3
Lumping

Where will an orbiting planet be 6 months from now? To predict its new location, we cannot simply multiply the 6 months by the planet's current velocity, for its velocity constantly varies. Such calculations are the reason that calculus was invented. Its fundamental idea is to divide the time into tiny intervals over which the velocity is constant, to multiply each tiny time by the corresponding velocity to compute a tiny distance, and then to add the tiny distances.

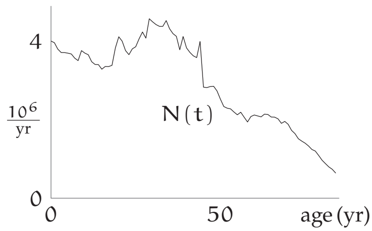
Amazingly, this computation can often be done exactly, even when the intervals have infinitesimal width and are therefore infinite in number. However, the symbolic manipulations can be lengthy and, worse, are often rendered impossible by even small changes to the problem. Using calculus methods, for example, we can exactly calculate the area under the Gaussian e^{-x^2} between $x = 0$ and ∞; yet if either limit is any value except zero or infinity, an exact calculation becomes impossible.

In contrast, approximate methods are robust: They almost always provide a reasonable answer. And the least accurate but most robust method is lumping. Instead of dividing a changing process into many tiny pieces, group or lump it into one or two pieces. This simple approximation and its advantages are illustrated using examples ranging from demographics (Section 3.1) to nonlinear differential equations (Section 3.5).

3.1 Estimating populations: How many babies?

The first example is to estimate the number of babies in the United States. For definiteness, call a child a baby until he or she turns 2 years old. An exact calculation requires the birth dates of every person in the United States. This, or closely similar, information is collected once every decade by the US Census Bureau.

As an approximation to this voluminous data, the Census Bureau [47] publishes the number of people at each age. The data for 1991 is a set of points lying on a wiggly line $N(t)$, where t is age. Then



$$N_{\text{babies}} = \int_0^{2\,\text{yr}} N(t)\,dt. \tag{3.1}$$

Problem 3.1 Dimensions of the vertical axis

Why is the vertical axis labeled in units of people per year rather than in units of people? Equivalently, why does the axis have dimensions of T^{-1}?

This method has several problems. First, it depends on the huge resources of the US Census Bureau, so it is not usable on a desert island for back-of-the-envelope calculations. Second, it requires integrating a curve with no analytic form, so the integration must be done numerically. Third, the integral is of data specific to this problem, whereas mathematics should be about generality. An exact integration, in short, provides little insight and has minimal transfer value. Instead of integrating the population curve exactly, approximate it—lump the curve into one rectangle.

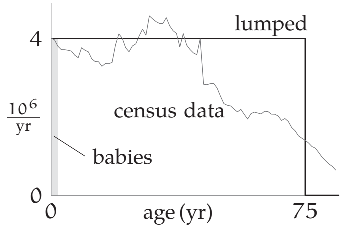
▶ *What are the height and width of this rectangle?*

The rectangle's width is a time, and a plausible time related to populations is the life expectancy. It is roughly 80 years, so make 80 years the width by pretending that everyone dies abruptly on his or her 80th birthday. The rectangle's height can be computed from the rectangle's area, which is the US population—conveniently 300 million in 2008. Therefore,

$$\text{height} = \frac{\text{area}}{\text{width}} \sim \frac{3 \times 10^8}{75\,\text{yr}}. \tag{3.2}$$

▶ *Why did the life expectancy drop from 80 to 75 years?*

Fudging the life expectancy simplifies the mental division: 75 divides easily into 3 and 300. The inaccuracy is no larger than the error made by lumping, and it might even cancel the lumping error. Using 75 years as the width makes the height approximately $4 \times 10^6\,\text{yr}^{-1}$.



Integrating the population curve over the range $t = 0 \ldots 2\,\text{yr}$ becomes just multiplication:

$$N_{\text{babies}} \sim \underbrace{4 \times 10^6\,\text{yr}^{-1}}_{\text{height}} \times \underbrace{2\,\text{yr}}_{\text{infancy}} = 8 \times 10^6. \quad (3.3)$$

The Census Bureau's figure is very close: 7.980×10^6. The error from lumping canceled the error from fudging the life expectancy to 75 years!

Problem 3.2 Landfill volume
Estimate the US landfill volume used annually by disposable diapers.

Problem 3.3 Industry revenues
Estimate the annual revenue of the US diaper industry.

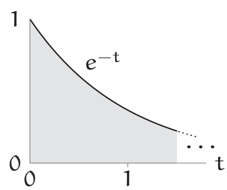
3.2 Estimating integrals

The US population curve (Section 3.1) was difficult to integrate partly because it was unknown. But even well-known functions can be difficult to integrate. In such cases, two lumping methods are particularly useful: the $1/e$ heuristic (Section 3.2.1) and the full width at half maximum (FWHM) heuristic (Section 3.2.2).

3.2.1 $1/e$ heuristic

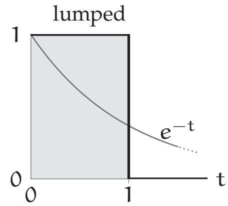
Electronic circuits, atmospheric pressure, and radioactive decay contain the ubiquitous exponential and its integral (given here in dimensionless form)

$$\int_0^\infty e^{-t}\,dt. \quad (3.4)$$

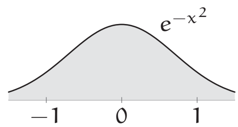


To approximate its value, let's lump the e^{-t} curve into one rectangle.

▶ *What values should be chosen for the width and height of the rectangle?*

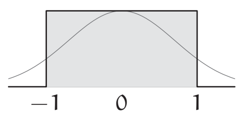


A reasonable height for the rectangle is the maximum of e^{-t}, namely 1. To choose its width, use significant change as the criterion (a method used again in Section 3.3.3): Choose a significant change in e^{-t}; then find the width Δt that produces this change. In an exponential decay, a simple and natural significant change is when e^{-t} becomes a factor of e closer to its final value (which is 0 here because t goes to ∞). With this criterion, $\Delta t = 1$. The lumping rectangle then has unit area—which is the exact value of the integral!



