The multiple gene duplication problem revisited
Mukul S. Bansal and Oliver Eulenstein*
Department of Computer Science, Iowa State University, Ames, IA 50011, USA

ABSTRACT
Motivation: Deciphering the location of gene duplications and multiple gene duplication episodes on the Tree of Life is fundamental to understanding the way gene families and genomes evolve. The multiple gene duplication problem provides a framework for placing gene duplication events onto nodes of a given species tree, and detecting episodes of multiple gene duplication. One version of the multiple gene duplication problem was defined by Guigó et al. in 1996. Several heuristic solutions have since been proposed for this problem, but no exact algorithms were known.

Results: In this article we solve this longstanding open problem by providing the first exact and efficient solution. We also demonstrate the improvement offered by our algorithm over the best heuristic approaches, by applying it to several simulated as well as empirical datasets.

Contact: oeulenst@cs.iastate.edu

1 INTRODUCTION
Gene duplication is known to have played a major role in the evolution of almost all life on Earth. For example, analyses of genomic data from numerous plants such as grasses (Guyot and Keller, 2004; Paterson et al., 2004; Schlüter et al., 2004; Vandepoele et al., 2003; Wang et al., 2005; Yu et al., 2005), Arabidopsis or other Brassicaceae (Blanc et al., 2003; Bowers et al., 2003; Cannon et al., 2006; Schlüter et al., 2004; Schranz and Mitchell-Olds, 2006; Simillion et al., 2002; Vision et al., 2000), poplar (Sterck et al., 2005), cotton (Blanc and Wolfe, 2004; Rong et al., 2004) and Physcomitrella (Rensing et al., 2007), among others, have revealed evidence of ancient gene duplications. Complex evolutionary processes such as gene duplication and loss, recombination and horizontal gene transfer generate gene trees that differ from species trees. One approach to infer gene duplications is to reconcile the conflicting gene trees with respect to a trusted species tree (Bonizzoni et al., 2005; Chen et al., 2006; Goodman et al., 1979; Górecki and Tiuryn, 2004; Guigó et al., 1996; Mirkin et al., 1995; Page, 1994; Zhang, 1997). Existing techniques that reconcile gene trees to species trees can identify gene duplications but cannot, in general, accurately locate them on the species tree. Other approaches make use of sequence similarity to reconstruct the underlying evolutionary history of genes (see, for example, Wapinski et al., 2007a,b). Probabilistic models for gene/species tree reconciliation as well as gene sequence evolution have also been developed (Arvestad et al., 2003, 2004).

There is evidence that many gene duplications are part of larger multiple gene duplication episodes, during which a large portion of an organism’s genome is duplicated. In fact, it is known that the entire genomes of numerous species (many eukaryotes for example) have been entirely duplicated one or more times. However, the rapid gene loss and gene rearrangements that follow a multiple gene duplication episode can make them difficult or even impossible to detect; and there is often no clear consensus on the number of ancient multiple gene duplication episodes or their precise location in evolutionary history.

Deciphering the location of gene duplications and multiple gene duplication episodes on the Tree of Life is a fundamental problem in understanding the way gene families and genomes evolve. The multiple gene duplication problem provides a framework for (i) mapping gene duplication events onto a given species tree and (ii) inferring and locating multiple gene duplication episodes. Informally, the multiple gene duplication problem is to assign duplication events to nodes in a species tree in such a way that the total number of multiple gene duplication episodes (or simply episodes in short) is minimized. This allows for a ‘parsimonious’ reconciliation of the gene trees with respect to a trusted species tree, which helps to locate gene duplications, as well as to detect multiple gene duplication episodes, more accurately.

Guigó et al. (1996) were the first to address a comprehensive phylogenetic problem that maps duplication events from a collection of rooted, binary gene trees onto a larger rooted binary species tree. They presented a heuristic that could be used to trace back the identified gene duplications to a few multiple gene duplication episodes. Later on, this heuristic approach was refined and restated in more formal terms, and used to study multiple gene duplication episodes in vertebrates by Page and Cotton (2002).

