3 Universal Constructions

In which we learn about the decisive category-theoretic notion of universality (universal properties), *familiarize ourselves with a variety of notable examples of this phenomenon, present definitions of other important notions making use of these properties, and consider the broader significance of the notion of universality developed here.*

3.1 Limits and Colimits

In category theory there is a very important notion of limits and colimits. These terms really codify and powerfully generalize certain decisive constructions that had been noticed in many concrete cases all over mathematics well before category theory was born. Throughout mathematics, we are constantly building new mathematical objects from old ones. Categorical limits and colimits are, in an important sense, the *most* efficient way of doing so. Examples include many familiar constructions—like taking disjoint unions or the intersections of sets, the least upper bound (supremum) or greatest lower bound (infimum) of a set of numbers, forming Cartesian products, direct sums, kernels and cokernels, forming the coarsest topology making a map continuous, and more. Limits capture a wide array of constructions where a certain subcollection of given objects is isolated, while colimits capture something like the amalgamation or gluing together of given objects. What specific instances of each construction have in common—usually indicated by the use of a superlative adjective, like the *largest*, *least*, *coarsest*, and so on—is that the construction or object satisfies a certain *universal property* in relation to other components of the category.

To carry out these constructions, we need to know *what we are taking the (co)limit of*. Rather than confining our attention to special objects—like numbers whose maximum or whose least common multiple we are interested in taking—the right way to develop this is to give the most general account, allowing for potentially complex input data. The input data for both notions will be a *diagram*, that is, some collection of objects in a category and some morphisms between them. In chapter 2, we met and began to explore the notion of a *diagram* (as a functor), where this was thought of as involving an instantiation of a particular template supplied by the shape or generic figures of an indexing category. There, you were invited to regard the indexing category as a template consisting of some nodes together with directed edges between certain nodes, that is, as a directed graph, on which the instantiations were patterned. A diagram instantiates (within the target category) each node of the template with an object of the target category and each edge with an arrow of the target category, thus yielding a diagram built out of shapes whose generic form is provided by the template category.

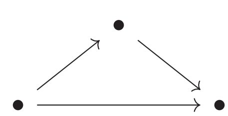
A diagram in a given category can be thought of as posing two problems, the left and right problems, the solutions to which are supplied by certain objects, together with a collection of morphisms, that “complete” the diagrams at either end. Such gadgets formed by the object and arrows of the left solution—that is, a special object C together with a collection of arrows, one for each object in the diagram, such that for any arrow between objects of the diagram there are arrows from C that make the resulting triangles commute—are called *cones*. Similarly, for a right solution in which all arrows terminate, such a thing is often called a *cocone*.

In general, on any particular side, a solution need not exist at all (or it may exist on one side but not on the other); on the other hand, each problem may have many solutions. A *universal* solution is one through which each (left or right) solution must pass by means of a (fundamentally) unique mediating arrow. In other words, if there are solutions (of the relevant handedness), then the universal solution is one that is abstractly “nearest” to the diagram and, as such, is the best or most optimal solution to the problem (“better” than any other object that could be used to complete the diagram). A limit is just a universal left solution, a colimit a universal right solution. If a diagram has a (co)limit, this (co)limit will be essentially unique, so that whenever such a solution does exist, we can in fact speak of *the* (co)limit. In short, and in the general case, a limit and a colimit are given by nominated objects among the (co)cones, that “universally complete” the diagram on the left and on the right (respectively).

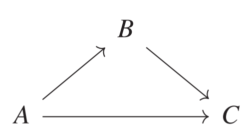
Let us be more explicit about all this. Recall the notion of $\mathbf{J}$-shaped diagrams $F : \mathbf{J} \to \mathbf{C}$, first introduced in example 35. Using this construction and natural transformations, we can introduce the concepts of cones and cocones of a diagram, thereby characterizing the limit and colimit of a diagram as the universal such (co)cone.

3.1.1 Cones and Cocones: Limits and Colimits Defined

Suppose



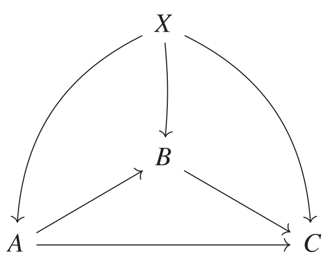
is the template for the diagram

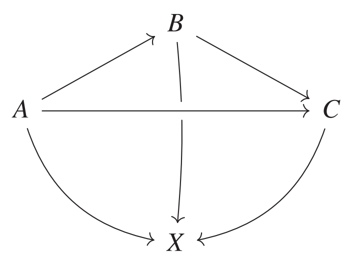


in a category $\mathbf{C}$. Observe that, in general, for any given object X in a category $\mathbf{C}$, there will be a *constant functor*—one that we will just call by the name X—from $\mathbf{J}$ to $\mathbf{C}$. This functor acts to send every object to X and every morphism to the identity map on X. This lets us view the object X itself as a diagram, $X : \mathbf{J} \to \mathbf{C}$.

Thus, given any functor (diagram) $F : \mathbf{J} \to \mathbf{C}$, we can consider a natural transformation between X (now viewed as a functor) and the diagram F. This natural transformation will just consist of a collection of morphisms between X and the objects found in the diagram F, such that these morphisms moreover commute with the morphisms found in the diagram. When the arrows go from X to the diagram F, this construction is called a *cone over D*; when the arrows go the other way, now from the diagram F to X, then the construction is called a *cone under D* (or a *cocone*). In this way, using the diagram introduced above, a natural transformation $X \Rightarrow F$ will look like the image on the left, while a natural transformation $F \Rightarrow X$ will look like the image on the right:

cone over





cone under

Now, there many be many such (co)cones over (under) a diagram. The punch line toward which we will build is that

> the *limit* of a diagram F is just a special (optimal) cone over F, in that it is the cone that "gets as close as possible" to the diagram F (where this means that any other cone will have to factor or pass through it);

and

> the *colimit* of a diagram F is just a special (optimal) cone under (cocone for) F, in that it is the cocone that "gets as close as possible" to the diagram F (where this means that any other cocone will have to factor or pass through it).