Encouraged by this result, let's try the heuristic on the difficult integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx. \tag{3.5}$$



Again lump the area into a single rectangle. Its height is the maximum of e^{-x^2}, which is 1. Its width is enough that e^{-x^2} falls by a factor of e. This drop happens at $x = \pm 1$, so the width is $\Delta x = 2$ and its area is 1×2. The exact area is $\sqrt{\pi} \approx 1.77$ (Section 2.1), so lumping makes an error of only 13%: For such a short derivation, the accuracy is extremely high.

Problem 3.4 General exponential decay
Use lumping to estimate the integral

$$\int_{0}^{\infty} e^{-\alpha t}\,dt. \tag{3.6}$$

Use dimensional analysis and easy cases to check that your answer makes sense.

Problem 3.5 Atmospheric pressure
Atmospheric density ρ decays roughly exponentially with height z:

$$\rho \sim \rho_0 e^{-z/H}, \tag{3.7}$$

where ρ_0 is the density at sea level, and H is the so-called scale height (the height at which the density falls by a factor of e). Use your everyday experience to estimate H.

Then estimate the atmospheric pressure at sea level by estimating the weight of an infinitely high cylinder of air.

Problem 3.6 Cone free-fall distance

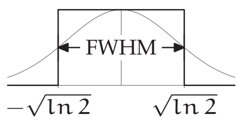
Roughly how far does a cone of Section 2.4 fall before reaching a significant fraction of its terminal velocity? How large is that distance compared to the drop height of 2 m? *Hint:* Sketch (very roughly) the cone's acceleration versus time and make a lumping approximation.

3.2.2 Full width at half maximum

Another reasonable lumping heuristic arose in the early days of spectroscopy. As a spectroscope swept through a range of wavelengths, a chart recorder would plot how strongly a molecule absorbed radiation of that wavelength. This curve contains many peaks whose location and area reveal the structure of the molecule (and were essential in developing quantum theory [14]). But decades before digital chart recorders existed, how could the areas of the peaks be computed?

They were computed by lumping the peak into a rectangle whose height is the height of the peak and whose width is the full width at half maximum (FWHM). Where the $1/e$ heuristic uses a factor of e as the significant change, the FWHM heuristic uses a factor of 2.

Try this recipe on the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.



The maximum height of e^{-x^2} is 1, so the half maxima are at $x = \pm\sqrt{\ln 2}$ and the full width is $2\sqrt{\ln 2}$. The lumped rectangle therefore has area $2\sqrt{\ln 2} \approx 1.665$. The exact area is $\sqrt{\pi} \approx 1.77$ (Section 2.1): The FWHM heuristic makes an error of only 6%, which is roughly one-half the error of the $1/e$ heuristic.

Problem 3.7 Trying the FWHM heuristic

Make single-rectangle lumping estimates of the following integrals. Choose the height and width of the rectangle using the FWHM heuristic. How accurate is each estimate?

a. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ [exact value: π].

b. $\displaystyle\int_{-\infty}^{\infty} e^{-x^4}\,dx$ [exact value: $\Gamma(1/4)/2 \approx 1.813$].

3.2.3 Stirling's approximation

The $1/e$ and FWHM lumping heuristics next help us approximate the ubiquitous factorial function $n!$; this function's uses range from probability theory to statistical mechanics and the analysis of algorithms. For positive integers, $n!$ is defined as $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. In this discrete form, it is difficult to approximate. However, the integral representation for $n!$,

$$n! \equiv \int_0^\infty t^n e^{-t}\, dt, \tag{3.8}$$

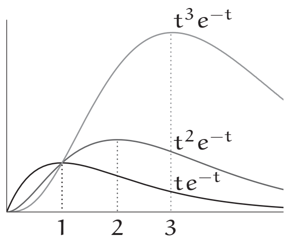
provides a definition even when n is not a positive integer—and this integral can be approximated using lumping.

The lumping analysis will generate almost all of Stirling's famous approximation formula

$$n! \approx n^n e^{-n} \sqrt{2\pi n}. \tag{3.9}$$

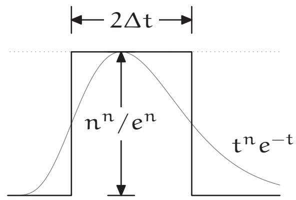
Lumping requires a peak, but does the integrand $t^n e^{-t}$ have a peak?

To understand the integrand $t^n e^{-t}$ or t^n/e^t, examine the extreme cases of t. When $t = 0$, the integrand is 0. In the opposite extreme, $t \to \infty$, the polynomial factor t^n makes the product infinity while the exponential factor e^{-t} makes it zero. Who wins that struggle? The Taylor series for e^t contains every power of t (and with positive coefficients), so it is an increasing, infinite-degree polynomial. Therefore, as t goes to infinity, e^t outruns any polynomial t^n and makes the integrand t^n/e^t equal 0 in the $t \to \infty$ extreme. Being zero at both extremes, the integrand must have a peak in between. In fact, it has exactly one peak. (Can you show that?)



Increasing n strengthens the polynomial factor t^n, so t^n survives until higher t before e^t outruns it. Therefore, the peak of t^n/e^t shifts right as n increases. The graph confirms this prediction and suggests that the peak occurs at $t = n$. Let's check by using calculus to maximize $t^n e^{-t}$ or, more simply, to maximize its logarithm $f(t) = n \ln t - t$. At a peak, a function has zero slope. Because $df/dt = n/t - 1$, the peak occurs at $t_{peak} = n$, when the integrand $t^n e^{-t}$ is $n^n e^{-n}$—thus reproducing the largest and most important factor in Stirling's formula.

What is a reasonable lumping rectangle?