Essentially, this heuristic approach sought to solve the multiple gene duplication problem of Guigó et al. by solving instead a similar problem which we refer to as the ‘episode clustering’ problem. An alternative version of the multiple gene duplication problem was introduced by Fellows et al. (1998b) which they proved to be intrinsically difficult. Hence, we direct our focus to the work of Guigó et al. and Page and Cotton. The episode clustering problem determines duplication events using the Gene Duplication (GD) model from Goodman et al. (1979). Each duplication can be placed on any species on a path between the two (not necessarily distinct) most recent species that could have contained the duplication and its parent, respectively. In case the parent does not exist, the path runs between the most recent species for the duplication and the root of the species tree. An example is depicted in Figure 1. The duplications in gene tree $G$ are represented by the three bold vertices. Associated with each bold vertex is its path represented by an interval. For example, the interval $[5,3]$ represents the path $(5,4,3)$ in the species trees $S$. Let $g$ denote the node corresponding to the interval $[5,3]$. Species $5$ is the most recent species that could have contained $g$ and the parent of species $3$, i.e. $2$, is the most recent.
species that could have contained the parent of \( g \). The Episode Clustering (EC) problem is, given a collection of gene trees and a species tree, find a minimum number of locations in the species tree where all duplications in the gene trees can be placed. For example, all three duplications in Figure 1 can be placed on species nodes 2 and 3. 

The EC problem itself has a long and interesting history. Guigó et al. (1996) presented a heuristic approach to solve this problem. This heuristic was somewhat imprecise, and there were hints, but no formal algorithm, on how to deal with certain optimization steps. Page and Cotton (2002) observed that the EC problem can be efficiently and cleanly reduced to the Set Cover problem. Recently, Burleigh et al. (2008) gave a formal algorithm, on how to deal with certain optimization steps. Page and Cotton (2002) also implemented the algorithm and demonstrated the improvement it offers over the best heuristic approaches experimentally by applying it to several simulated as well as empirical datasets (see Section 4).

2 BASIC DEFINITIONS, NOTATION AND PRELIMINARIES

In this section we first introduce basic definitions and notation that we shall use and then define the preliminaries required for this work.

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Fig. 1. A gene tree \( G \) and a comparable species tree \( S \) is depicted. The bold vertices in \( G \) are duplications and their intervals represent their allowed locations in the species tree \( S \). In this section we first introduce basic definitions and notation that we shall use and then define the preliminaries required for this work. The EC problem itself has a long and interesting history. Guigó et al. (1996) presented a heuristic approach to solve this problem. This heuristic was somewhat imprecise, and there were hints, but no formal algorithm, on how to deal with certain optimization steps. Page and Cotton (2002) observed that the EC problem can be efficiently and cleanly reduced to the Set Cover problem. Recently, Burleigh et al. (2008) gave a formal algorithm, on how to deal with certain optimization steps. Page and Cotton (2002) also implemented the algorithm and demonstrated the improvement it offers over the best heuristic approaches experimentally by applying it to several simulated as well as empirical datasets (see Section 4).

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2.1 Basic definitions and notation

A tree \( T \) is a connected graph with no cycles, consisting of a node set \( V(T) \) and an edge set \( E(T) \). \( T \) is rooted if it has exactly one distinguished node called the root which we denote by \( \text{Ro}(T) \). Let \( T \) be a rooted tree. We define \( \leq_T \) to be the partial order on \( V(T) \) where \( x \leq_T y \) if \( y \) is a node on the path between \( \text{Ro}(T) \) and \( x \). The set of minima under \( \leq_T \) is denoted by \( \text{Le}(T) \) and its elements are called leaves. If \( \{x,y\} \in E(T) \) and \( x \leq_T y \) then we call \( y \) the parent of \( x \) denoted by \( \text{Pa}_T(x) \) and we call \( x \) a child of \( y \). The set of all children of \( y \) is denoted by \( \text{Ch}_T(y) \). If two nodes in \( T \) have the same parent, they are called siblings. The least common ancestor of a non-empty subset \( L \subseteq V(T) \), denoted as \( \text{lca}_T(L) \), is the unique smallest upper bound of \( L \) under \( \leq_T \). A subtree of \( T \) rooted at node \( y \in V(T) \), denoted by \( T_y \), is the tree induced by \( \{x \in V(T) \mid x \leq_T y \} \). The height of \( T \), denoted by \( h(T) \), is the number of nodes on a maximal length path from \( \text{Ro}(T) \) to a leaf node of \( T \). Thus, a rooted binary tree with three leaves has height three.

