This is a portion of the eBook at doi:10.7551/mitpress/7728.001.0001

2
Easy cases

A correct solution works in all cases, including the easy ones. This maxim underlies the second tool—the method of easy cases. It will help us guess integrals, deduce volumes, and solve exacting differential equations.

2.1 Gaussian integral revisited

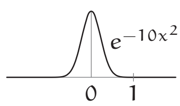
As the first application, let's revisit the Gaussian integral from Section 1.3,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx. \tag{2.1}$$

▶ *Is the integral* $\sqrt{\pi\alpha}$ *or* $\sqrt{\pi/\alpha}$*?*

The correct choice must work for all $\alpha \geqslant 0$. At this range's endpoints ($\alpha = \infty$ and $\alpha = 0$), the integral is easy to evaluate.

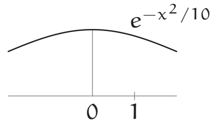
▶ *What is the integral when* $\alpha = \infty$*?*



As the first easy case, increase α to ∞. Then $-\alpha x^2$ becomes very negative, even when x is tiny. The exponential of a large negative number is tiny, so the bell curve narrows to a sliver, and its area shrinks to zero. Therefore, as $\alpha \to \infty$ the integral shrinks to zero. This result refutes the option

$\sqrt{\pi\alpha}$, which is infinite when $\alpha = \infty$; and it supports the option $\sqrt{\pi/\alpha}$, which is zero when $\alpha = \infty$.

What is the integral when $\alpha = 0$?



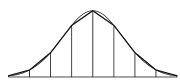
In the $\alpha = 0$ extreme, the bell curve flattens into a horizontal line with unit height. Its area, integrated over the infinite range, is infinite. This result refutes the $\sqrt{\pi\alpha}$ option, which is zero when $\alpha = 0$; and it supports the $\sqrt{\pi/\alpha}$ option, which is infinity when $\alpha = 0$. Thus the $\sqrt{\pi\alpha}$ option fails both easy-cases tests, and the $\sqrt{\pi/\alpha}$ option passes both easy-cases tests.

If these two options were the only options, we would choose $\sqrt{\pi/\alpha}$. However, if a third option were $\sqrt{2/\alpha}$, how could you decide between it and $\sqrt{\pi/\alpha}$? Both options pass both easy-cases tests; they also have identical dimensions. The choice looks difficult.

To choose, try a third easy case: $\alpha = 1$. Then the integral simplifies to

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx. \tag{2.2}$$

This classic integral can be evaluated in closed form by using polar coordinates, but that method also requires a trick with few other applications (textbooks on multivariable calculus give the gory details). A less elegant but more general approach is to evaluate the integral numerically and to use the approximate value to guess the closed form.



Therefore, replace the smooth curve e^{-x^2} with a curve having n line segments. This piecewise-linear approximation turns the area into a sum of n trapezoids. As n approaches infinity, the area of the trapezoids more and more closely approaches the area under the smooth curve.

The table gives the area under the curve in the range $x = -10 \ldots 10$, after dividing the curve into n line segments. The areas settle onto a stable value, and it looks familiar. It begins with 1.7, which might arise from $\sqrt{3}$. However, it continues as 1.77, which is too large to be $\sqrt{3}$. Fortunately, π is slightly larger than 3, so the area might be converging to $\sqrt{\pi}$.

n	Area
10	2.07326300569564
20	1.77263720482665
30	1.77245385170978
40	1.77245385090552
50	1.77245385090552

Let's check by comparing the squared area against π:

$$\begin{aligned} 1.77245385090552^2 &\approx 3.14159265358980, \\ \pi &\approx 3.14159265358979. \end{aligned} \tag{2.3}$$

The close match suggests that the $\alpha = 1$ Gaussian integral is indeed $\sqrt{\pi}$:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{2.4}$$

Therefore the general Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx \tag{2.5}$$

must reduce to $\sqrt{\pi}$ when $\alpha = 1$. It must also behave correctly in the other two easy cases $\alpha = 0$ and $\alpha = \infty$.