The rectangle's height is the peak height $n^n e^{-n}$. For the rectangle's width, use either the $1/e$ or the FWHM heuristics. Because both heuristic require approximating $t^n e^{-t}$, expand its logarithm $f(t)$ in a Taylor series around its peak at $t = n$:

$$f(n + \Delta t) = f(n) + \Delta t \frac{df}{dt}\bigg|_{t=n} + \frac{(\Delta t)^2}{2} \frac{d^2 f}{dt^2}\bigg|_{t=n} + \cdots. \tag{3.10}$$

The second term of the Taylor expansion vanishes because $f(t)$ has zero slope at the peak. In the third term, the second derivative d^2f/dt^2 at $t = n$ is $-n/t^2$ or $-1/n$. Thus,

$$f(n + \Delta t) \approx f(n) - \frac{(\Delta t)^2}{2n}. \tag{3.11}$$

To decrease $t^n e^{-t}$ by a factor of F requires decreasing $f(t)$ by $\ln F$. This choice means $\Delta t = \sqrt{2n \ln F}$. Because the rectangle's width is $2\Delta t$, the lumped-area estimate of $n!$ is

$$n! \sim n^n e^{-n} \sqrt{n} \times \begin{cases} \sqrt{8} & (1/e \text{ criterion: } F = e) \\ \sqrt{8 \ln 2} & (\text{FWHM criterion: } F = 2). \end{cases} \tag{3.12}$$

For comparison, Stirling's formula is $n! \approx n^n e^{-n} \sqrt{2\pi n}$. Lumping has explained almost every factor. The $n^n e^{-n}$ factor is the height of the rectangle, and the $\sqrt{n}$ factor is from the width of the rectangle. Although the exact $\sqrt{2\pi}$ factor remains mysterious (Problem 3.9), it is approximated to within 13% (the $1/e$ heuristic) or 6% (the FWHM heuristic).

Problem 3.8 Coincidence?
The FWHM approximation for the area under a Gaussian (Section 3.2.2) was also accurate to 6%. Coincidence?

Problem 3.9 Exact constant in Stirling's formula
Where does the more accurate constant factor of $\sqrt{2\pi}$ come from?

3.3 Estimating derivatives

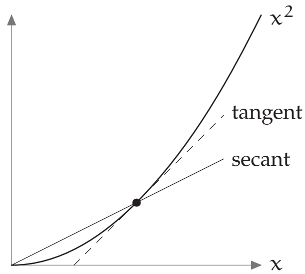
In the preceding examples, lumping helped estimate integrals. Because integration and differentiation are closely related, lumping also provides a method for estimating derivatives. The method begins with a dimensional observation about derivatives. A derivative is a ratio of differentials; for example, df/dx is the ratio of df to dx. Because d is dimensionless (Section 1.3.2), the dimensions of df/dx are the dimensions of f/x. This useful, surprising conclusion is worth testing with a familiar example: Differentiating height y with respect to time t produces velocity dy/dt, whose dimensions of LT^{-1} are indeed the dimensions of y/t.

Problem 3.10 Dimensions of a second derivative
What are the dimensions of d^2f/dx^2?

3.3.1 Secant approximation

As df/dx and f/x have identical dimensions, perhaps their magnitudes are similar:

$$\frac{df}{dx} \sim \frac{f}{x}. \tag{3.13}$$



Geometrically, the derivative df/dx is the slope of the tangent line, whereas the approximation f/x is the slope of the secant line. By replacing the curve with the secant line, we make a lumping approximation.

Let's test the approximation on an easy function such as $f(x) = x^2$. Good news—the secant and tangent slopes differ only by a factor of 2:

$$\frac{df}{dx} = 2x \qquad \text{and} \qquad \frac{f(x)}{x} = x. \tag{3.14}$$

Problem 3.11 Higher powers
Investigate the secant approximation for $f(x) = x^n$.

Problem 3.12 Second derivatives
Use the secant approximation to estimate d^2f/dx^2 with $f(x) = x^2$. How does the approximation compare to the exact second derivative?

▶ *How accurate is the secant approximation for* $f(x) = x^2 + 100$?

The secant approximation is quick and useful but can make large errors. When $f(x) = x^2 + 100$, for example, the secant and tangent at $x = 1$

have dramatically different slopes. The tangent slope df/dx is 2, whereas the secant slope $f(1)/1$ is 101. The ratio of these two slopes, although dimensionless, is distressingly large.

Problem 3.13 Investigating the discrepancy
With $f(x) = x^2 + 100$, sketch the ratio

$$\frac{\text{secant slope}}{\text{tangent slope}} \tag{3.15}$$

as a function of x. The ratio is not constant! Why is the dimensionless factor not constant? (That question is tricky.)

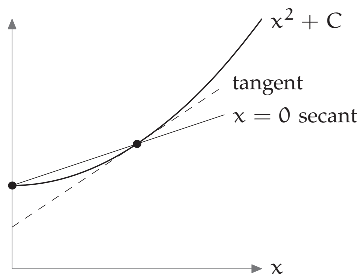
The large discrepancy in replacing the derivative df/dx, which is

$$\lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{\Delta x}, \tag{3.16}$$

with the secant slope $f(x)/x$ is due to two approximations. The first approximation is to take $\Delta x = x$ rather than $\Delta x = 0$. Then $df/dx \approx (f(x) - f(0))/x$. This first approximation produces the slope of the line from $(0, f(0))$ to $(x, f(x))$. The second approximation replaces $f(0)$ with 0, which produces $df/dx \approx f/x$; that ratio is the slope of the secant from $(0, 0)$ to $(x, f(x))$.

3.3.2 Improved secant approximation

The second approximation is fixed by starting the secant at $(0, f(0))$ instead of $(0, 0)$.



▶ *With that change, what are the secant and tangent slopes when $f(x) = x^2 + C$?*

Call the secant starting at $(0, 0)$ the origin secant; call the new secant the $x = 0$ secant. Then the $x = 0$ secant always has one-half the slope of the tangent, no matter the constant C. The $x = 0$ secant approximation is robust against—is unaffected by—vertical translation.

▶ *How robust is the $x = 0$ secant approximation against horizontal translation?*

To investigate how the $x = 0$ secant handles horizontal translation, translate $f(x) = x^2$ rightward by 100 to make $f(x) = (x-100)^2$. At the parabola's vertex $x = 100$, the $x = 0$ secant, from $(0, 10^4)$ to $(100, 0)$, has slope -100; however, the tangent has zero slope. Thus the $x = 0$ secant, although an improvement on the origin secant, is affected by horizontal translation.

3.3.3 Significant-change approximation

The derivative itself is unaffected by horizontal and vertical translation, so a derivative suitably approximated might be translation invariant. An approximate derivative is

$$\frac{df}{dx} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{3.17}$$

where Δx is not zero but is still small.