2.2 The ME problem

In this section we formally define the ME problem. The ME problem seeks to assign duplication events to nodes in a species tree, where each duplication event is associated with an interval in the species tree describing the locations where that duplication can be placed. The definition of duplication is based on the (GD) model introduced by Goodman et al. (1979). Guigó et al. (1996) extended this model and defined the associated intervals for each gene duplication. Here we only provide definitions necessary to state the ME problem.

The GD model is based on a gene and species tree from which gene duplications can be derived. A species tree is a tree that depicts the evolutionary relationships of a set of species. Given a gene family for a set of species, a gene tree is a tree that depicts the evolutionary relationships among the sequences encoding only that gene family in the given species. Thus the vertices in a gene tree represent genes. In order to compare a gene tree \( G \) with a species tree \( S \) a mapping from each gene \( g \in V(G) \) to the most recent species in \( S \) that could have contained \( g \) is required.

Definition 2.1 (LCA Mapping). A leaf-mapping \( \xi_{G,S} : V(G) \rightarrow V(S) \) specifies, for each gene \( g \), the species from which it was sampled. The extension \( \xi_{G,S} : V(G) \rightarrow V(S) \) of \( \xi_{G,S} \) is the LCA mapping defined by \( \text{LCA}_{G,S}(x) = \text{lca}_G(G(x) \xi_{G,S}(x)) \).

Definition 2.2 (Comparability). The trees \( G \) and \( S \) are comparable if there exists a leaf-mapping \( \xi_{G,S} \). A set of gene trees \( G \) and \( S \) are comparable if each gene tree in \( G \) is comparable with \( S \).

Throughout the remainder of this article, \( G \) denotes a collection of input gene trees, \( S \) a comparable species tree, and \( G \) denotes an arbitrary gene tree in \( G \).

Note that mathematically speaking such a leaf-mapping always exists. However, in the current context, we are only concerned with biologically relevant leaf-mappings.
DEFINITION 2.3 (Duplication). A node \( v \in V(G) \) is a (gene)
duplication if \( M_{G,S}(v) = M_{G,S}(u) \) for some \( u \in Ch(v) \) and we define
\( \text{Dup}(G,S) = \{ g \in V(G) : g \text{ is a duplication} \} \).

DEFINITION 2.4 (Interval \( I(g) \)). For every \( g \in \text{Dup}(G,S) \), the
interval \( I(g) \) is defined to be:

\[ I(g) = \left[ \text{LCA}(g), \text{Ro}(g) \right] \]

DEFINITION 2.5 (Valid Mapping). A mapping \( \mathcal{F}_{G,S} : V(G) \rightarrow V(S) \)
is called valid if for each \( g \in G \),
\[ \mathcal{F}_{G,S}(g) = \begin{cases} s, & \text{where } s \in I(g) \text{, if } g \in \text{Dup}(G,S), \\ M_{G,S}(g), & \text{otherwise}. \end{cases} \]

Note: (i) \( \mathcal{F} \) is used to denote the mapping \( \cup_{G \in \mathcal{G}} \mathcal{F}_{G,S} \), and we say \( \mathcal{F} \) is valid if \( \mathcal{F} \) is valid for each \( G \in \mathcal{G} \), (ii) given a mapping \( \mathcal{F} \),
and a node \( s \in V(S) \), we write \( \mathcal{F}^{-1}(s) \) to denote the set \( \{ g : \mathcal{F}(g) = s \} \)
and (iii) in the remainder of this article, we denote by \( \mathcal{F}_M \) the valid
mapping \( \cup_{G \in \mathcal{G}} M_{G,S} \).

DEFINITION 2.6 (H(\( F \), s)). Given \( \mathcal{G} \) and \( S \), a valid mapping \( \mathcal{F} \), and
a node \( s \in V(S) \), we define \( H(\mathcal{F}, s) \) to be the sub-graph of \( G \)
defined by the induced node set \( \mathcal{F}^{-1}(s) \).

Note that \( H(\mathcal{F}, s) \) must be a forest.

Throughout this article we abbreviate the term ‘multiple gene
duplication episode’ simply to ‘episode.’ Given any valid mapping
\( \mathcal{F} \), the following definition defines (i) the number of episodes,
\( \Delta(\mathcal{F}, s) \), at each node \( s \in V(S) \) and (ii) the total number of episodes
\( \Delta(\mathcal{F}) \). For the actual definition of an episode itself, we refer the
reader to Gagné et al. (1996).