Let us now do this more formally, starting with the limit. There are a few steps. First, let us be more precise about how to regard any given object of a category in terms of a cone. In a category $\mathbf{C}$, a *terminal object* is a special object, usually denoted 1 (owing to the fact that in **Set**, it is just a one-element set), satisfying a certain universal property:

> for every object X of $\mathbf{C}$, there exists a unique morphism $! : X \to 1$.

If such a terminal object exists, it will be unique (up to unique isomorphism). Dually, an *initial object* in a category $\mathbf{C}$ is an object $\emptyset$ such that for any object X of $\mathbf{C}$, there is a unique morphism $! : \emptyset \to X$. Similarly, an initial object, if it exists, will be unique up to unique isomorphism, letting us speak of *the* initial object. Note that an initial object in $\mathbf{C}$ is the same as a terminal object in $\mathbf{C}^{op}$.

But **Cat** is a category, and we thus speak of the terminal object in **Cat** as the *terminal category*. This is just the category with a single object and a single morphism (necessarily the identity morphism on that object). We denote this $\underline{1}$ (or sometimes $\mathbf{1}$).

Let $t: \mathbf{J} \to \underline{1}$ denote the unique functor to the terminal category. Suppose, given a category $\mathbf{C}$, we are given an object $X \in \mathrm{Ob}(\mathbf{C})$. Such an object can be represented by the functor $X: \underline{1} \to \mathbf{C}$. Then, precomposing this functor with t to get $X \circ t: \mathbf{J} \to \mathbf{C}$ will just give us the *constant functor* at X, where this sends each object in $\mathbf{J}$ to the same $\mathbf{C}$-object X and every morphism in $\mathbf{J}$ to the identity id_X on that object. In other words, composing with t induces a functor $\mathbf{C} \cong Fun(\underline{1}, \mathbf{C}) \to Fun(\mathbf{J}, \mathbf{C})$, which is commonly denoted $\Delta_t: \mathbf{C} \to Fun(\mathbf{J}, \mathbf{C}) = \mathbf{C}^{\mathbf{J}}$. Altogether, this actually gives $\Delta: \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ that takes an object X to the constant functor at X and a morphism $f: X \to Y$ to the *constant natural transformation*, where each component is defined to be the morphism f. One can observe that each arrow $f: X \to Y$ in $\mathbf{C}$ will induce a natural transformation $\Delta(X) \xrightarrow{\Delta(f)} \Delta(Y)$ such that

$$\begin{array}{ccccc} (\Delta X)(i) & \xrightarrow{\Delta(f)_i} & (\Delta Y)(i) & \qquad & i \\ {\scriptstyle (\Delta X)(e)}\downarrow & & \downarrow{\scriptstyle (\Delta Y)(e)} & & \downarrow{\scriptstyle e} \\ (\Delta X)(j) & \xrightarrow[\Delta(f)_j]{} & (\Delta Y)(j) & & j \end{array}$$

commutes for each edge e of the indexing category $\mathbf{J}$. But recall that the constant functor just sends every object to itself and assigns the identity map to each edge, so the previous diagram in fact just reduces to

$$\begin{array}{ccc} X & \xrightarrow{\Delta(f)_i} & Y \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\ X & \xrightarrow[\Delta(f)_j]{} & Y \end{array}$$

which obviously commutes.

If we now consider, for an arbitrary $\mathbf{J}$-diagram $F: \mathbf{J} \to \mathbf{C}$ and for $X \in \mathbf{C}$, the arrows (which are in fact natural transformations)

$$\Delta X \longrightarrow F \qquad\qquad F \longrightarrow \Delta X,$$

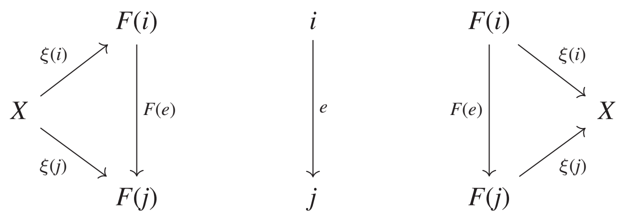
we get that a typical arrow in $\mathbf{C}^{\mathbf{J}}$ corresponding to these arrows is just a natural transformation, that is, a family of arrows of $\mathbf{C}$,

$$(\Delta X)(i) \xrightarrow{\xi(i)} F(i) \qquad\qquad F(i) \xrightarrow{\xi(i)} (\Delta X)(i)$$

indexed by the various objects or nodes of $\mathbf{J}$ and such that

$$\begin{array}{ccccccccc} (\Delta X)(i) & \xrightarrow{\xi(i)} & F(i) & \qquad & i & \qquad & F(i) & \xrightarrow{\xi(i)} & (\Delta X)(i) \\ {\scriptstyle (\Delta X)(e)}\downarrow & & \downarrow{\scriptstyle F(e)} & & \downarrow{\scriptstyle e} & & {\scriptstyle F(e)}\downarrow & & \downarrow{\scriptstyle (\Delta X)(e)} \\ (\Delta X)(j) & \xrightarrow[\xi(j)]{} & F(j) & & j & & F(j) & \xrightarrow[\xi(j)]{} & (\Delta X)(j) \end{array}$$

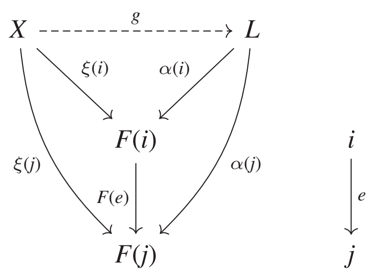
commute for each such edge $e: i \to j$ in $\mathbf{J}$. But when we apply the functor Δ, these commutative squares collapse to the commutative triangles



The definitions guarantee that whenever the indexing category has composable edges, the corresponding composite triangles commute. The natural transformations represented by the triangles on the left give a *left solution* for the diagram in **C**, sometimes also called a *cone over* the diagram F with *summit* vertex X. The natural transformations represented by the triangles on the right give a *right solution* for the diagram, also called a *cocone for* (or *cone under*) the diagram F with *nadir* X.[52]

We can then take such gadgets and use them to form the category of cones, where an object in the category of cones over F will be a cone over F, with some summit, while a morphism from a cone $\xi : X \Rightarrow F$ to a cone $\mu : Z \Rightarrow F$ is a morphism $f : X \to Z$ in **C** such that for each index $j \in \mathbf{J}$, $\mu_j \circ f = \xi_j$, that is, a map between the summits such that each leg of the domain cone factors through the corresponding leg of the codomain cone.