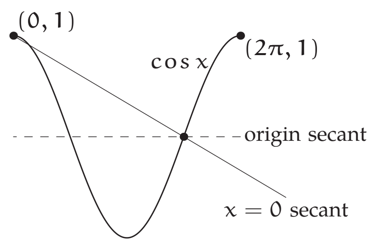
▶ *How small should Δx be? Is $\Delta x = 0.01$ small enough?*

The choice $\Delta x = 0.01$ has two defects. First, it cannot work when x has dimensions. If x is a length, what length is small enough? Choosing $\Delta x = 1\,\mathrm{mm}$ is probably small enough for computing derivatives related to the solar system, but is probably too large for computing derivatives related to falling fog droplets. Second, no fixed choice can be scale invariant. Although $\Delta x = 0.01$ produces accurate derivatives when $f(x) = \sin x$, it fails when $f(x) = \sin 1000x$, the result of simply rescaling x to $1000x$.

These problems suggest trying the following significant-change approximation:

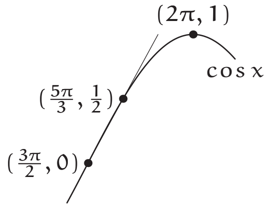
$$\frac{df}{dx} \sim \frac{\text{significant } \Delta f \text{ (change in } f\text{) at } x}{\Delta x \text{ that produces a significant } \Delta f}. \tag{3.18}$$

Because the Δx here is defined by the properties of the curve at the point of interest, without favoring particular coordinate values or values of Δx, the approximation is scale and translation invariant.



To illustrate this approximation, let's try $f(x) = \cos x$ and estimate df/dx at $x = 3\pi/2$ with the three approximations: the origin secant, the $x = 0$ secant, and the significant-change approximation. The origin secant goes from $(0, 0)$ to $(3\pi/2, 0)$, so it has zero slope. It is a poor approximation to the exact slope of 1. The $x = 0$

secant goes from $(0, 1)$ to $(3\pi/2, 0)$, so it has a slope of $-2/3\pi$, which is worse than predicting zero slope because even the sign is wrong!



The significant-change approximation might provide more accuracy. What is a significant change in $f(x) = \cos x$? Because the cosine changes by 2 (from −1 to 1), call 1/2 a significant change in $f(x)$. That change happens when x changes from $3\pi/2$, where $f(x) = 0$, to $3\pi/2 + \pi/6$, where $f(x) = 1/2$. In other words, Δx is $\pi/6$. The approximate derivative is therefore

$$\frac{df}{dx} \sim \frac{\text{significant } \Delta f \text{ near } x}{\Delta x} \sim \frac{1/2}{\pi/6} = \frac{3}{\pi}. \tag{3.19}$$

This estimate is approximately 0.955—amazingly close to the true derivative of 1.

Problem 3.14 Derivative of a quadratic

With $f(x) = x^2$, estimate df/dx at $x = 5$ using three approximations: the origin secant, the $x = 0$ secant, and the significant-change approximation. Compare these estimates to the true slope.

Problem 3.15 Derivative of the logarithm

Use the significant-change approximation to estimate the derivative of $\ln x$ at $x = 10$. Compare the estimate to the true slope.

Problem 3.16 Lennard–Jones potential

The Lennard–Jones potential is a model of the interaction energy between two nonpolar molecules such as N_2 or CH_4. It has the form

$$V(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right], \tag{3.20}$$

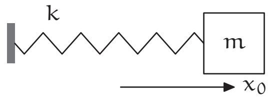
where r is the distance between the molecules, and ϵ and σ are constants that depend on the molecules. Use the origin secant to estimate r_0, the separation r at which $V(r)$ is a minimum. Compare the estimate to the true r_0 found using calculus.

Problem 3.17 Approximate maxima and minima

Let $f(x)$ be an increasing function and $g(x)$ a decreasing function. Use the origin secant to show, approximately, that $h(x) = f(x) + g(x)$ has a minimum where $f(x) = g(x)$. This useful rule of thumb, which generalizes Problem 3.16, is often called the balancing heuristic.

3.4 Analyzing differential equations: The spring–mass system

Estimating derivatives reduces differentiation to division (Section 3.3); it thereby reduces differential equations to algebraic equations.



To produce an example equation to analyze, connect a block of mass m to an ideal spring with spring constant (stiffness) k, pull the block a distance x_0 to the right relative to the equilibrium position $x = 0$, and release it at time $t = 0$. The block oscillates back and forth, its position x described by the ideal-spring differential equation

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{3.21}$$

Let's approximate the equation and thereby estimate the oscillation frequency.

3.4.1 Checking dimensions

Upon seeing any equation, first check its dimensions (Chapter 1). If all terms do not have identical dimensions, the equation is not worth solving—a great savings of effort. If the dimensions match, the check has prompted reflection on the meaning of the terms; this reflection helps prepare for solving the equation and for understanding any solution.

▶ *What are the dimensions of the two terms in the spring equation?*

Look first at the simple second term kx. It arises from Hooke's law, which says that an ideal spring exerts a force kx where x is the extension of the spring relative to its equilibrium length. Thus the second term kx is a force. Is the first term also a force?

The first term $m(d^2x/dt^2)$ contains the second derivative d^2x/dt^2, which is familiar as an acceleration. Many differential equations, however, contain unfamiliar derivatives. The Navier–Stokes equations of fluid mechanics (Section 2.4),

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \nu\boldsymbol{\nabla}^2\mathbf{v}, \tag{3.22}$$

contain two strange derivatives: $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ and $\boldsymbol{\nabla}^2\mathbf{v}$. What are the dimensions of those terms?

To practice for later handling such complicated terms, let's now find the dimensions of d^2x/dt^2 by hand. Because d^2x/dt^2 contains two exponents of 2, and x is length and t is time, d^2x/dt^2 might plausibly have dimensions of L^2T^{-2}.

▶ *Are* L^2T^{-2} *the correct dimensions?*

To decide, use the idea from Section 1.3.2 that the differential symbol d means "a little bit of." The numerator d^2x, meaning d of dx, is "a little bit of a little bit of x." Thus, it is a length. The denominator dt^2 could plausibly mean $(dt)^2$ or $d(t^2)$. [It turns out to mean $(dt)^2$.] In either case, its dimensions are T^2. Therefore, the dimensions of the second derivative are LT^{-2}:

$$\left[\frac{d^2x}{dt^2}\right] = LT^{-2}. \tag{3.23}$$

This combination is an acceleration, so the spring equation's first term $m(d^2x/dt^2)$ is mass times acceleration—giving it the same dimensions as the kx term.

