, wave growth still eventually saturates because of the growing importance of the tension in the magnetic field lines. This effect is already evident in the linear dispersion relation. In the linear dispersion relation, the term $k^2 v_A^2 \zeta (\propto k^2 B_j^2)$, which represents field-line tension, opposes wave growth, which is driven by the term $\zeta v_i^2 k/r_{gl}$. As the amplitude of the turbulence increases beyond $\delta B/B \sim 1$, the Alfvén speed increases, and the term opposing growth grows more rapidly than the driving term. In practical terms this means that the saturation in the magnetic field lines becomes sufficiently strong to oppose the $j \wedge B$ force. This transition occurs when the current $|\nabla \wedge B|/\mu_0 k B_i$ becomes comparable with the CR current $j_1$. This implies that magnetic fluctuations on a scale $L \sim 1/k$ saturate at a value $B \sim \mu_0 j_1/k$. The fastest growing modes, occurring close to the wavenumber boundary between regimes II and III, saturate at a relatively low amplitude, whereas longer-wavelength modes, which grow more slowly, are able to continue growing to a larger amplitude. The range of wavenumbers in our non-linear simulation is not sufficient for this effect to be readily quantified, but the results show a clear lengthening of the characteristic scalelength as the turbulence develops. The longest-wavelength in the simulation has $k = \pi/32$, for which the estimated saturated magnetic field $\mu_0 j_1/k$ is $\sim 26$ in our dimensionless units. Fig. 3 shows evidence of saturation at around this value at $t \approx 18$, and the magnetic field increased very little after this time even though the calculation was continued to $t = 40$.

In a real plasma, the smallest wavenumber is determined by the size of the system, or, more likely, by the intervention of kinetic effects at the boundary $kr_{gl} \sim 1$ between regimes I and II.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Evolution of magnetic field and energy densities.}
\end{figure}
$r_{\parallel}$ decreases as the magnetic field increases non-linearly, thereby effectively moving the boundary between regimes I and II to higher $k$. If the maximum scalelength for wave growth is determined by $r_{\parallel}$, the saturation magnetic field can be estimated to be $B \sim \mu_0 j_\parallel r_{\parallel} \sim \mu_0 j_\parallel p_1/eB$ with a scalelength $k^{-1} \sim r_{\parallel}$. The corresponding magnetic energy density is $B^2/2\mu_0 \sim p_1 j_\parallel/2c \sim \rho v^2/2 \sim v_s U/c \ln (p_2/m_\rho c)$. For our typical parameters, the estimated saturation field driven by CR at $10^{15}$ eV in the upstream plasma is $100 \mu G$ with a scalelength $3 \times 10^{14}$ m, and the magnetic energy density is $4 \times 10^{-11}$ J m$^{-3}$. A favourable aspect of this saturation process is that the dominant scale-length automatically becomes equal to the Larmor radius $r_{\parallel}$ as required for Bohm diffusion.

This estimated saturation field is that driven only by the highest energy CR. Magnetic field on smaller scales will be generated by lower energy CR in the plasma closer to the shock. Immediately upstream of the shock the total turbulent magnetic energy density will be an integral over the energy density on all scalelengths. If we speculate that the energy density in each unit bandwidth ($\Delta k/k = 1$) is $v_s U/c \ln (p_2/m_\rho c)$ at all $k$, then the total saturated magnetic energy density is

$$\frac{B_{\text{sat}}^2}{2\mu_0} \sim \frac{v_s U}{2 c}.$$

This estimate for the total saturation magnetic field needs the support of non-linear simulations in which the CR are treated kinetically over a large range of scalelengths, but if our argument is correct, it shows that a large saturation magnetic field is favoured by a large shock velocity and a large upstream density as $B_{\text{sat}}^2 \propto \rho v^3$. Once again, this points to very young SNR as the site for CR acceleration to high energy. Similarly, if the shock velocity is significantly lower than our typical value of $10^7$ m s$^{-1}$, saturation effects may stop the generation of magnetic field much beyond the typical interstellar value.