Using these notions, we can define the *limit* of F in terms of a universal cone, where a cone $\alpha : L \to F$ with vertex L is universal with respect to F provided for every cone $\Delta X \to F$, there is a unique map $g : \Delta X \to L$ making



commute. In such a case, one usually refers (somewhat improperly) to the universal cone by just the vertex L, and calls this the limit of F.

For reasons that will become clearer after chapter 6 on the Yoneda results and representability, we can also see a limit for a diagram $F : \mathbf{J} \to \mathbf{C}$ as a representation for the corresponding functor $Cone(-, F) : \mathbf{C}^{op} \to \mathbf{Set}$, sending $X \in \mathbf{C}$ to the set of cones over F with summit X.[53] The limiting cone will fundamentally be *universal* in the sense that for any other cone over F, there will exist a unique arrow from the summit of that cone to the summit of the limiting cone, that is, any other cone must pass uniquely through the limiting cone if it wants to pass down to F. In short,

52. I hope this is already clear, but in case it is not: the terminology of "over" and "under" has to do with the fact that the above triangles can be presented as rotated clockwise 90 degrees, as we did earlier.

53. While we could consider limits and colimits in any category, something called the Yoneda lemma (discussed in chapter 6) assures us that the constructions of (co)limits of diagrams valued in the category **Set** suffice to provide formulae for (co)limits in any category. To ensure that we have a *set* of cones, we need only assume that the diagram is indexed by a small category **J** and that **C** is locally small, thereby guaranteeing that the functor category $\mathbf{C}^{\mathbf{J}}$ is locally small. Under certain conditions, we can weaken these restrictions.

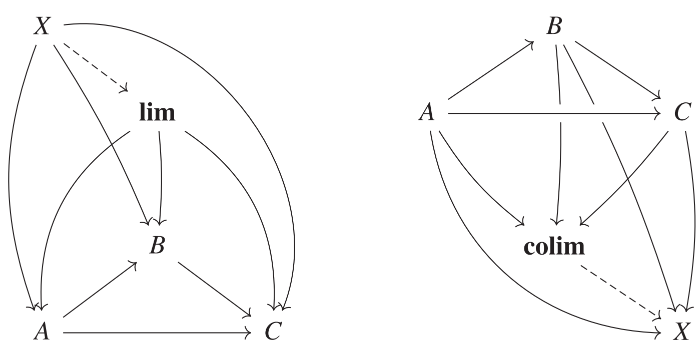
Definition 57 The *limit* of a diagram $F:\mathbf{J}\to\mathbf{C}$ is an object $\lim F$ in $\mathbf{C}$ together with a natural transformation $\eta:\lim F\Rightarrow F$ satisfying the following universal property:

> for any object X and for any natural transformation $\alpha: X\Rightarrow F$, there is a unique morphism $g: X\to \lim F$ such that $\alpha=\eta\circ g$.

The dual construction leaves us with a category of cocones **CoCones**(F), wherein the universal cocone emerges as the *colimit* of the diagram F, denoted colim F, where this can be seen as a representation for $Cone(F,-)$, forcing all cocones to receive maps from the colimiting cone if they want to receive maps from F. Explicitly,

Definition 58 The *colimit* of a diagram $F:\mathbf{J}\to\mathbf{C}$ is an object colim F in $\mathbf{C}$ together with a natural transformation $\epsilon: F\Rightarrow \text{colim } F$ that satisfies that for any object X and any natural transformation $\beta: F\Rightarrow X$, there is a unique morphism $h:\text{colim } F\to X$ such that $\beta=h\circ\epsilon$.

Altogether, in terms of where we began this discussion, for such a diagram, we can picture the limit and colimit as follows:



But while this gives us a nice and literal "cone-like" way of picturing things, there is no reason why we should have to restrict ourselves to diagrams of precisely such a shape—by considering diagrams of different shapes, taking limits and colimits of such diagrams will again recover a number of important constructions found across mathematics. Let us start with some examples of limits.

3.1.2 Examples of Limits

Example 59 An extreme case, where we take the limit of the *empty diagram*—the diagram that has no objects and no morphisms—yields a construction we have in fact already met: the *terminal object*. One can check that the limit of the empty diagram will be just one object, if one exists, that has the property that there is a unique morphism to it from every object in the category. While not every category has a terminal object, if it does, then it is unique.

Concrete instances of this include the following:

- In **Set**, the terminal object is given by the *singleton set* $*$, for if X is any set, then there is only one possible function $X\to *$, the one that just takes everything to $*$.

- In a poset $\mathcal{P}$, the terminal object will be its *greatest element*, when such a thing exists. In other words, it will be an element l such that $p \leq l$ for all $p \in \mathcal{P}$. In particular, then, in the poset $[0, \infty]$, the terminal object is ∞, as $x \leq \infty$ for all $x \in [0, \infty]$.
- For bouquets, the terminal object will be given by a single loop stationed at a vertex.
- In **Top**, the terminal object is the *one-point space*, that is, the one-point set $*$ equipped with the indiscrete topology (described explicitly in chapter 4); note that if X is any space, then there will be only one possible continuous function from X to this one-point space.
- In **Group**, the terminal object is given by the *group with one element* $\{e\}$; note that if G is any group, then there will be only one possible group homomorphism $G \to \{e\}$.

Example 60 Suppose we instead use a *discrete diagram*, where this is one that consists of just objects (dots), with only identity morphisms. Taking the limit of such a diagram yields some very familiar "product-like" constructions, when specialized to familiar categories. Again, products need not exist—for instance, in particular, not every poset has all products. For some concrete examples of this construction, we have the following:

- In **Set**, the limit of a discrete diagram consisting of sets $X_1, X_2, \ldots$, is the *Cartesian product* $\prod_{i \in I} X_i$, where this construction comes with projection maps $\prod_{i \in I} X_i \to X_i$ onto each of the factors.
- In a poset $\mathcal{P}$, the limit of a discrete diagram (set of elements $p_1, p_2, \ldots$) is their *infimum* (or *greatest lower bound*) $\bigwedge_{i \in I} p_i$.[54]
- In **Top**, the limit is given by the Cartesian product $\prod_{i \in I} X_i$ equipped with the product topology.