Problem 3.18 Dimensions of spring constant
What are the dimensions of the spring constant k?

3.4.2 Estimating the magnitudes of the terms

The spring equation passes the dimensions test, so it is worth analyzing to find the oscillation frequency. The method is to replace each term with its approximate magnitude. These replacements will turn a complicated differential equation into a simple algebraic equation for the frequency.

To approximate the first term $m(d^2x/dt^2)$, use the significant-change approximation (Section 3.3.3) to estimate the magnitude of the acceleration d^2x/dt^2.

$$\frac{d^2x}{dt^2} \sim \frac{\text{significant } \Delta x}{(\Delta t \text{ that produces a significant } \Delta x)^2}. \tag{3.24}$$

Problem 3.19 Explaining the exponents
The numerator contains only the first power of Δx, whereas the denominator contains the second power of Δt. How can that discrepancy be correct?

To evaluate this approximate acceleration, first decide on a significant Δx—on what constitutes a significant change in the mass's position. The mass moves between the points $x = -x_0$ and $x = +x_0$, so a significant change in position should be a significant fraction of the peak-to-peak amplitude $2x_0$. The simplest choice is $\Delta x = x_0$.

Now estimate Δt: the time for the block to move a distance comparable to Δx. This time—called the characteristic time of the system—is related to the oscillation period T. During one period, the mass moves back and forth and travels a distance $4x_0$—much farther than x_0. If Δt were, say, $T/4$ or $T/2\pi$, then in the time Δt the mass would travel a distance comparable to x_0. Those choices for Δt have a natural interpretation as being approximately $1/\omega$, where the angular frequency ω is connected to the period by the definition $\omega \equiv 2\pi/T$. With the preceding choices for Δx and Δt, the $m(d^2x/dt^2)$ term is roughly $mx_0\omega^2$.

▶ *What does "is roughly" mean?*

The phrase cannot mean that $mx_0\omega^2$ and $m(d^2x/dt^2)$ are within, say, a factor of 2, because $m(d^2x/dt^2)$ varies and mx_0/τ^2 is constant. Rather, "is roughly" means that a typical or characteristic magnitude of $m(d^2x/dt^2)$—for example, its root-mean-square value—is comparable to $mx_0\omega^2$. Let's include this meaning within the twiddle notation $\sim$. Then the typical-magnitude estimate can be written

$$m\frac{d^2x}{dt^2} \sim mx_0\omega^2. \tag{3.25}$$

With the same meaning of "is roughly", namely that the typical magnitudes are comparable, the spring equation's second term kx is roughly kx_0. The two terms must add to zero—a consequence of the spring equation

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{3.26}$$

Therefore, the magnitudes of the two terms are comparable:

$$mx_0\omega^2 \sim kx_0. \tag{3.27}$$

The amplitude x_0 divides out! With x_0 gone, the frequency ω and oscillation period $T = 2\pi/\omega$ are independent of amplitude. [This reasoning uses several approximations, but this conclusion is exact (Problem 3.20).] The approximated angular frequency ω is then $\sqrt{k/m}$.

For comparison, the exact solution of the spring differential equation is, from Problem 3.22,

$$x = x_0 \cos \omega t, \tag{3.28}$$

where ω is $\sqrt{k/m}$. The approximated angular frequency is also exact!

Problem 3.20 Amplitude independence
Use dimensional analysis to show that the angular frequency ω cannot depend on the amplitude x_0.

Problem 3.21 Checking dimensions in the alleged solution
What are the dimensions of ωt? What are the dimensions of $\cos \omega t$? Check the dimensions of the proposed solution $x = x_0 \cos \omega t$, and the dimensions of the proposed period $2\pi\sqrt{m/k}$.

Problem 3.22 Verification
Show that $x = x_0 \cos \omega t$ with $\omega = \sqrt{k/m}$ solves the spring differential equation

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{3.29}$$

3.4.3 Meaning of the Reynolds number

As a further example of lumping—in particular, of the significant-change approximation—let's analyze the Navier–Stokes equations introduced in Section 2.4,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{v}, \tag{3.30}$$

and extract from them a physical meaning for the Reynolds number $r v/\nu$.

To do so, we estimate the typical magnitude of the inertial term $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ and of the viscous term $\nu\boldsymbol{\nabla}^2\mathbf{v}$.

▶ *What is the typical magnitude of the inertial term?*

The inertial term $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$ contains the spatial derivative $\boldsymbol{\nabla}\mathbf{v}$. According to the significant-change approximation (Section 3.3.3), the derivative $\boldsymbol{\nabla}\mathbf{v}$ is roughly the ratio

$$\frac{\text{significant change in flow velocity}}{\text{distance over which flow velocity changes significantly}}. \tag{3.31}$$

The flow velocity (the velocity of the air) is nearly zero far from the cone and is comparable to v near the cone (which is moving at speed v). Therefore, v, or a reasonable fraction of v, constitutes a significant change in flow velocity. This speed change happens over a distance comparable to the size of the cone: Several cone lengths away, the air hardly knows about the falling cone. Thus $\nabla \mathbf{v} \sim v/r$. The inertial term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ contains a second factor of $\mathbf{v}$, so $(\mathbf{v}\cdot\nabla)\mathbf{v}$ is roughly v^2/r.

▶ *What is the typical magnitude of the viscous term?*

The viscous term $\nu\nabla^2\mathbf{v}$ contains two spatial derivatives of $\mathbf{v}$. Because each spatial derivative contributes a factor of $1/r$ to the typical magnitude, $\nu\nabla^2\mathbf{v}$ is roughly $\nu v/r^2$. The ratio of the inertial term to the viscous term is then roughly $(v^2/r)/(\nu v/r^2)$. This ratio simplifies to rv/ν—the familiar, dimensionless, Reynolds number.