DEFINITION 2.7 (\( \Delta(\mathcal{F}, s) \) and \( \Delta(\mathcal{F}) \)). Given a valid mapping \( \mathcal{F} \),
and a node \( s \in V(S) \), we denote by \( \Delta(\mathcal{F}, s) \) the number of episodes
at \( s \) caused by the mapping \( \mathcal{F} \). Then, \( \Delta(\mathcal{F}, s) = \max(T \in H(\mathcal{F}, s), \Delta(\mathcal{F}, T)) \).

DEFINITION 2.8 (\( \Delta_{\text{opt}} \)). \( \Delta_{\text{opt}} \) is the minimum valid mapping Hansen et al. (1986).

\( \mathcal{G} \) and \( S \) form the input for the ME problem. The output is a valid
mapping \( \mathcal{F}_{\text{opt}} : \cup_{G \in \mathcal{G}} V(G) \rightarrow V(S) \), such that \( \Delta(\mathcal{F}_{\text{opt}}) \) is minimized.

More formally,

PROBLEM 1. Minimum Episodes (ME)

Instance: A collection of gene trees \( \mathcal{G} \) and a comparable
species tree \( S \).

Find: A valid mapping \( \mathcal{F}_{\text{opt}} \) such that \( \Delta(\mathcal{F}_{\text{opt}}) = \Delta_{\text{opt}} \).

3 THE ME PROBLEM

It is not hard to see that the number of distinct valid mappings can be
extremely large (exponential in the size of the input). It is therefore
infeasible to solve the ME problem by considering all possible valid
mappings and then picking the one that causes the fewest episodes.

In this section we give a simple and extremely efficient algorithm to
solve the ME problem. The main idea of the algorithm is to traverse
the species tree \( S \) in post-order, and perform greedy optimization
steps at each node. As we shall prove, this leads us to a globally
optimal mapping.

In order to state the algorithm, we must first define a few terms.

DEFINITION 3.1 (Leading node). Let \( \mathcal{F} \) be a valid mapping and
let \( s \in V(S) \). Then, we say a node \( g \in \mathcal{F}^{-1}(s) \) is a leading node if and
only if \( g = \text{Ro}(T) \) where \( T \in H(\mathcal{F}, s) \) and \( h(T) = s \).

DEFINITION 3.2 (Free node). Given a valid mapping \( \mathcal{F} \), and
a node \( s \in V(S) \) such that \( s \neq \text{Ro}(S) \), a node \( g \in \mathcal{F}^{-1}(s) \) is called free
if and only if \( \text{Pa}(g) \in \mathcal{F}(g) \).

For convenience, we refer to each node \( s \in V(S) \) for which
\( \mathcal{F}^{-1}(s) \neq \emptyset \), as a relevant node. Also recall that \( \mathcal{F}_M \) denotes the
mapping \( \cup_{G \in \mathcal{G}} M_{G,S} \).

We begin by stating the intuitive idea behind our algorithm.
Consider any valid mapping \( \mathcal{F} : \cup_{G \in \mathcal{G}} V(G) \rightarrow V(S) \). Given any node
\( s \in V(S) \), let \( \mathcal{F}^* \) be a new mapping constructed from \( \mathcal{F} \) by moving
the mapping of all the free nodes in \( \mathcal{F}^{-1}(s) \) to \( \text{Pa}(s) \). Clearly, \( \mathcal{F}^* \)
must be a valid mapping. Now, if all the leading nodes in \( \mathcal{F}^{-1}(s) \) are free,
then we can show that \( \Delta(\mathcal{F}^*) \leq \Delta(\mathcal{F}) \). On the other hand,
if not all the leading nodes in \( \mathcal{F}^{-1}(s) \) are free, then we must have
\( \Delta(\mathcal{F}^*) > \Delta(\mathcal{F}) \). This simple observation forms the basis of our
greedy algorithm. If these greedy optimizations are carried out in
a particular order, then it can be shown that the resulting mapping
will be an optimal one.

We are now ready to state our algorithm. The algorithm starts
with the LCA mapping from the gene trees to the species tree,
and progressively modifies it so that when the algorithm terminates,
we have an optimal valid mapping. First, a valid mapping \( \mathcal{F} : \cup_{G \in \mathcal{G}} V(G) \rightarrow V(S) \)
is initialized such that \( \mathcal{F} = \mathcal{F}_M \). Next, we traverse \( S \) in post-order,
and at each node, say \( s \), we check if it is relevant and if all the leading
nodes in \( \mathcal{F}^{-1}(s) \) are free. If they are, then we modify the mapping \( \mathcal{F} \)
by changing the mapping of all the leading nodes in \( \mathcal{F}^{-1}(s) \) to \( \text{Pa}(s) \). It can be shown that when the
post-order traversal is finished, the mapping \( \mathcal{F} \) must be a solution
to the ME problem (see Theorem 3.1). This algorithm is described
more formally in Algorithm 1.