Example 61 Suppose we have a diagram of shape

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

In a category **C**, the limit of a diagram of such a shape is called the *pullback* (or *fibered product*)—this will consist of an object together with morphisms satisfying the stipulated universal property. Explicitly, we can define this as follows:

Definition 62 Given any two maps with common codomain, as in

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z, \end{array}$$

by their *pullback* (or *fibered product*) we mean a pair of maps π_0, π_1 with common domain P that

54. In general, in the context of posets, given a subset $S \subseteq P$, we say that $p \in P$ is a *lower bound* for S if for all $s \in S$, $p \leq s$. The *infimum* $\inf(S)$ of S, provided it exists, is then an element p such that (1) p is a lower bound for S and (2) if p' if another lower bound for S, then $p' \leq p$. It is common to write $\bigwedge_{a \in A} a$ for $\inf(A)$; we also use $\wedge$, especially when considering the infimum of a pair of elements x and y, written $x \wedge y$, where this is referred to as the *meet* of x and y.

1. forms a commutative square $f \circ \pi_0 = g \circ \pi_1$,

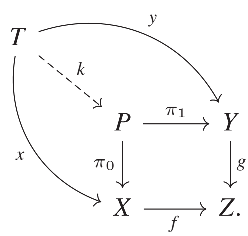
$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & Y \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

   and also

2. is universal among all such commutative squares, that is, for any T, x, y

$$\begin{array}{ccc} T & \xrightarrow{y} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

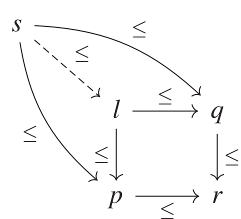
   if $f \circ x = g \circ y$, then there exists a unique map $k : T \to P$ such that $x = \pi_0 \circ k$ and $y = \pi_1 \circ k$, as in



In a poset $\mathscr{P}$, the pullback of a diagram

$$\begin{array}{ccc} & & q \\ & & \downarrow{\scriptstyle \leq} \\ p & \xrightarrow[\leq]{} & r \end{array}$$

will be given by an element l, with $l \leq p$ and $l \leq q$, that moreover satisfies that for any element s for which it is also the case that $s \leq p$ and $s \leq q$, we have $s \leq l$.



In other words, l will be the *greatest lower bound* of p and q.

In a poset, it turns out that the pullback reduces to the product. However, in more general categories, the two need not coincide.

Working in **Set**, **Top**, **Group**, in particular, the pullback will consist of (1) the subset (or subspace, or subgroup) of the product $X \times Y$ that comprises pairs (x, y) such that $f(x) = g(y)$, together with (2) two projection morphisms $X \times_Z Y \to X$ and $X \times_Z Y \to Y$, the first mapping (x, y) onto the first factor x, and the second onto the second factor y. What, then, will this pullback be like? The short answer is that it depends not just on the category we are working within but also on how the data of this category is defined: in particular, on what the given objects are and how the morphisms are defined. By adopting certain objects, or choosing certain given morphisms, this construction recovers a number of other familiar constructions. For instance, even just confining our attention to **Set**, the following can all be constructed using the pullback construction:

- Suppose $Z = *$, the singleton set. Then, as $*$ is the terminal object in **Set**, both $X \xrightarrow{f} *$ and $Y \xrightarrow{g} *$ are the unique functions taking everything to $*$. The pullback of such a diagram would then be all pairs (x, y) such that both x and y are sent to $*$ under the (unique) functions f and g—but there is no pair that does not satisfy this requirement. Thus, we recover *all* pairs (x, y), making the pullback of such a diagram the entire set $X \times Y$, the usual binary Cartesian product.
- Now suppose $Y = *$, while Z is any set. A function $* \xrightarrow{g} Z$ just picks out an element $z \in Z$. Then, for any function $X \xrightarrow{f} Z$, the pullback will be the subset of elements in X that are sent to z by f, recovering the usual *preimage* (or *fiber*) of f over g construction.
- Now suppose that X and Y are subsets of Z, making f and g inclusions,

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle g} \\ X & \xhookrightarrow[f]{} & Z \end{array}$$

  In such a case, the pullback will consist of pairs (x, y) such that x and y are *equal* on being included into Z. In other words, the pullback will consist of the elements $x = y$ of X that are also in Y (and vice versa)—and this is just the *intersection* $X \cap Y$.

Example 63 Suppose we have a diagram of shape

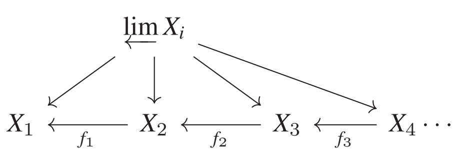
$$\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \cdots$$

In a category **C**, the limit of such a diagram is called an *inverse limit*. It will comprise an object together with morphisms from that object to each $\bullet$ such that each of the resulting triangles commutes, and the universal property of the limit is satisfied.

Explicitly, supposing $X_1, X_2, \ldots$ are objects in **C**, then the inverse limit, denoted $\varprojlim X_i$, of the diagram

$$X_1 \xleftarrow[f_1]{} X_2 \xleftarrow[f_2]{} X_3 \xleftarrow[f_3]{} X_4 \cdots$$

would amount to an object with maps into each of the X_i, making the resulting triangles commute. In **Set**, for instance, this would be a subset $\varprojlim X_i$ of the product $\prod_{i \in I} X_i$ containing all sequences $(x_1, x_2, x_3, \ldots)$ where the i-th factor is such that $f_i(x_{i+1}) = x_i$.



In **Top** and **Group**, among other categories, the inverse limit would also look just like this. Note that in **Set**, if we have sets related by inclusion, so that $X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \cdots$, then the inverse limit is just the intersection $\bigcap_{i \in I} X_i$.