Among the three choices $\sqrt{2/\alpha}$, $\sqrt{\pi/\alpha}$, and $\sqrt{\pi\alpha}$, only $\sqrt{\pi/\alpha}$ passes all three tests $\alpha = 0$, 1, and ∞. Therefore,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}. \tag{2.6}$$

Easy cases are not the only way to judge these choices. Dimensional analysis, for example, can also restrict the possibilities (Section 1.3). It even eliminates choices like $\sqrt{\pi}/\alpha$ that pass all three easy-cases tests. However, easy cases are, by design, simple. They do not require us to invent or deduce dimensions for x, α, and dx (the extensive analysis of Section 1.3). Easy cases, unlike dimensional analysis, can also eliminate choices like $\sqrt{2/\alpha}$ with correct dimensions. Each tool has its strengths.

Problem 2.1 Testing several alternatives

For the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx, \tag{2.7}$$

use the three easy-cases tests to evaluate the following candidates for its value.

(a) $\sqrt{\pi}/\alpha$ (b) $1 + (\sqrt{\pi} - 1)/\alpha$ (c) $1/\alpha^2 + (\sqrt{\pi} - 1)/\alpha$.

Problem 2.2 Plausible, incorrect alternative

Is there an alternative to $\sqrt{\pi/\alpha}$ that has valid dimensions and passes the three easy-cases tests?

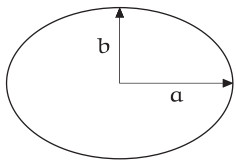
Problem 2.3 Guessing a closed form

Use a change of variable to show that

$$\int_0^\infty \frac{dx}{1+x^2} = 2\int_0^1 \frac{dx}{1+x^2}. \tag{2.8}$$

The second integral has a finite integration range, so it is easier than the first integral to evaluate numerically. Estimate the second integral using the trapezoid approximation and a computer or programmable calculator. Then guess a closed form for the first integral.

2.2 Plane geometry: The area of an ellipse



The second application of easy cases is from plane geometry: the area of an ellipse. This ellipse has semimajor axis a and semiminor axis b. For its area A consider the following candidates:

(a) ab^2 (b) $a^2 + b^2$ (c) a^3/b (d) $2ab$ (e) πab.

▶ *What are the merits or drawbacks of each candidate?*

The candidate $A = ab^2$ has dimensions of L^3, whereas an area must have dimensions of L^2. Thus ab^2 must be wrong.

The candidate $A = a^2 + b^2$ has correct dimensions (as do the remaining candidates), so the next tests are the easy cases of the radii a and b. For a, the low extreme $a = 0$ produces an infinitesimally thin ellipse with zero area. However, when $a = 0$ the candidate $A = a^2 + b^2$ reduces to $A = b^2$ rather than to 0; so $a^2 + b^2$ fails the $a = 0$ test.

The candidate $A = a^3/b$ correctly predicts zero area when $a = 0$. Because $a = 0$ was a useful easy case, and the axis labels a and b are almost interchangeable, its symmetric counterpart $b = 0$ should also be a useful easy case. It too produces an infinitesimally thin ellipse with zero area; alas, the candidate a^3/b predicts an infinite area, so it fails the $b = 0$ test. Two candidates remain.

The candidate $A = 2ab$ shows promise. When $a = 0$ or $b = 0$, the actual and predicted areas are zero, so $A = 2ab$ passes both easy-cases tests. Further testing requires the third easy case: $a = b$. Then the ellipse becomes a circle with radius a and area πa^2. The candidate $2ab$, however, reduces to $A = 2a^2$, so it fails the $a = b$ test.

The candidate $A = \pi ab$ passes all three tests: $a = 0$, $b = 0$, and $a = b$. With each passing test, our confidence in the candidate increases; and πab is indeed the correct area (Problem 2.4).

Problem 2.4 Area by calculus
Use integration to show that $A = \pi ab$.

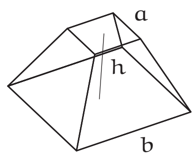
Problem 2.5 Inventing a passing candidate
Can you invent a second candidate for the area that has correct dimensions and passes the $a = 0$, $b = 0$, and $a = b$ tests?

Problem 2.6 Generalization
Guess the volume of an ellipsoid with principal radii a, b, and c.

2.3 Solid geometry: The volume of a truncated pyramid

The Gaussian-integral example (Section 2.1) and the ellipse-area example (Section 2.2) showed easy cases as a method of analysis: for checking whether formulas are correct. The next level of sophistication is to use easy cases as a method of synthesis: for constructing formulas.

As an example, take a pyramid with a square base and slice a piece from its top using a knife parallel to the base. This truncated pyramid (called the frustum) has a square base and square top parallel to the base. Let h be its vertical height, b be the side length of its base, and a be the side length of its top.