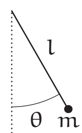
Thus, the Reynolds number measures the importance of viscosity. When $\mathrm{Re} \gg 1$, the viscous term is small, and viscosity has a negligible effect. It cannot prevent nearby pieces of fluid from acquiring significantly different velocities, and the flow becomes turbulent. When $\mathrm{Re} \ll 1$, the viscous term is large, and viscosity is the dominant physical effect. The flow oozes, as when pouring cold honey.

3.5 Predicting the period of a pendulum

Lumping not only turns integration into multiplication, it turns nonlinear into linear differential equations. Our example is the analysis of the period of a pendulum, for centuries the basis of Western timekeeping.

▶ *How does the period of a pendulum depend on its amplitude?*

The amplitude θ_0 is the maximum angle of the swing; for a lossless pendulum released from rest, it is also the angle of release. The effect of amplitude is contained in the solution to the pendulum differential equation (see [24] for the equation's derivation):



$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{3.32}$$

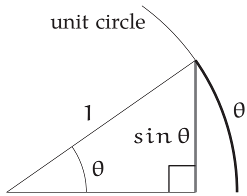
The analysis will use all our tools: dimensions (Section 3.5.2), easy cases (Section 3.5.1 and Section 3.5.3), and lumping (Section 3.5.4).

Problem 3.23 Angles
Explain why angles are dimensionless.

Problem 3.24 Checking and using dimensions
Does the pendulum equation have correct dimensions? Use dimensional analysis to show that the equation cannot contain the mass of the bob (except as a common factor that divides out).

3.5.1 Small amplitudes: Applying extreme cases

The pendulum equation is difficult because of its nonlinear factor $\sin\theta$. Fortunately, the factor is easy in the small-amplitude extreme case $\theta \to 0$. In that limit, the height of the triangle, which is $\sin\theta$, is almost exactly the arclength θ. Therefore, for small angles, $\sin\theta \approx \theta$.



Problem 3.25 Chord approximation
The $\sin\theta \approx \theta$ approximation replaces the arc with a straight, vertical line. To make a more accurate approximation, replace the arc with the chord (a straight but nonvertical line). What is the resulting approximation for $\sin\theta$?

In the small-amplitude extreme, the pendulum equation becomes linear:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{3.33}$$

Compare this equation to the spring–mass equation (Section 3.4)

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{3.34}$$

The equations correspond with x analogous to θ and k/m analogous to g/l. The frequency of the spring–mass system is $\omega = \sqrt{k/m}$, and its period is $T = 2\pi/\omega = 2\pi\sqrt{m/k}$. For the pendulum equation, the corresponding period is

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{(for small amplitudes)}. \tag{3.35}$$

(This analysis is a preview of the method of analogy, which is the subject of Chapter 6.)

Problem 3.26 Checking dimensions

Does the period $2\pi\sqrt{l/g}$ have correct dimensions?

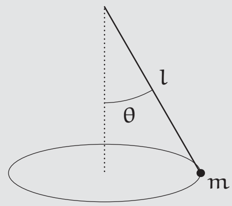
Problem 3.27 Checking extreme cases

Does the period $T = 2\pi\sqrt{l/g}$ make sense in the extreme cases $g \to \infty$ and $g \to 0$?

Problem 3.28 Possible coincidence

Is it a coincidence that $g \approx \pi^2\,\mathrm{m\,s^{-2}}$? (For an extensive historical discussion that involves the pendulum, see [1] and more broadly also [4, 27, 42].)

Problem 3.29 Conical pendulum for the constant



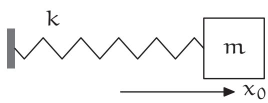
The dimensionless factor of 2π can be derived using an insight from Huygens [15, p. 79]: to analyze the motion of a pendulum moving in a horizontal circle (a conical pendulum). Projecting its two-dimensional motion onto a vertical screen produces one-dimensional pendulum motion, so the period of the two-dimensional motion is the same as the period of one-dimensional pendulum motion! Use that idea along with Newton's laws of motion to explain the 2π.

3.5.2 Arbitrary amplitudes: Applying dimensional analysis

The preceding results might change if the amplitude θ_0 is no longer small.

▶ *As θ_0 increases, does the period increase, remain constant, or decrease?*

Any analysis becomes cleaner if expressed using dimensionless groups (Section 2.4.1). This problem involves the period T, length l, gravitational strength g, and amplitude θ_0. Therefore, T can belong to the dimensionless group $T/\sqrt{l/g}$. Because angles are dimensionless, θ_0 is itself a dimensionless group. The two groups $T/\sqrt{l/g}$ and θ_0 are independent and fully describe the problem (Problem 3.30).



An instructive contrast is the ideal spring–mass system. The period T, spring constant k, and mass m can form the dimensionless group $T/\sqrt{m/k}$; but the amplitude x_0, as the only quantity containing a length, cannot be part of any dimensionless group (Problem 3.20) and cannot therefore affect the period of the spring–mass system. In contrast, the pendulum's amplitude θ_0 is already a dimensionless group, so it can affect the period of the system.

> **Problem 3.30 Choosing dimensionless groups**
> Check that period T, length l, gravitational strength g, and amplitude θ_0 produce two independent dimensionless groups. In constructing useful groups for analyzing the period, why should T appear in only one group? And why should θ_0 not appear in the same group as T?

Two dimensionless groups produce the general dimensionless form

$$\text{one group} = \text{function of the other group}, \tag{3.36}$$

so

$$\frac{T}{\sqrt{l/g}} = \text{function of } \theta_0. \tag{3.37}$$

Because $T/\sqrt{l/g} = 2\pi$ when $\theta_0 = 0$ (the small-amplitude limit), factor out the 2π to simplify the subsequent equations, and define a dimensionless period h as follows:

$$\frac{T}{\sqrt{l/g}} = 2\pi\, h(\theta_0). \tag{3.38}$$

The function h contains all information about how amplitude affects the period of a pendulum. Using h, the original question about the period becomes the following: Is h an increasing, constant, or decreasing function of amplitude? This question is answered in the following section.

3.5.3 Large amplitudes: Extreme cases again

For guessing the general behavior of h as a function of amplitude, useful clues come from evaluating h at two amplitudes. One easy amplitude is the extreme of zero amplitude, where $h(0) = 1$. A second easy amplitude is the opposite extreme of large amplitudes.