Example 64 Suppose we have a diagram indexed by the category consisting of two objects and two parallel nonidentity morphisms,

$$\bullet \rightrightarrows \bullet$$

Of course, a diagram in **C** of this shape will just be a parallel pair of morphisms

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

living in the target category **C**. What is a cone over this diagram? Well, a cone with summit C will consist of a pair of morphisms $h: C \to X$ and $i: C \to Y$ such that $f \circ h = i$ and $g \circ h = i$, which together just assert that $f \circ h = g \circ h$. In short, a cone over such a parallel pair of arrows f, g can be represented by a single morphism $h: C \to X$ such that $f \circ h = g \circ h$. We can then define E together with $e: E \to X$, called the *equalizer of* f and g, as the universal arrow with this same property, that is, $f \circ e = g \circ e$. To be more explicit, the universal property in question asserts that given any $h: C \to X$ such that $f \circ h = g \circ h$, there exists a unique $k: C \to E$ that factors the morphism h through e in the sense that $e \circ k = h$, as summed up in the diagram

$$\begin{array}{ccccc} C & & & & \\ {\scriptstyle k}\downarrow & \searrow^{h} & & & \\ E & \xrightarrow{e} & X & \underset{g}{\overset{f}{\rightrightarrows}} & Y. \end{array}$$

In **Set**, the equalizer of f, g is then a subset of elements of X for which the two given functions coincide,

$$E = Eq(f, g) := \{x \in X \mid f(x) = g(x)\},$$

which set is accompanied by the natural *inclusion* $e: Eq(f, g) \to X$ into X.

In terms of graphs, since a graph can be defined using a pair of functions $A \underset{t}{\overset{s}{\rightrightarrows}} V$, where A stands for arrows and V for vertices, and where s just picks out the source vertex of an arrow and t the target vertex, consider that for each graph G we can find its set of length one loops via the equalizer construction $Eq(G)$:

$$Eq(s, t) \xrightarrow{e} A \underset{t}{\overset{s}{\rightrightarrows}} V.$$

Observe that the equalizer assignment is in fact functorial, thus incidentally furnishing us with another example of a functor, since given a graph homomorphism $G \to G'$, there will be an induced function $Eq(G) \to Eq(G')$.

Let us also take this opportunity to define the following notion, capturing when arrows can be canceled on one side:

Definition 65 A morphism $i: B \to C$ in a category is called a *monomorphism* (or *monic* morphism) provided for any A with parallel morphisms f, g as in

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{i} C,$$

$i \circ f = i \circ g$ implies $f = g$.[55]

55. This is a categorical generalization of the set theoretic notion of an injective function.

Observe that an equalizer of two morphisms is automatically a monomorphism. To emphasize that a particular morphism is a monomorphism, it is common to use the decorated arrow $\rightarrowtail$ (or $\hookrightarrow$ when it is an inclusion).

Certain conditions can ensure that a given mapping is a monomorphism. One such condition is that the mapping has something called a *retraction*, another useful notion we record here (together with its dual notion of a *section*):

Definition 66 For mappings r, s in any category, r is said to be a *retraction* for s provided $r \circ s$ is an identity mapping. In such a situation, s is said to be a *section* for r.

So if i is a morphism from X to Y, r is a retraction provided we have

$$X \underset{i}{\overset{r}{\rightleftarrows}} Y \qquad \text{with } r \circ i = \mathrm{id}_X.$$

A given i may or may not have a retraction, and if there is such a map, there may in fact be many retractions. Likewise, a given morphism need not have sections. However, if there is at least one retraction r for a given i, then i will be a monomorphism.

Finally, observe also that any section will be a monomorphism. A section for a mapping f might be thought of as a procedure that picks out an element from each of the fibers of f. We will have more to say on this in subsequent chapters.

3.1.3 Examples of Colimits

Limits and colimits are dual notions, meaning that colimits in a category are just limits in the opposite category, so by dualizing the above constructions (same definitions, but with all arrows reversed), we can thus expect to get specific examples of colimits. These will include: initial objects, coproducts (disjoint unions), pushouts, direct limits (denoted $\varinjlim$), and coequalizers. We will focus on explicitly constructing just a few of these.

Example 67 First consider the coproduct. This is the colimit of the discrete diagram of shape

$$\bullet \qquad \bullet$$

and is usually written $\coprod$, so that with a diagram with objects $X(i)$ and $X(j)$, the colimit of such a diagram is written $X(i) \coprod X(j)$. This binary case can be generalized to more objects, $\coprod_i X(i)$. For some concrete examples of this construction, we have the following:

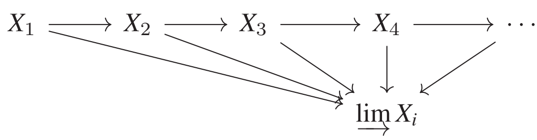
- In **Set**, the colimit of a discrete diagram consisting of sets $X_1, X_2, \ldots,$ is the *disjoint union* $\bigsqcup_{i \in I} X_i$, where this construction comes with injective functions of each X_i into the coproduct set $X = \bigsqcup_{i \in I} X_i$ such that each element of X belongs to exactly one of the images of the injections. If the sets being summed are pairwise disjoint, the disjoint union just becomes the standard union $\bigcup$.
- In a poset $\mathcal{P}$, the colimit of a discrete diagram (set of elements $p_1, p_2, \ldots$) is their *supremum* (or *least upper bound*) $\bigvee_{i \in I} p_i$, where this construction also consists of the

inequalities $p_i \leq \bigvee_{i \in I} p_i$.[56] Guided by this, and considering the importance of order theory as a microcosm for the more general categorical notions, one can see colimits as a generalization of suprema or joins (just as limits generalize infima or meets).

Example 68 Dual to inverse limits are *direct limits*, where this is the colimit of a diagram indexed by the ordinal category ω. In other words, for a diagram

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow X_4 \longrightarrow \cdots$$

its colimit is the direct limit $\varinjlim X_n$, defining a diagram of shape $\omega + 1$:



The term "direct limit" is occasionally used to designate colimits of any shape.

Observe then that the colimit of a sequence of sets with the inclusions

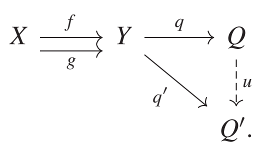
$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

recovers their union $\bigcup_{n \geq 0} X_n$.