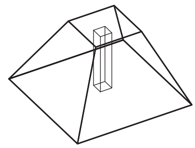
▶ *What is the volume of the truncated pyramid?*

Let's synthesize the formula for the volume. It is a function of the three lengths h, a, and b. These lengths split into two kinds: height and base lengths. For example, flipping the solid on its head interchanges the meanings of a and b but preserves h; and no simple operation interchanges height with a or b. Thus the volume probably has two factors, each containing a length or lengths of only one kind:

$$V(h, a, b) = f(h) \times g(a, b). \tag{2.9}$$

Proportional reasoning will determine f; a bit of dimensional reasoning and a lot of easy-cases reasoning will determine g.

▶ *What is* f*: How should the volume depend on the height?*



To find f, use a proportional-reasoning thought experiment. Chop the solid into vertical slivers, each like an oil-drilling core; then imagine doubling h. This change doubles the volume of each sliver and therefore doubles the whole volume V. Thus $f \sim h$ and $V \propto h$:

$$V = h \times g(a, b). \tag{2.10}$$

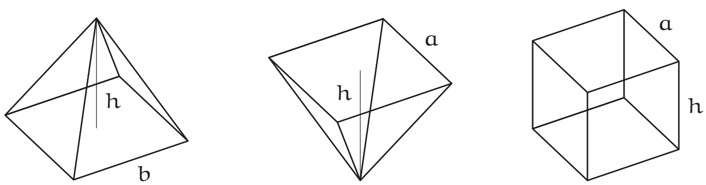
▶ *What is* g*: How should the volume depend on* a *and* b*?*

Because V has dimensions of L^3, the function $g(a, b)$ has dimensions of L^2. That constraint is all that dimensional analysis can say. Further constraints are needed to synthesize g, and these constraints are provided by the method of easy cases.

2.3.1 Easy cases

▶ *What are the easy cases of* a *and* b*?*

The easiest case is the extreme case $a = 0$ (an ordinary pyramid). The symmetry between a and b suggests two further easy cases, namely $a = b$ and the extreme case $b = 0$. The easy cases are then threefold:



$a = 0$ $b = 0$ $a = b$

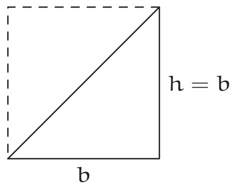
When $a = 0$, the solid is an ordinary pyramid, and g is a function only of the base side length b. Because g has dimensions of L^2, the only possibility for g is $g \sim b^2$; in addition, $V \propto h$; so, $V \sim hb^2$. When $b = 0$, the solid is an upside-down version of the $b = 0$ pyramid and therefore has volume $V \sim ha^2$. When $a = b$, the solid is a rectangular prism having volume $V = ha^2$ (or hb^2).

▶ *Is there a volume formula that satisfies the three easy-cases constraints?*

The $a = 0$ and $b = 0$ constraints are satisfied by the symmetric sum $V \sim h(a^2 + b^2)$. If the missing dimensionless constant is 1/2, making $V = h(a^2 + b^2)/2$, then the volume also satisfies the $a = b$ constraint, and the volume of an ordinary pyramid ($a = 0$) would be $hb^2/2$.

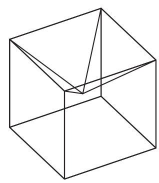
▶ *When $a = 0$, is the prediction $V = hb^2/2$ correct?*

Testing the prediction requires finding the exact dimensionless constant in $V \sim hb^2$. This task looks like a calculus problem: Slice a pyramid into thin horizontal sections and add (integrate) their volumes. However, a simple alternative is to apply easy cases again.



The easy case is easier to construct after we solve a similar but simpler problem: to find the area of a triangle with base b and height h. The area satisfies $A \sim hb$, but what is the dimensionless constant? To find it, choose b and h to make an easy triangle: a right triangle with $h = b$. Two such triangles make an easy shape: a square with area b^2. Thus each right triangle has area $A = b^2/2$; the dimensionless constant is 1/2. Now extend this reasoning to three dimensions—find an ordinary pyramid (with a square base) that combines with itself to make an easy solid.

▶ *What is the easy solid?*



A convenient solid is suggested by the pyramid's square base: Perhaps each base is one face of a cube. The cube then requires six pyramids whose tips meet in the center of the cube; thus the pyramids have the aspect ratio $h = b/2$. For numerical simplicity, let's meet this condition with $b = 2$ and $h = 1$.