### 10 General Application

The process of magnetic field amplification described in this paper may be applied to other sites of diffusive shock acceleration, and also wherever there is strong CR streaming even if the CR are accelerated elsewhere. Equation (19), which gives the condition for waves to be strongly driven and regime II to exist, can be rewritten as the condition that $j_\parallel > B_0/\mu_0 r_{\parallel}$ where $r_{\parallel} = p_1/eB_1$ and $p_1$ is the momentum of the lowest energy CR driving the waves. Furthermore, using $j_\parallel = (v_s/c) e (U_\alpha/p_1)/\ln (p_2/m_\rho c)$ as given in Section 3, the condition can be rewritten as $v_s U_\alpha > cB_1^2/\mu_0 \ln (p_2/m_\rho c)$. Since $v_s$ is the CR drift velocity relative to the upstream plasma, the CR energy flux is $I_\alpha = v_s U_\alpha$, and the condition takes the form

$$I_\alpha > \ln(\epsilon_{\text{max}}/m_\rho c^2) \frac{cB_1^2}{\mu_0},$$

where $\epsilon_{\text{max}} = c p_2$ is the maximum CR energy. This condition may be used to assess the likelihood of magnetic field amplification by strong CR-streaming in other scenarios, although the factor $\ln(\epsilon_{\text{max}}/m_\rho c^2)$, which arises from the shape of the CR distribution, will vary from case to case.

To within factors of $\ln(\epsilon_{\text{max}}/m_\rho c^2)$, equation (28) shows that the magnetic field is amplified to a level at which equation (29) is marginally satisfied. This expresses a form of equipartition, but of energy fluxes rather than energy densities.

### 11 Conclusions

Previous work has assumed that the magnetic fluctuations that scatter CR during diffusive shock acceleration are Alfvén waves with a wavelength in resonance with the CR Larmor radius. We have shown here that, during acceleration at the outer shocks of SNR, the magnetic fluctuations are more usually strongly driven, non-resonant, nearly purely growing modes at shorter wavelengths. The modes have a circular polarization contrary to that of the Larmor rotation.

---

**Figure 4.** Magnitude of the magnetic field in the $(x, y)$ plane; slices at $z = 0$. The grey-scale minima (black) and maxima (white) at each time as bracketed pairs (minimum, maximum) are: $(0.81, 1.22)$ at $t = 0$, $(0.69, 1.35)$ at $t = 2$, $(0.40, 2.30)$ at $t = 4$, $(0.20, 12.01)$ at $t = 6$, $(0.09, 39.88)$ at $t = 10$, $(0.24, 79.72)$ at $t = 20$.


---

Downloaded from https://academic.oup.com/mnras/article-abstract/353/2/550/1108803/Turbulent-amplification-of-magnetic-field-and
of the CR flowing away from the shock. A non-linear MHD simulation indicates that CR-excited turbulence can amplify the magnetic field well beyond its initial seed value. Field amplification saturates when \( \nabla \wedge U B / \mu_0 \) becomes comparable with the electric current carried by the CR.

The linear growth rate has a strong positive dependence on the shock velocity and a weaker, although maybe important, dependence on the plasma density. The condition that the growth time should be shorter than the age of the SNR places a tight constraint on CR acceleration to high energy. It appears that acceleration to the spectral knee at 10\(^{15}\) eV normally ceases as SNR enter the Sedov phase. It suggests that acceleration beyond the knee may be possible in very young SNR expanding at high velocity into dense circumstellar material. Similar conclusions are reached from our approximate estimates of the saturation value of the amplified magnetic field.

The process of magnetic field amplification described here may find application in other environments in which there are strong CR fluxes.

ACKNOWLEDGMENTS

I gratefully acknowledge Professors HJ Völk and JG Kirk and their colleagues at the Max Planck Institut für Kernphysik at Heidelberg for their hospitality during 2004 February and March and for their thoughtful encouragement and questioning.

REFERENCES

Völk H.J., 1984, Proc. 19th Rencontres de Moriond Astrophysics Meeting
Wentzel D.G., 1974, ARA&A, 12, 71

This paper has been typeset from a \( \TeX / \LaTeX \) file prepared by the author.