Example 69 We have already met the construction called an *equalizer*, where this is the limit of a diagram of shape

$$\bullet \rightrightarrows \bullet.$$

In a similar fashion, we can form the dual notion of a *coequalizer* by taking the colimit of the diagram consisting of two objects X, Y and two parallel morphisms $f, g: X \to Y$, which gives us something like the categorical generalization of taking a quotient by an equivalence relation. More explicitly, a coequalizer is defined as an object Q (sometimes denoted $Coeq(f, g)$, wanting to stress the arrows it coequalizes) together with a morphism $q: Y \to Q$ such that $q \circ f = q \circ g$, where the pair (Q, q) must moreover be universal in the sense that given any other pair (Q', q') there exists a unique morphism $u: Q \to Q'$ so that $u \circ q = q'$, as in the diagram



In **Set**, the coequalizer of two functions $f, g: X \to Y$ is just the *quotient* of Y by the smallest equivalence relation $\sim$ such that for all $x \in X, f(x) \sim g(x)$.

56. In general, just as with the notion of greatest lower bounds, given a subset $S \subseteq P$ of a poset, we say that $p \in P$ is an *upper bound* for S if for all $s \in S$, $s \leq p$. The *supremum* $\sup(S)$ of S, provided it exists, is then an element p such that (1) p is an upper bound for S and (2) if p' if another upper bound for S, then $p \leq p'$. It is common to write $\bigvee_{a \in A} a$ for $\sup(A)$; we also use $\vee$, especially when considering the supremum of a pair of elements x and y, written $x \vee y$, where this is referred to as the *join* of x and y. Incidentally, we can use this notion (together with that of meets) to define an important entity we will meet throughout the book: a *lattice* is a poset for which every pair of elements has a join and a meet. We will wait until chapter 7 to more formally introduce lattices and explore them in greater depth.

In the context of bouquets, the coequalizer of two inclusion morphisms of a vertex into an object consisting of individual loops stationed at different vertices would be given by an object that glued those loops together onto a single (the same) vertex—an example that makes it evident how colimits can be seen as involving some sort of gluing.

Various categories—including graphs, reflexive graphs, discrete dynamical systems, the category of elements (see below, definition 71)—support a construction that allows us to count the *connected components* in that category. For concreteness, we stick for now with the case of the category of graphs, **Grph**, and consider the "connected components" functor $\Pi_0 : \mathbf{Grph} \to \mathbf{Set}$. This is actually obtained via the coequalizer construction:

$$A \underset{s}{\overset{t}{\rightrightarrows}} V \overset{q}{\dashrightarrow} Coeq(s,t).$$

$Coeq(s,t)$ is defined as $V/\sim$, where $\sim$ is an equivalence relation on V, that is, $s(x) \sim t(x)$ for all $x \in A$, and where q is the quotient function $q : V \to V/\sim$. This construction accordingly acts to identify all arrows where the source of one arrow is equal to the target of the other. In other words, all we are doing is picking out the connected components of the graph. This assignment of the set of connected components of a graph can be shown to be functorial as well. In a moment, we will see more of this construction in action.

Finally, let us also take the opportunity to define the dual of the notion of a monomorphism.

Definition 70 $f : X \to Y$ is said to be an *epimorphism* (or *epi* for short) if for all B, $h, h' : Y \to B$,

$$X \xrightarrow{f} Y \underset{h'}{\overset{h}{\rightrightarrows}} B$$

$h \circ f = h' \circ f$ implies $h = h'$.

Observe that every coequalizer is automatically an epimorphism, and that moreover every retraction is automatically an epimorphism. For graphical emphasis, epimorphisms are sometimes displayed using $\twoheadrightarrow$.

Before moving on, let us now consider an alternative way to view things.

Definition 71 Let $\mathbf{C}$ be a category, and let $F : \mathbf{C} \to \mathbf{Set}$ be a (covariant) functor. Then the *category of elements of* F, denoted $\int_{\mathbf{C}} F$ (or just $\int F$ if the context is clear), is defined:

$$\text{Ob}\Big(\int F\Big) = \{(c,x) \mid c \in \mathbf{C}, x \in F(c)\}$$

$$\text{Hom}_{\int F}\big((c,x),(c',x')\big) = \{f : c \to c' \mid F(f)(x) = x'\}.$$

Similarly for the contravariant case: for $F : \mathbf{C}^{op} \to \mathbf{Set}$ the *category of elements of* F, denoted $\int_{\mathbf{C}^{op}} F$ (or just $\int F$), is defined:

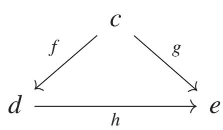
$$\text{Ob}\Big(\int F\Big) = \{(c,x) \mid c \in \mathbf{C}, x \in F(c)\}$$

$$\text{Hom}_{\int F}\big((c,x),(c',x')\big) = \{f : c \to c' \mid F(f)(x') = x\}.$$

Associated with these constructions are the functors $\pi_F : \int F \to \mathbf{C}$, called the projection functors, sending each object $(c,x) \in \text{Ob}(\int F)$ to the object $c \in \text{Ob}(\mathbf{C})$ or $\text{Ob}(\mathbf{C}^{op})$, and each morphism $f : (c,x) \to (c',x')$ to the morphism $f : c \to c'$, that is, $\pi(f,(c,x),(c',x')) = f$.

As a concrete instance of this, recall the vertex coloring functor *nColor* from example 37 (chapter 2). An object in the category of elements $\int nColor$ of this functor *nColor* will be a graph together with a chosen n-coloring, that is, objects are n-colored graphs. A morphism $\phi : G \to G'$ between a pair of n-colored graphs will be a graph homomorphism $\phi : G \to G'$ so that the induced function $nColor(\phi) : nColor(G') \to nColor(G)$ takes the chosen coloring of G' to the chosen coloring of G, that is, the graph homomorphism ϕ will preserve the chosen colorings in the sense that each red vertex of G will be carried to a red vertex of G'. In short, then, $\int nColor$ is the category of n-colored graphs and the color-preserving graph homomorphisms between them.