Six such pyramids form a cube with volume $b^3 = 8$, so the volume of one pyramid is 4/3. Because each pyramid has volume $V \sim hb^2$, and $hb^2 = 4$ for these pyramids, the dimensionless constant in $V \sim hb^2$ must be 1/3. The volume of an ordinary pyramid (a pyramid with $a = 0$) is therefore $V = hb^2/3$.

Problem 2.7 Triangular base
Guess the volume of a pyramid with height h and a triangular base of area A. Assume that the top vertex lies directly over the centroid of the base. Then try Problem 2.8.

Problem 2.8 Vertex location

The six pyramids do not make a cube unless each pyramid's top vertex lies directly above the center of the base. Thus the result $V = hb^2/3$ might apply only with this restriction. If instead the top vertex lies above one of the base vertices, what is the volume?

The prediction from the first three easy-cases tests was $V = hb^2/2$ (when $a = 0$), whereas the further easy case $h = b/2$ alongside $a = 0$ just showed that $V = hb^2/3$. The two methods are making contradictory predictions.

How can this contradiction be resolved?

The contradiction must have snuck in during one of the reasoning steps. To find the culprit, revisit each step in turn. The argument for $V \propto h$ looks correct. The three easy-case requirements—that $V \sim hb^2$ when $a = 0$, that $V \sim ha^2$ when $b = 0$, and that $V = h(a^2 + b^2)/2$ when $a = b$—also look correct. The mistake was leaping from these constraints to the prediction $V \sim h(a^2 + b^2)$ for any a or b.

Instead let's try the following general form that includes an ab term:

$$V = h(\alpha a^2 + \beta ab + \gamma b^2). \tag{2.11}$$

Then solve for the coefficients α, β, and γ by reapplying the easy-cases requirements.

The $b = 0$ test along with the $h = b/2$ easy case, which showed that $V = hb^2/3$ for an ordinary pyramid, require that $\alpha = 1/3$. The $a = 0$ test similarly requires that $\gamma = 1/3$. And the $a = b$ test requires that $\alpha + \beta + \gamma = 1$. Therefore $\beta = 1/3$ and voilà,

$$V = \frac{1}{3} h(a^2 + ab + b^2). \tag{2.12}$$

This formula, the result of proportional reasoning, dimensional analysis, and the method of easy cases, is exact (Problem 2.9)!

Problem 2.9 Integration

Use integration to show that $V = h(a^2 + ab + b^2)/3$.

Problem 2.10 Truncated triangular pyramid

Instead of a pyramid with a square base, start with a pyramid with an equilateral triangle of side length b as its base. Then make the truncated solid by slicing a piece from the top using a knife parallel to the base. In terms of the height h

and the top and bottom side lengths a and b, what is the volume of this solid? (See also Problem 2.7.)

Problem 2.11 Truncated cone
What is the volume of a truncated cone with a circular base of radius r_1 and circular top of radius r_2 (with the top parallel to the base)? Generalize your formula to the volume of a truncated pyramid with height h, a base of an arbitrary shape and area A_{base}, and a corresponding top of area A_{top}.

2.4 Fluid mechanics: Drag

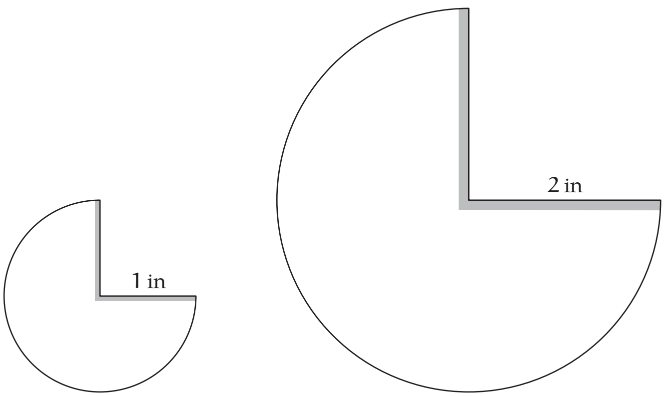
The preceding examples showed that easy cases can check and construct formulas, but the examples can be done without easy cases (for example, with calculus). For the next equations, from fluid mechanics, no exact solutions are known in general, so easy cases and other street-fighting tools are almost the only way to make progress.