▶ *How does the period behave at large amplitudes? As part of that question, what is a large amplitude?*

An interesting large amplitude is $\pi/2$, which means releasing the pendulum from horizontal. However, at $\pi/2$ the exact h is the following awful expression (Problem 3.31):

$$h(\pi/2) = \frac{\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}. \tag{3.39}$$

Is this integral less than, equal to, or more than 1? Who knows? The integral is likely to have no closed form and to require numerical evaluation (Problem 3.32).

Problem 3.31 General expression for h
Use conservation of energy to show that the period is

$$T(\theta_0) = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{3.40}$$

Confirm that the equivalent dimensionless statement is

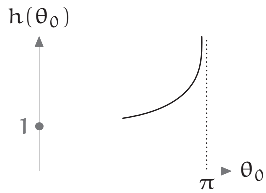
$$h(\theta_0) = \frac{\sqrt{2}}{\pi}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{3.41}$$

For horizontal release, $\theta_0 = \pi/2$, and

$$h(\pi/2) = \frac{\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta}}. \tag{3.42}$$

Problem 3.32 Numerical evaluation for horizontal release
Why do the lumping recipes (Section 3.2) fail for the integrals in Problem 3.31? Compute $h(\pi/2)$ using numerical integration.

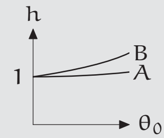
Because $\theta_0 = \pi/2$ is not a helpful extreme, be even more extreme. Try $\theta_0 = \pi$, which means releasing the pendulum bob from vertical. If the bob is connected to the pivot point by a string, however, a vertical release would mean that the bob falls straight down instead of oscillating. This novel behavior is neither included in nor described by the pendulum differential equation.



Fortunately, a thought experiment is cheap to improve: Replace the string with a massless steel rod. Balanced perfectly at $\theta_0 = \pi$, the pendulum bob hangs upside down forever, so $T(\pi) = \infty$ and $h(\pi) = \infty$. Thus, $h(\pi) > 1$ and $h(0) = 1$. From these data, the most likely conjecture is that h increases monotonically with amplitude. Although h could first decrease and then increase, such twists and turns would be surprising behavior from such a clean differential equation. (For the behavior of h near $\theta_0 = \pi$, see Problem 3.34).

Problem 3.33 Small but nonzero amplitude

As the amplitude approaches π, the dimensionless period h diverges to infinity; at zero amplitude, $h = 1$. But what about the derivative of h? At zero amplitude ($\theta_0 = 0$), does $h(\theta_0)$ have zero slope (curve A) or positive slope (curve B)?



Problem 3.34 Nearly vertical release

Imagine releasing the pendulum from almost vertical: an initial angle $\pi - \beta$ with β tiny. As a function of β, roughly how long does the pendulum take to rotate by a significant angle—say, by 1 rad? Use that information to predict how $h(\theta_0)$ behaves when $\theta_0 \approx \pi$. Check and refine your conjectures using the tabulated values. Then predict $h(\pi - 10^{-5})$.

β	$h(\pi - \beta)$
10^{-1}	2.791297
10^{-2}	4.255581
10^{-3}	5.721428
10^{-4}	7.187298

3.5.4 Moderate amplitudes: Applying lumping

The conjecture that h increases monotonically was derived using the extremes of zero and vertical amplitude, so it should apply at intermediate amplitudes. Before taking that statement on faith, recall a proverb from arms-control negotiations: “Trust, but verify.”

▶ *At moderate (small but nonzero) amplitudes, does the period, or its dimensionless cousin h, increase with amplitude?*

In the zero-amplitude extreme, $\sin\theta$ is close to θ. That approximation turned the nonlinear pendulum equation

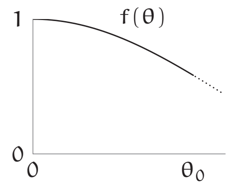
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \tag{3.43}$$

into the linear, ideal-spring equation—in which the period is independent of amplitude.

At nonzero amplitude, however, θ and $\sin\theta$ differ and their difference affects the period. To account for the difference and predict the period, split $\sin\theta$ into the tractable factor θ and an adjustment factor $f(\theta)$. The resulting equation is

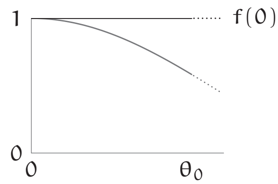
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta\underbrace{\frac{\sin\theta}{\theta}}_{f(\theta)} = 0. \tag{3.44}$$

The nonconstant $f(\theta)$ encapsulates the nonlinearity of the pendulum equation. When θ is tiny, $f(\theta) \approx 1$: The pendulum behaves like a linear, ideal-spring system. But when θ is large, $f(\theta)$ falls significantly below 1, making the ideal-spring approximation significantly inaccurate. As is often the case, a changing process is difficult to analyze—for example, see the awful integrals in Problem 3.31. As a countermeasure, make a lumping approximation by replacing the changing $f(\theta)$ with a constant.



The simplest constant is $f(0)$. Then the pendulum differential equation becomes

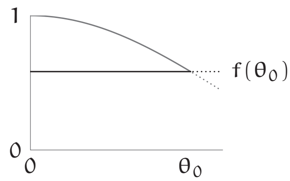
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{3.45}$$



This equation is, again, the ideal-spring equation. In this approximation, period does not depend on amplitude, so $h = 1$ for all amplitudes. For determining how the period of an unapproximated pendulum depends on amplitude, the $f(\theta) \to f(0)$ lumping approximation discards too much information.

Therefore, replace $f(\theta)$ with the other extreme $f(\theta_0)$. Then the pendulum equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta f(\theta_0) = 0. \tag{3.46}$$



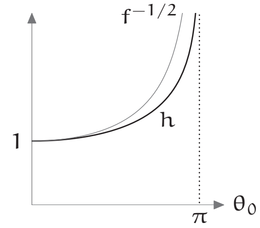
Is this equation linear? What physical system does it describe?