For another example, recall the hom-functors, first introduced in example 42 (chapter 2). Objects in the category of elements of $\mathrm{Hom}_{\mathbf{C}}(c, -)$ are the morphisms $f : c \to d$ in $\mathbf{C}$. A morphism from $f : c \to d$ to $g : c \to e$ is then a morphism $h : d \to e$ such that $g = h \circ f$. h is said to be a morphism *under* c because of the diagram attached to this condition:



This category is none other than the *co-slice category* of objects *under* the $c \in \mathbf{C}$. Note that the forgetful functor $U : c/\mathbf{C} \to \mathbf{C}$ sends an object $f : c \to d$ to the codomain, and takes a morphism (a commutative triangle) to the arrow opposite the object c, that is, to h in the above instance. We could also construct the dual category of elements $\int \mathrm{Hom}_{\mathbf{C}}(-, c)$ in terms of the *slice category* $\mathbf{C}/c$ *over* the object $c \in \mathbf{C}$.[57]

The category of elements is rather significant because any universal property can be seen as defining an initial or terminal object in this category. In particular, it turns out that for any small functor[58] $F : \mathbf{C} \to \mathbf{Set}$, we have

$$\mathrm{colim}\, F \cong \Pi_0\Big(\int F\Big),$$

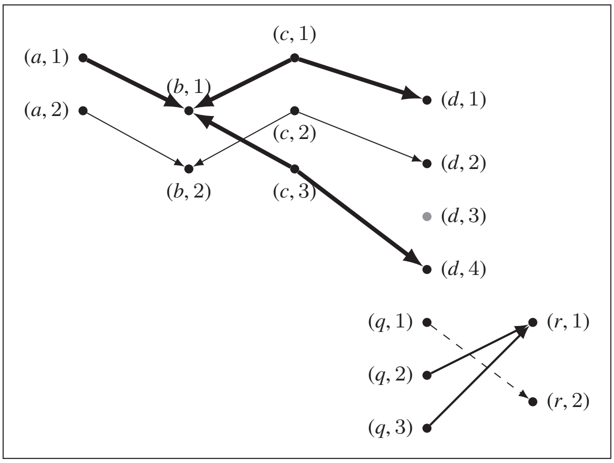
where Π_0 operates by picking out the connected components and $\int F$ is the category of elements of F. So, in other words, the set of connected components of the category of elements of a functor F, $\Pi_0(\int F)$, is isomorphic to the colimit of F. Alternatively, in general, we could just have said that a colimit is an initial object in the category $\int Cone(F, -)$, and we note that the forgetful functor $\int Cone(F, -) \to \mathbf{C}$ will take a cone to its nadir.

To see this in action, recall the functor (diagram)

57. In this connection, we can mention the important result that for $\mathbf{C}$ small and P a presheaf on $\mathbf{C}$, one can show an equivalence of categories

$$\mathbf{Set}^{\mathbf{C}^{op}}/P \simeq \mathbf{Set}^{(\int_{\mathbf{C}} P)^{op}}.$$

58. A functor or diagram is small if its indexing category is small.



from example 36 (chapter 2). The various thicknesses or shadings in this picture can now be explained. The picture above is in fact a representation of the category of elements of F. The various thicknesses or shadings depict the result of taking its connected components. By inspection, one can verify that this is just the set

$$\{[(a,1)],[(a,2)],[(d,3)],[(q,1)],[(q,2)]\},$$

where each element is a representative of one of the components, which is in turn isomorphic to a set of cardinality 5, entailing that $\mathrm{colim}_{\mathbf{J}}F \cong$ a set with 5 elements.

As for limits, we could also show that the *limit* of any small functor $F:\mathbf{C}\to\mathbf{Set}$ is isomorphic to the set of functors $\mathbf{C}\to\int F$ that define a *section* to the canonical projection $\pi:\int F\to\mathbf{C}$. Alternatively, we can define the limit as a terminal object in the category of elements of cones over F, that is, in $\int Cone(-,F)$. Note also that the forgetful functor $\int Cone(-,F)\to\mathbf{C}$ will send a given cone to its summit.

3.1.4 Further Notions: (Co)Complete, (Co)Continuous

We have seen how both the limiting and the colimiting cones are universal in the sense of acting as a kind of doorkeeper or special mediator for all other cones. Such universal objects need not exist. However, we define a category $\mathbf{C}$ as *complete* if it admits limits of all small diagrams valued in $\mathbf{C}$, and as *cocomplete* if it admits all colimits of all small diagrams valued in $\mathbf{C}$.

We should also take the opportunity to supply the important definition:

Definition 72 A functor is *(co)continuous* if it preserves all small (co)limits,

where *preservation* is defined as follows: for any class of diagrams $K:\mathbf{J}\to\mathbf{C}$ valued in $\mathbf{C}$, a functor $F:\mathbf{C}\to\mathbf{D}$ is said to *preserve* limits if for any diagram K and limit cone over K, the image of this cone under the action of the functor defines a limit cone over the (composite) diagram $F\circ K:\mathbf{J}\to\mathbf{D}$.[59]

59. Importantly, in chapter 6 on the Yoneda results, we will learn about *representable functors*, and how covariant representable functors preserve all limits, taking limits in $\mathbf{C}$ to limits in $\mathbf{Set}$; while contravariant representable functors preserve all limits in $\mathbf{C}^{op}$, taking colimits in $\mathbf{C}$ to limits in $\mathbf{Set}$. We will learn about something called the Yoneda embedding $\mathbf{y}:\mathbf{C}\hookrightarrow\mathbf{Set}^{\mathbf{C}^{op}}$, which preserves all limits that exist in $\mathbf{C}$; and the dual embedding $\mathbf{y}:\mathbf{C}^{op}\hookrightarrow$

Intuitively, this concept of a (co)continuous functor as a special sort of functor that takes universal objects in the source category to universal objects in the target category can be thought of as follows: whatever else the functor does to objects as it sends them from one category to another, a (co)continuous functor will send the entity that acted as a privileged mediator or doorkeeper ((co)limit) in relation to the rest of the objects of the source category to an entity that plays the similar role of privileged mediator or doorkeeper for its fellows in the target category. We could label this characterization of continuity via the preservation of the roles of privileged mediators or doorkeepers/gatekeepers as *katholic continuity* (after the Greek word *katholou* meaning "universal," and perhaps also evoking connections with *katechon*, sometimes identified in theological contexts with the figure of the gatekeeper who indirectly enforces lawfulness by restraining chaos or lawlessness).

3.2 Philosophical Pass: Universality and Mediation

The katholic understanding of (co)continuity as a preservation of (co)limits is very important for certain definitions of sheaves to follow. The pivotal notion of a *(co)continuous functor* as a special sort of functor that takes universal objects in the source category to universal objects in the target category, moreover enables us to regard *continuity* as a special kind of passage or translation from one world of objects and relations to another world of objects and relations—*special* in that, in passing between worlds, it takes care to preserve the role of those that act as privileged or optimal mediators or gatekeepers for the rest of the entities of their world. Finally, the universal constructions introduced in this chapter seem to be of some philosophical interest in their own right in light of the particular conception of universality that is ushered in.