Algorithm 1 This algorithm solves the ME problem

Input: A set of gene trees \( \mathcal{G} \) and the species tree \( S \).

1: Initially all gene nodes map according to the LCA mapping.
2: Compute the interval \( I(g) \) for each gene tree node \( g \).
3: Define a mapping \( \mathcal{F}_M : \cup_{G \in \mathcal{G}} V(G) \rightarrow V(S) \) as follows:
\[ \mathcal{F}_M(g) = \begin{cases} \text{Pa}(g), & \text{if } g \text{ is a leading node in } \mathcal{F}_M^{-1}(s), \\ \mathcal{F}(g), & \text{otherwise}. \end{cases} \]

4: for each node \( s \) in a post-order traversal of \( S \) do
5: if \( s \) is a relevant node then
6: if all leading nodes in \( \mathcal{F}_M^{-1}(s) \) are free then
7: Define a mapping \( \mathcal{F} : \cup_{G \in \mathcal{G}} V(G) \rightarrow V(S) \) as follows:
\[ \mathcal{F}(g) = \begin{cases} \text{Pa}(g), & \text{if } g \text{ is a leading node in } \mathcal{F}^{-1}(s), \\ \mathcal{F}(g), & \text{otherwise}. \end{cases} \]

9: return \( \mathcal{F} \)
We denote the final mapping output by Algorithm 1 by $F_{\text{opt}}$. In the remainder of this section we first show that $F_{\text{opt}}$ is a valid mapping and $\Delta (F_{\text{opt}}) = \Delta_{\text{opt}}$, and then study the complexity of Algorithm 1.

**Lemma 3.1.** $F_{\text{opt}}$ is a valid mapping.

**Proof.** Algorithm 1 starts with the mapping $F = F_M$ which is valid by definition. During each iteration of the loop, the mapping $F$ may be modified according to Equation (1). However, Equation (1) only modifies the mapping of those nodes that are free, and hence produces a valid mapping. Therefore, each mapping produced by the algorithm, including the mapping $F_{\text{opt}}$, is valid. ■

**Lemma 3.2.** Let $F : \mathcal{V}(G) \rightarrow \mathcal{V}(S)$ be any valid mapping. Then we have the following:

1. If $x \in \mathcal{V}(S)$ is a relevant node under mapping $F_M$, then $\Delta (F_M, s) - 1 \leq \Delta (F, s) \leq \Delta (F_M, s)$.
2. If $x \in \mathcal{V}(S)$ is not a relevant node under mapping $F_M$, then $0 \leq \Delta (F, s) \leq 1$.

**Proof.** Part 1: Here $s$ is a relevant node i.e. $F^{-1}_M(s) \neq \emptyset$. Let $A$ denote the set of nodes that are present in $F^{-1}_M(s)$ but not in $F^{-1}(x)$ and $B$ denote the set of nodes that are present in $F^{-1}(x)$ but not in $F^{-1}_M(s)$. Observe that all the nodes in $A$ must be leading nodes in $F^{-1}_M(s)$. Relocating all the leading nodes in $F^{-1}_M(s)$ reduces $\Delta (F_M, s)$ by exactly 1. Therefore, relocating the nodes in $A$ reduces $\Delta (F_M, s)$ by at most 1. This proves that $\Delta (F_M, s) - 1 \leq \Delta (F, s)$.

Consider now the set $B$. Let $a$ and $b$ be two gene duplication nodes from some gene tree in $G$ such that one is an ancestor of the other, and $F_M(a) \neq F_M(b)$. Then, Definition 2.4 implies that $F(a) \neq F(b)$. Therefore, none of the nodes in set $B$ is an ancestor of another, and hence none of them is a leading node in $F^{-1}(s)$. This proves that $\Delta (F, s) \leq \Delta (F_M, s)$.