Here then are the Navier–Stokes equations of fluid mechanics:

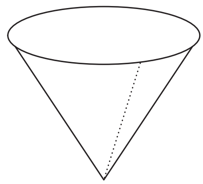
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{v}, \tag{2.13}$$

where $\mathbf{v}$ is the velocity of the fluid (as a function of position and time), ρ is its density, p is the pressure, and ν is the kinematic viscosity. These equations describe an amazing variety of phenomena including flight, tornadoes, and river rapids.

Our example is the following home experiment on drag. Photocopy this page while magnifying it by a factor of 2; then cut out the following two templates:



With each template, tape together the shaded areas to make a cone. The two resulting cones have the same shape, but the large cone has twice the height and width of the small cone.



When the cones are dropped point downward, what is the approximate ratio of their terminal speeds (the speeds at which drag balances weight)?

The Navier–Stokes equations contain the answer to this question. Finding the terminal speed involves four steps.

Step 1. Impose boundary conditions. The conditions include the motion of the cone and the requirement that no fluid enters the paper.

Step 2. Solve the equations, together with the continuity equation $\nabla \cdot \mathbf{v} = 0$, in order to find the pressure and velocity at the surface of the cone.

Step 3. Use the pressure and velocity to find the pressure and velocity gradient at the surface of the cone; then integrate the resulting forces to find the net force and torque on the cone.

Step 4. Use the net force and torque to find the motion of the cone. This step is difficult because the resulting motion must be consistent with the motion assumed in step 1. If it is not consistent, go back to step 1, assume a different motion, and hope for better luck upon reaching this step.

Unfortunately, the Navier–Stokes equations are coupled and nonlinear partial-differential equations. Their solutions are known only in very simple cases: for example, a sphere moving very slowly in a viscous fluid, or a sphere moving at any speed in a zero-viscosity fluid. There is little hope of solving for the complicated flow around an irregular, quivering shape such as a flexible paper cone.

Problem 2.12 Checking dimensions in the Navier–Stokes equations

Check that the first three terms of the Navier–Stokes equations have identical dimensions.

Problem 2.13 Dimensions of kinematic viscosity

From the Navier–Stokes equations, find the dimensions of kinematic viscosity ν.

2.4.1 Using dimensions

Because a direct solution of the Navier–Stokes equations is out of the question, let's use the methods of dimensional analysis and easy cases. A direct approach is to use them to deduce the terminal velocity itself. An indirect approach is to deduce the drag force as a function of fall speed and then to find the speed at which the drag balances the weight of the cones. This two-step approach simplifies the problem. It introduces only one new quantity (the drag force) but eliminates two quantities: the gravitational acceleration and the mass of the cone.

Problem 2.14 Explaining the simplification
Why is the drag force independent of the gravitational acceleration g and of the cone's mass m (yet the force depends on the cone's shape and size)?

The principle of dimensions is that all terms in a valid equation have identical dimensions. Applied to the drag force F, it means that in the equation $F = f(\text{quantities that affect } F)$ both sides have dimensions of force. Therefore, the strategy is to find the quantities that affect F, find their dimensions, and then combine the quantities into a quantity with dimensions of force.

▶ *On what quantities does the drag depend, and what are their dimensions?*

The drag force depends on four quantities: two parameters of the cone and two parameters of the fluid (air). (For the dimensions of ν, see Problem 2.13.)

v	speed of the cone	LT^{-1}
r	size of the cone	L
ρ	density of air	ML^{-3}
ν	viscosity of air	L^2T^{-1}

▶ *Do any combinations of the four parameters v, r, ρ, and ν have dimensions of force?*

The next step is to combine v, r, ρ, and ν into a quantity with dimensions of force. Unfortunately, the possibilities are numerous—for example,

$$\begin{aligned} F_1 &= \rho v^2 r^2, \\ F_2 &= \rho \nu v r, \end{aligned} \tag{2.14}$$

or the product combinations $\sqrt{F_1 F_2}$ and F_1^2/F_2. Any sum of these ugly products is also a force, so the drag force F could be $\sqrt{F_1 F_2} + F_1^2/F_2$, $3\sqrt{F_1 F_2} - 2F_1^2/F_2$, or much worse.

Narrowing the possibilities requires a method more sophisticated than simply guessing combinations with correct dimensions. To develop the sophisticated approach, return to the first principle of dimensions: All terms in an equation have identical dimensions. This principle applies to any statement about drag such as

$$A + B = C \tag{2.15}$$

where the blobs A, B, and C are functions of F, v, r, ρ, and ν.