Because $f(\theta_0)$ is a constant, this equation is linear! It describes a zero-amplitude pendulum on a planet with gravity g_{eff} that is slightly weaker than earth gravity—as shown by the following slight regrouping:

$$\frac{d^2\theta}{dt^2} + \frac{\overbrace{gf(\theta_0)}^{g_{\text{eff}}}}{l}\theta = 0. \tag{3.47}$$

Because the zero-amplitude pendulum has period $T = 2\pi\sqrt{l/g}$, the zero-amplitude, low-gravity pendulum has period

$$T(\theta_0) \approx 2\pi\sqrt{\frac{l}{g_{\text{eff}}}} = 2\pi\sqrt{\frac{l}{gf(\theta_0)}}. \tag{3.48}$$



Using the dimensionless period h avoids writing the factors of 2π, l, and g, and it yields the simple prediction

$$h(\theta_0) \approx f(\theta_0)^{-1/2} = \left(\frac{\sin\theta_0}{\theta_0}\right)^{-1/2}. \quad (3.49)$$

At moderate amplitudes the approximation closely follows the exact dimensionless period (dark curve). As a bonus, it also predicts $h(\pi) = \infty$, so it agrees with the thought experiment of releasing the pendulum from upright (Section 3.5.3).

▶ *How much larger than the period at zero amplitude is the period at* $10°$ *amplitude?*

A $10°$ amplitude is roughly 0.17 rad, a moderate angle, so the approximate prediction for h can itself accurately be approximated using a Taylor series. The Taylor series for $\sin\theta$ begins $\theta - \theta^3/6$, so

$$f(\theta_0) = \frac{\sin\theta_0}{\theta_0} \approx 1 - \frac{\theta_0^2}{6}. \quad (3.50)$$

Then $h(\theta_0)$, which is roughly $f(\theta_0)^{-1/2}$, becomes

$$h(\theta_0) \approx \left(1 - \frac{\theta_0^2}{6}\right)^{-1/2}. \quad (3.51)$$

Another Taylor series yields $(1+x)^{-1/2} \approx 1 - x/2$ (for small x). Therefore,

$$h(\theta_0) \approx 1 + \frac{\theta_0^2}{12}. \quad (3.52)$$

Restoring the dimensioned quantities gives the period itself.

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{12}\right). \quad (3.53)$$

Compared to the period at zero amplitude, a $10°$ amplitude produces a fractional increase of roughly $\theta_0^2/12 \approx 0.0025$ or 0.25%. Even at moderate amplitudes, the period is nearly independent of amplitude!

Problem 3.35 Slope revisited

Use the preceding result for $h(\theta_0)$ to check your conclusion in Problem 3.33 about the slope of $h(\theta_0)$ at $\theta_0 = 0$.

Does our lumping approximation underestimate or overestimate the period?

The lumping approximation simplified the pendulum differential equation by replacing $f(\theta)$ with $f(\theta_0)$. Equivalently, it assumed that the mass always remained at the endpoints of the motion where $|\theta| = \theta_0$. Instead, the pendulum spends much of its time at intermediate positions where $|\theta| < \theta_0$ and $f(\theta) > f(\theta_0)$. Therefore, the average f is greater than $f(\theta_0)$. Because h is inversely related to f ($h = f^{-1/2}$), the $f(\theta) \to f(\theta_0)$ lumping approximation overestimates h and the period.

The $f(\theta) \to f(0)$ lumping approximation, which predicts $T = 2\pi\sqrt{l/g}$, underestimates the period. Therefore, the true coefficient of the θ_0^2 term in the period approximation

$$T \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{12}\right) \tag{3.54}$$

lies between 0 and $1/12$. A natural guess is that the coefficient lies halfway between these extremes—namely, $1/24$. However, the pendulum spends more time toward the extremes (where $f(\theta) = f(\theta_0)$) than it spends near the equilibrium position (where $f(\theta) = f(0)$). Therefore, the true coefficient is probably closer to $1/12$—the prediction of the $f(\theta) \to f(\theta_0)$ approximation—than it is to 0. An improved guess might be two-thirds of the way from 0 to $1/12$, namely $1/18$.

In comparison, a full successive-approximation solution of the pendulum differential equation gives the following period [13, 33]:

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\,\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots\right). \tag{3.55}$$

Our educated guess of $1/18$ is very close to the true coefficient of $1/16$!

3.6 Summary and further problems

Lumping turns calculus on its head. Whereas calculus analyzes a changing process by dividing it into ever finer intervals, lumping simplifies a changing process by combining it into one unchanging process. It turns curves into straight lines, difficult integrals into multiplication, and mildly nonlinear differential equations into linear differential equations.

…*the crooked shall be made straight, and the rough places plain.* (Isaiah 40:4)

Problem 3.36 FWHM for another decaying function
Use the FWHM heuristic to estimate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}. \tag{3.56}$$

Then compare the estimate with the exact value of $\pi/\sqrt{2}$. For an enjoyable additional problem, derive the exact value.

Problem 3.37 Hypothetical pendulum equation
Suppose the pendulum equation had been

$$\frac{d^2\theta}{d\theta^2} + \frac{g}{l}\tan\theta = 0. \tag{3.57}$$

How would the period T depend on amplitude θ_0? In particular, as θ_0 increases, would T decrease, remain constant, or increase? What is the slope $dT/d\theta_0$ at zero amplitude? Compare your results with the results of Problem 3.33.

For small but nonzero θ_0, find an approximate expression for the dimensionless period $h(\theta_0)$ and use it to check your previous conclusions.

Problem 3.38 Gaussian 1-sigma tail
The Gaussian probability density function with zero mean and unit variance is

$$p(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}. \tag{3.58}$$

The area of its tail is an important quantity in statistics, but it has no closed form. In this problem you estimate the area of the 1-sigma tail

$$\int_{1}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx. \tag{3.59}$$

a. Sketch the above Gaussian and shade the 1-sigma tail.

b. Use the $1/e$ lumping heuristic (Section 3.2.1) to estimate the area.

c. Use the FWHM heuristic to estimate the area.

d. Compare the two lumping estimates with the result of numerical integration:

$$\int_{1}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \frac{1-\mathrm{erf}(1/\sqrt{2})}{2} \approx 0.159, \tag{3.60}$$

where $\mathrm{erf}(z)$ is the error function.

Problem 3.39 Distant Gaussian tails
For the canonical probability Gaussian, estimate the area of its n-sigma tail (for large n). In other words, estimate

$$\int_{n}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx. \tag{3.61}$$