**Part 2:** In this case $F^{-1}_M(s) = \emptyset$, and therefore $B = F^{-1}(x)$. Following the argument from the previous paragraph, we can conclude that none of the nodes in set $B$ is an ancestor of another. This implies that $\Delta (F, s) \leq 1$. ■

The following three lemmas are required for the proof of Theorem 3.1.

**Lemma 3.3.** Let $F : \mathcal{V}(G) \rightarrow \mathcal{V}(S)$ be a valid mapping and $x \in \mathcal{V}(S)$ be a node such that $\Delta (F_{\text{opt}}, x) > \Delta (F, x)$. Then, $\Delta (F_{\text{opt}}, x) = 1$.

**Proof.** There are two possible cases: (i) $x$ is a relevant node under $F_M$ or (ii) $x$ is a relevant node under $F_{\text{opt}}$. We analyze these cases separately.

**Case (i):** In this case, by Part 2 of Lemma 3.2, we must have $\Delta (F, x) = 0$ and $\Delta (F_{\text{opt}}, x) = 1$.

**Case (ii):** If $\Delta (F_M, x) < 2$, then by Part 1 of Lemma 3.2 the result follows immediately; therefore, let us assume that $\Delta (F_M, x) \geq 2$.

Part 1 of Lemma 3.2 implies that we must have $\Delta (F, x) = \Delta (F_M, x) - 1$ and $\Delta (F_{\text{opt}}, x) = \Delta (F_M, x)$.

Let $A$ denote the set of nodes that are present in $F^{-1}_M(s)$ but not in $F^{-1}(x)$, and $B$ denote the set of nodes that are present in $F_{\text{opt}}(x)$ but not in $F^{-1}_M(s)$. All the nodes in $A$ must be leading nodes in $F^{-1}_M(s)$, and since these nodes are not present in $F^{-1}(x)$, all the nodes in $A$ must be free as well. Also, none of the nodes in $B$ can be a leading node in $F_{\text{opt}}(x)$ (see the proof of Part 1 of Lemma 3.2).

Therefore, all of the leading nodes in $F_{\text{opt}}(x)$ must be present in $A$, which implies that all the leading nodes in $F_{\text{opt}}(x)$ are free. Thus, during the execution of Algorithm 1, the mapping for these nodes would have been changed. This is a contradiction, and hence we cannot have $\Delta (F_M, x) \geq 2$. ■

**Lemma 3.4.** Let node $a$ be such that $F_M(a) < F_{\text{opt}}(a)$. If $F : \mathcal{V}(G) \rightarrow \mathcal{V}(S)$ is a valid mapping such that $F(a) = F_M(a)$, then $\Delta (F_{\text{opt}}, F_M(a)) = \Delta (F, F_M(a)) = 0$.

**Proof.** Since $F(a) < F_{\text{opt}}(a)$, $a$ must be a leading node in $F^{-1}_M(a)$. This implies that $\Delta (F_{\text{opt}}, F_M(a)) = \Delta (F_M(a), F_{\text{opt}}(a))$. Moreover, since $F_{\text{opt}}(a) \neq F_M(a)$, we have $\Delta (F_{\text{opt}}, F_M(a)) = \Delta (F_M(a), F_{\text{opt}}(a)) = 0$ (see Algorithm 1). The lemma follows. ■

**Lemma 3.5.** Let node $a$ be such that $F_M(a) = F_{\text{opt}}(a)$. If $F : \mathcal{V}(G) \rightarrow \mathcal{V}(S)$ is a valid mapping, then $F_{\text{opt}}(a) = F(a)$ for all $a \in \Gamma$.

**Proof.** Consider any node $x \in \Gamma$. There must exist some valid mapping $F_{\text{opt}}$, realized during the execution of Algorithm 1, for which $F(a) = x$. However, as the execution of Algorithm 1 progresses, the mapping of $a$ changes. This implies that $a$ must be a leading node in $F^{-1}(a)$. Observe that a could be a leading node in $F^{-1}(a)$ only if $\Delta (F, a) = 1$. Furthermore, for the mapping of $a$ to be changed, all the nodes in $F^{-1}(a)$ must be free, and therefore would not map to node $x$ when the algorithm terminates. Thus, $F_{\text{opt}}(a) = x$. ■

**Theorem 3.1.** Algorithm 1 solves the ME problem.

**Proof.** In Lemma 3.1 we have already established that $F_{\text{opt}}$ is a valid mapping. Therefore, to establish the correctness of our algorithm, it is sufficient to show that $\Delta (F_{\text{opt}}) = \Delta_{\text{opt}}$. Let us assume, for the sake of contradiction, that there exists some valid mapping $F$ for which $\Delta (F_{\text{opt}}) > \Delta (F)$. This implies that there must be at least one node $x \in \mathcal{V}(S)$ for which $\Delta (F_{\text{opt}}, x) > \Delta (F, x)$. We may assume, without any loss of generality, that $x$ has the following property: there does not exist any other node $t \in V(S)$ for which $\Delta (F_{\text{opt}}, t) > \Delta (F, t)$.