Although the blobs can be absurdly complex functions, they have identical dimensions. Therefore, dividing each term by A, which produces the equation

$$\frac{A}{A} + \frac{B}{A} = \frac{C}{A}, \tag{2.16}$$

makes each term dimensionless. The same method turns any valid equation into a dimensionless equation. Thus, any (true) equation describing the world can be written in a dimensionless form.

Any dimensionless form can be built from dimensionless groups: from dimensionless products of the variables. Because any equation describing the world can be written in a dimensionless form, and any dimensionless form can be written using dimensionless groups, any equation describing the world can be written using dimensionless groups.

▶ *Is the free-fall example (Section 1.2) consistent with this principle?*

Before applying this principle to the complicated problem of drag, try it in the simple example of free fall (Section 1.2). The exact impact speed of an object dropped from a height h is $v = \sqrt{2gh}$, where g is the gravitational acceleration. This result can indeed be written in the dimensionless form $v/\sqrt{gh} = \sqrt{2}$, which itself uses only the dimensionless group $v/\sqrt{gh}$. The new principle passes its first test.

This dimensionless-group analysis of formulas, when reversed, becomes a method of synthesis. Let's warm up by synthesizing the impact speed v. First, list the quantities in the problem; here, they are v, g, and h. Second, combine these quantities into dimensionless groups. Here, all dimensionless groups can be constructed just from one group. For that group, let's choose v^2/gh (the particular choice does not affect the conclusion). Then the only possible dimensionless statement is

This is a portion of the eBook at doi:10.7551/mitpress/7728.001.0001

$$\frac{v^2}{gh} = \text{dimensionless constant.} \tag{2.17}$$

(The right side is a dimensionless constant because no second group is available to use there.) In other words, $v^2/gh \sim 1$ or $v \sim \sqrt{gh}$.

This result reproduces the result of the less sophisticated dimensional analysis in Section 1.2. Indeed, with only one dimensionless group, either analysis leads to the same conclusion. However, in hard problems—for example, finding the drag force—the less sophisticated method does not provide its constraint in a useful form; then the method of dimensionless groups is essential.

Problem 2.15 Fall time

Synthesize an approximate formula for the free-fall time t from g and h.

Problem 2.16 Kepler's third law

Synthesize Kepler's third law connecting the orbital period of a planet to its orbital radius. (See also Problem 1.15.)

▶ *What dimensionless groups can be constructed for the drag problem?*

One dimensionless group could be $F/\rho v^2 r^2$; a second group could be rv/ν. Any other group can be constructed from these groups (Problem 2.17), so the problem is described by two *independent* dimensionless groups. The most general dimensionless statement is then

$$\text{one group} = f(\text{second group}), \tag{2.18}$$

where f is a still-unknown (but dimensionless) function.

▶ *Which dimensionless group belongs on the left side?*

The goal is to synthesize a formula for F, and F appears only in the first group $F/\rho v^2 r^2$. With that constraint in mind, place the first group on the left side rather than wrapping it in the still-mysterious function f. With this choice, the most general statement about drag force is

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right). \tag{2.19}$$

The physics of the (steady-state) drag force on the cone is all contained in the dimensionless function f.

Problem 2.17 Only two groups
Show that F, v, r, ρ, and ν produce only two independent dimensionless groups.

Problem 2.18 How many groups in general?
Is there a general method to predict the number of independent dimensionless groups? (The answer was given in 1914 by Buckingham [9].)

The procedure might seem pointless, having produced a drag force that depends on the unknown function f. But it has greatly improved our chances of finding f. The original problem formulation required guessing the four-variable function h in $F = h(v, r, \rho, \nu)$, whereas dimensional analysis reduced the problem to guessing a function of only one variable (the ratio vr/ν). The value of this simplification was eloquently described by the statistician and physicist Harold Jeffreys (quoted in [34, p. 82]):

> A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Problem 2.19 Dimensionless groups for the truncated pyramid
The truncated pyramid of Section 2.3 has volume

$$V = \frac{1}{3}h(a^2 + ab + b^2). \qquad (2.20)$$

Make dimensionless groups from V, h, a, and b, and rewrite the volume using these groups. (There are many ways to do so.)