By Lemma 3.3 we know that node $x$ must be such that $\Delta (F_{\text{opt}}, x) = 1$. This implies that $\Delta (F, x) = 0$, i.e. $F^{-1}(x) = \emptyset$. We will now show that there exists at least one node $t \in \mathcal{V}(S)$ for which $\Delta (F_{\text{opt}}, t) < \Delta (F, t)$. This would imply that $\sum_{x \in \mathcal{V}(S)} \Delta (F_{\text{opt}}, t) \leq \sum_{x \in \mathcal{V}(S)} \Delta (F, t)$.

Let $A = F^{-1}_M(x)$; clearly $A \neq \emptyset$. We now have two possible scenarios, exactly one of which must be true: (i) $F(x) > x$ for each $g \in A$ or (ii) there exists some $g \in A$ for which $F(x) < x$. If case (i) were to hold, it would imply that all nodes in $A$ are leading and free, and therefore Algorithm 1 would have already moved their mappings to nodes that are proper ancestors of $s$. Hence, case (ii) is the only possible scenario.

So far we have shown that if there exists some valid mapping $F$ for which $\Delta (F_{\text{opt}}) > \Delta (F)$, then there must exist some node, say $a$, where $a \in A$ and $F(a) < x$. Clearly, $F_M(a) < F(a) < x$. This leads us to two possible cases, exactly one of which —

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We evaluated the efficacy and efficiency of our novel algorithm \( \text{FOMD} \) against the program \( \text{ApproxMGD} \) of Burleigh et al. (2008), which implements the currently best known approach to solve the ME problem.

Recall that the objective of the ME problem is to produce a mapping which defines the fewest number of episodes. The smaller the number of episodes, the more accurate the mapping. Therefore, for each dataset we compared the programs by measuring three values: (i) The number of episodes defined by the initial LCA mapping (i.e., the unoptimized value), (ii) the number of episodes defined by the mapping produced by \( \text{ApproxMGD} \) and (iii) the number of episodes defined by the mapping produced by \( \text{EXACTMGD} \). All analyses were performed on a 3 Ghz Intel Pentium 4 CPU based PC with Windows XP operating system. A run of \( \text{EXACTMGD} \) on each of these datasets terminated in less than 1s!

4.1 Simulated datasets

Our simulated datasets consist of 50 randomly generated gene trees, all with the same set of taxa, along with a randomly generated species tree. We generated four such datasets, each with a different number of taxa (50, 100, 200 and 400) in the input trees. \( \text{EXACTMGD} \) shows a significant reduction in the number of episodes as compared to \( \text{ApproxMGD} \) for each of the four datasets (Table 1).

4.2 Empirical datasets

For the empirical study we evaluated two datasets from the literature. The first dataset consists of 53 gene trees, each representing the evolutionary history of different gene families from a set of 16 eukaryotes. This set was assembled and evaluated for episodes by Guigó et al. (1996). Subsequently, this dataset was reused and its evaluation refined by Page and Cotton (2002). The second dataset, assembled and evaluated for episodes by Burleigh et al. (2008), consists of 85 gene trees from the Phytome comparative plant genome database and contains genes from 136 plant taxa.

4 EXPERIMENTAL RESULTS

We evaluated the efficacy and efficiency of our novel algorithm for the ME problem through comparative studies on simulated and empirical datasets. For this evaluation we implemented our algorithm in the program \( \text{EXACTMGD} \). We compared our program against the program \( \text{ApproxMGD} \) of Burleigh et al. (2008) that implements the currently best known approach to solve the ME problem.