2.4.2 Using easy cases

Although improved, our chances do not look high: Even the one-variable drag problem has no exact solution. But it might have exact solutions in its easy cases. Because the easiest cases are often extreme cases, look first at the extreme cases.

▶ *Extreme cases of what?*

The unknown function f depends on only rv/ν,

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right), \qquad (2.21)$$

so try extremes of rv/ν. However, to avoid lapsing into mindless symbol pushing, first determine the meaning of rv/ν. This combination rv/ν, often denoted Re, is the famous Reynolds number. (Its physical interpretation requires the technique of lumping and is explained in Section 3.4.3.)

The Reynolds number affects the drag force via the unknown function f:

$$\frac{F}{\rho v^2 r^2} = f\,(\mathrm{Re})\,. \tag{2.22}$$

With luck, f can be deduced at extremes of the Reynolds number; with further luck, the falling cones are an example of one extreme.

▶ *Are the falling cones an extreme of the Reynolds number?*

The Reynolds number depends on r, v, and ν. For the speed v, everyday experience suggests that the cones fall at roughly $1\,\mathrm{m\,s^{-1}}$ (within, say, a factor of 2). The size r is roughly $0.1\,\mathrm{m}$ (again within a factor of 2). And the kinematic viscosity of air is $\nu \sim 10^{-5}\,\mathrm{m^2\,s^{-1}}$. The Reynolds number is

$$\frac{\overbrace{0.1\,\mathrm{m}}^{r} \times \overbrace{1\,\mathrm{m\,s^{-1}}}^{v}}{\underbrace{10^{-5}\,\mathrm{m^2\,s^{-1}}}_{\nu}} \sim 10^4. \tag{2.23}$$

It is significantly greater than 1, so the falling cones are an extreme case of high Reynolds number. (For low Reynolds number, try Problem 2.27 and see [38].)

Problem 2.20 Reynolds numbers in everyday flows

Estimate Re for a submarine cruising underwater, a falling pollen grain, a falling raindrop, and a 747 crossing the Atlantic.

The high-Reynolds-number limit can be reached many ways. One way is to shrink the viscosity ν to 0, because ν lives in the denominator of the Reynolds number. Therefore, in the limit of high Reynolds number, viscosity disappears from the problem and the drag force should not depend on viscosity. This reasoning contains several subtle untruths, yet its conclusion is mostly correct. (Clarifying the subtleties required two centuries of progress in mathematics, culminating in singular perturbations and the theory of boundary layers [12, 46].)

Viscosity affects the drag force only through the Reynolds number:

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right). \tag{2.24}$$

To make F independent of viscosity, F must be independent of Reynolds number! The problem then contains only one independent dimensionless group, $F/\rho v^2 r^2$, so the most general statement about drag is

$$\frac{F}{\rho v^2 r^2} = \text{dimensionless constant.} \tag{2.25}$$

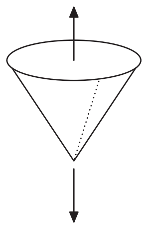
The drag force itself is then $F \sim \rho v^2 r^2$. Because r^2 is proportional to the cone's cross-sectional area A, the drag force is commonly written

$$F \sim \rho v^2 A. \tag{2.26}$$

Although the derivation was for falling cones, the result applies to any object as long as the Reynolds number is high. The shape affects only the missing dimensionless constant. For a sphere, it is roughly 1/4; for a long cylinder moving perpendicular to its axis, it is roughly 1/2; and for a flat plate moving perpendicular to its face, it is roughly 1.

2.4.3 Terminal velocities

The result $F \sim \rho v^2 A$ is enough to predict the terminal velocities of the cones. Terminal velocity means zero acceleration, so the drag force must balance the weight. The weight is $W = \sigma_{\text{paper}} A_{\text{paper}} g$, where σ_{paper} is the areal density of paper (mass per area) and A_{paper} is the area of the template after cutting out the quarter section. Because A_{paper} is comparable to the cross-sectional area A, the weight is roughly given by



$$W \sim \sigma_{\text{paper}} A g. \tag{2.27}$$

Therefore,

$$\underbrace{\rho v^2 A}_{\text{drag}} \sim \underbrace{\sigma_{\text{paper}} A g}_{\text{weight}}. \tag{2.28}$$

The area divides out and the terminal velocity becomes

$$v \sim \sqrt{\frac{g\sigma_{\text{paper}}}{\rho}}. \tag{2.29}$$

All cones constructed from the same paper and having the same shape, *whatever their size*, fall at the same speed!

To test this prediction, I constructed the small and large cones described on page 21, held one in each hand above my head, and let them fall. Their 2 m fall lasted roughly 2 s, and they landed within 0.1 s of one another. Cheap experiment and cheap theory agree!

Problem 2.21 Home experiment of a small versus a large cone
Try the cone home experiment yourself (page 21).

Problem 2.22 Home experiment of four stacked cones versus one cone
Predict the ratio

$$\frac{\text{terminal velocity of four small cones stacked inside each other}}{\text{terminal velocity of one small cone}}. \tag{2.30}$$

Test your prediction. Can you find a method not requiring timing equipment?

Problem 2.23 Estimating the terminal speed
Estimate or look up the areal density of paper; predict the cones' terminal speed; and then compare that prediction to the result of the home experiment.

2.5 Summary and further problems

A correct solution works in all cases, including the easy ones. Therefore, check any proposed formula in the easy cases, and guess formulas by constructing expressions that pass all easy-cases tests. To apply and extend these ideas, try the following problems and see the concise and instructive book by Cipra [10].

Problem 2.24 Fencepost errors
A garden has 10 m of horizontal fencing that you would like to divide into 1 m segments by using vertical posts. Do you need 10 or 11 vertical posts (including the posts needed at the ends)?

Problem 2.25 Odd sum
Here is the sum of the first n odd integers:

$$S_n = \underbrace{1 + 3 + 5 + \cdots + l_n}_{n \text{ terms}} \tag{2.31}$$

a. Does the last term l_n equal $2n + 1$ or $2n - 1$?

b. Use easy cases to guess S_n (as a function of n).

An alternative solution is discussed in Section 4.1.

Problem 2.26 Free fall with initial velocity

The ball in Section 1.2 was released from rest. Now imagine that it is given an initial velocity v_0 (where positive v_0 means an upward throw). Guess the impact velocity v_i.

Then solve the free-fall differential equation to find the exact v_i, and compare the exact result to your guess.

Problem 2.27 Low Reynolds number

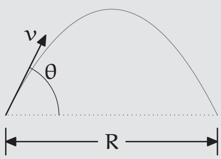
In the limit $\mathrm{Re} \ll 1$, guess the form of f in

$$\frac{F}{\rho v^2 r^2} = f\left(\frac{rv}{\nu}\right). \tag{2.32}$$

The result, when combined with the correct dimensionless constant, is known as Stokes drag [12].

Problem 2.28 Range formula

How far does a rock travel horizontally (no air resistance)? Use dimensions and easy cases to guess a formula for the range R as a function of the launch velocity v, the launch angle θ, and the gravitational acceleration g.



Problem 2.29 Spring equation

The angular frequency of an ideal mass–spring system (Section 3.4.2) is $\sqrt{k/m}$, where k is the spring constant and m is the mass. This expression has the spring constant k in the numerator. Use extreme cases of k or m to decide whether that placement is correct.

Problem 2.30 Taping the cone templates

The tape mark on the large cone template (page 21) is twice as wide as the tape mark on the small cone template. In other words, if the tape on the large cone is, say, 6 mm wide, the tape on the small cone should be 3 mm wide. Why?

This is a section of doi:10.7551/mitpress/7728.001.0001

Street-Fighting Mathematics

The Art of Educated Guessing and Opportunistic Problem Solving

By: Sanjoy Mahajan

Citation:

Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving
By: Sanjoy Mahajan
DOI: 10.7551/mitpress/7728.001.0001
ISBN (electronic): 9780262265881
Publisher: The MIT Press
Published: 2010

The MIT Press

For information about special quantity discounts, please email
special_sales@mitpress.mit.edu

Typeset in Palatino and Euler by the author using ConTEXt and PDFTEX

LIBRARY OF CONGRESS CATALOGING-IN-PUBLICATION DATA

Mahajan, Sanjoy, 1969–
Street-fighting mathematics : the art of educated guessing and opportunistic problem solving / Sanjoy Mahajan ; foreword by Carver A. Mead.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-262-51429-3 (pbk. : alk. paper) 1. Problem solving. 2. Hypothesis. 3. Estimation theory. I. Title.
QA63.M34 2010
510—dc22

2009028867

Printed and bound in the United States of America

10 9 8 7 6 5 4 3 2 1