Recall that the objective of the ME problem is to produce a mapping which defines the fewest number of episodes. The smaller the number of episodes, the more accurate the mapping. Therefore, for each dataset we compared the programs by measuring three values: (i) The number of episodes defined by the initial LCA mapping (i.e., the unoptimized value), (ii) the number of episodes defined by the mapping produced by \( \text{ApproxMGD} \) and (iii) the number of episodes defined by the mapping produced by \( \text{EXACTMGD} \). All analyses were performed on a 3 Ghz Intel Pentium 4 CPU based PC with Windows XP operating system. A run of \( \text{EXACTMGD} \) on each of these datasets terminated in less than 1s!

4.1 Simulated datasets

Our simulated datasets consist of 50 randomly generated gene trees, all with the same set of taxa, along with a randomly generated species tree. We generated four such datasets, each with a different number of taxa (50, 100, 200 and 400) in the input trees. \( \text{EXACTMGD} \) shows a significant reduction in the number of episodes as compared to \( \text{ApproxMGD} \) for each of the four datasets (Table 1).

4.2 Empirical datasets

For the empirical study we evaluated two datasets from the literature. The first dataset consists of 53 gene trees, each representing the evolutionary history of different gene families from a set of 16 eukaryotes. This set was assembled and evaluated for episodes by Guigó et al. (1996). Subsequently, this dataset was reused and its evaluation refined by Page and Cotton (2002). The second dataset, assembled and evaluated for episodes by Burleigh et al. (2008), consists of 85 gene trees from the Phytome comparative plant genome database and contains genes from 136 plant taxa.

4 EXPERIMENTAL RESULTS

We evaluated the efficacy and efficiency of our novel algorithm for the ME problem through comparative studies on simulated and empirical datasets. For this evaluation we implemented our algorithm in the program \( \text{EXACTMGD} \). We compared our program against the program \( \text{ApproxMGD} \) of Burleigh et al. (2008) that implements the currently best known approach to solve the ME problem.

Recall that the objective of the ME problem is to produce a mapping which defines the fewest number of episodes. The smaller

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**Table 1. Performance of \( \text{EXACTMGD} \) on simulated datasets**

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Unoptimized</th>
<th>ApproxMGD</th>
<th>ExactMGD</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 taxa</td>
<td>30</td>
<td>28</td>
<td>25</td>
</tr>
<tr>
<td>100 taxa</td>
<td>47</td>
<td>38</td>
<td>35</td>
</tr>
<tr>
<td>200 taxa</td>
<td>64</td>
<td>54</td>
<td>49</td>
</tr>
<tr>
<td>400 taxa</td>
<td>65</td>
<td>45</td>
<td>40</td>
</tr>
</tbody>
</table>

**Table 2. Performance of \( \text{EXACTMGD} \) on empirical datasets**

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Unoptimized</th>
<th>ApproxMGD</th>
<th>ExactMGD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guigó et al. (1996)</td>
<td>9</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Burleigh et al. (2008)</td>
<td>1180</td>
<td>1152</td>
<td>1042</td>
</tr>
</tbody>
</table>

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Our randomly generated trees have a random (binary) topology and a random assignment of leaf labels.
algorithm are significantly more optimal compared to the mappings produced by the best current approaches. This leads to more accurate inference of the location of gene and genome duplications on the Tree of Life.

5 OUTLOOK AND CONCLUSION

In this article we have provided the first exact and efficient algorithm for a longstanding open problem. Traditionally, the multiple episodes problem has been used to infer the location of episodes of multiple gene duplication on a given species tree. Our new algorithm allows this to be done far more accurately. But another interesting and important fallout of our algorithm is that it may also allow us to infer the ‘correct’ species trees. Consider the problem of constructing a species tree from conflicting gene trees based on the gene duplication optimality criteria. The gene duplication problem is to find for a given set of gene trees a corresponding species tree with the minimum reconciliation cost (Fellows et al., 1998a; Hallett and Lagergren, 2000; Ma et al., 2000; Stege, 1999). However, since these gene duplication events are usually a part of larger multiple gene duplication episodes, it might be more helpful if we can infer species trees directly based on the multiple gene duplication optimality criteria (see also Fellows et al., 1998b). Our algorithm offers the first practical and effective means to do so. The idea is to use a local search based hill climbing heuristic to traverse through the search space of possible species trees. Our algorithm can be used to compute the number of episodes induced by each candidate species tree during the local search steps. The lower the number of episodes, the better the species tree.

In addition, it would be interesting to extend the multiple GD model of Guigó et al. (1996) by relaxing the constraints on the possible locations of gene duplications on the species tree.

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REFERENCES