Box 3.1
On Universality

One could argue that basic category theory is the study of universal properties. Within a category, for some structure of a given kind, there may indeed be a family of such structures, that is, a number of objects of the category related in such a way that they exhibit the structure in question. A very natural question to ask, in the presence of a family of structures of the same kind, is: *Which is the optimal (most efficient) one?* (And what does "optimal" here mean?) It is here that universal properties come into play. Broadly, universality has been understood here to mean that a certain gadget occupies a privileged position in relation to other gadgets of its world that are of the same type, in that it serves as a special gatekeeper or go-between: "You have to pass (factor) through me if you want to relate in this way to anything else of this type." If such universality is thus thought of in terms of a designated object's privileged role as mediator or gatekeeper for all other objects of the same type trying to relate or interact, katholic continuity can moreover be understood as explaining how, in passing from one network of objects and relations to another, those objects that occupy the position of mediators *preserve* their roles within their respective networks; that is, in passing from one world to another, the special mediator or gatekeeper in the one world is sent, or given a direct line, to the special mediator or gatekeeper in the other world.

$\mathbf{Set}^{\mathbf{C}}$ that preserves limits in $\mathbf{C}^{op}$. For our purposes, we can highlight that it will turn out that a sheaf is a special sort of presheaf that preserves limits in this way.

It is curious that, long before category theory, a number of philosophers—such as Aristotle, Hegel, and Charles Peirce—suggested, and attempted to think through, close connections between universality, mediation, and continuity. Charles Peirce was perhaps the most insistent on the connections between the general (universality) and mediation, brought together in his notion of "thirdness," one of his three "categories" (in his own sense of the word, having nothing to do with category theory). Peirce was a systematic philosopher who developed a theory of three main categories, first designed to help develop his theory of signs (semiotics) and classify the sciences and human knowledge, but gradually extended to be of grander and grander scope. Peirce came to have rather ambitious aspirations for these categories, such that, as he thought, they furnished a classification applicable to systems of all sorts, so that ultimately all conceptions and phenomena—even elements of cosmology and physiology—at the most fundamental level could be broken down in terms of these three general categories and their interplay. As such, Peirce would come to admit that these three categories "are excessively general ideas, so very uncommonly general that it is far from easy to get any but a vague apprehension of their meaning" (Peirce 1997, 4.3). Roughly, we could describe these three categories as follows:

- *Firstness*: being considered simply in itself, or independent of anything else, as a *unit*—this involves immediacy and a kind of naive realism (insofar as it relates to perceivers and knowers like us); the paradigm is brute quality "free of relations";
- *Secondness*: being correlative to, dependent on, an effect or result of, in reaction with, or limited by something else—this involves dyadic relations corresponding to some "brute action" or finite process of reaction or resistance;
- *Thirdness*: involving mediation bringing something into relation to another. (See, for instance, Peirce (1997, 6.32 [1891]).)

There is an element of givenness or sheer fact to phenomena that fall within the scope of either Firstness and Secondness. Secondness is a dyadic relation that is irreducible to a single part. Thirdness, for its part, is generally some ternary relation that (provided it is a "genuine Third") cannot be reduced to two terms, to two-part relations. (Peirce isolated two sorts of "degenerate Thirds," which could indeed be reduced to other categories; see, for instance, "A Guess at the Riddle" [Peirce 1997, 1.3.3].) Peirce held that all other more complex relations could be reduced ultimately to combinations of triads. While, again, the true scope of applicability of the categories was meant to be extremely broad, in terms of Thirdness in particular, Peirce would speak of a Third as "every kind of sign, representative, or deputy, everything which for any purpose stands instead of something else, whatever is helpful, or mediates." For Peirce, in short, "Thirdness is nothing but the character of an object which embodies Betweenness or Mediation in its simplest and most rudimentary form; and I use it as the name of that element of the phenomenon which is predominant wherever Mediation is predominant" (Peirce 1997, 5.77 [1903]).

Here is an initial example, of a somewhat phenomenological flavor, that might help to start to give some shape to these notions:

- Firstness: the visible light of the sun;
- Secondness: a child turning its eyes to the sun and the surprise and pain that is the event of being hurt;
- Thirdness: a child watching someone else look at the sun and forming the judgment, "Oh, that must have hurt!"

As such an example is meant to suggest, the "theory of mind" and empathy involved in what is described as Thirdness above is not reducible to Secondness—the child itself being painfully affected by the light and the other person themself being painfully affected by the light. Taking either of those individually, or even simply adding them together, cannot hope to account for anything involved in the capacity for forming the empathic judgment.

An example often used by Peirce to better illustrate Thirdness and its distinction from Secondness in particular is found in the act of *giving*:

> Analyze for instance the relation involved in "*A gives B to C*." Now what is giving? It does not consist in A's putting B away from him and C's subsequently taking B up. It is not necessary that any material transfer should take place. It consists in A's making C the possessor according to Law. There must be some kind of law before there can be any kind of giving—be it but the law of the strongest. But now suppose that giving did consist merely in A's laying down the B which C subsequently picks up. That would be a degenerate form of Thirdness in which the thirdness is externally appended. In A's putting away B, there is no thirdness. In C's taking B, there is no thirdness. But if you say that these two acts constitute a single operation by virtue of the identity of the B, you transcend the mere brute fact. (Peirce 1997, 8.331 [Letter to Lady Welby])

As he would stress elsewhere,

> [W]e cannot build up the fact that A presents C to B by any aggregate of dual relations between A and B, B and C, and C and A. A may enrich B, B may receive C, and A may part with C, and yet A need not necessarily *give* C to B. For that, it would be necessary that these three dual relations should not only coexist, but be welded into one fact. (Peirce 1997, 1.3 ["A Guess at the Riddle"])

Still another related example might be found in that of a legal contract. Such a thing cannot be accounted for just by the combination of two dyadic relations: the first being A's signature on document C and the second being B's signature on document C. As Peirce would stress, the nature of such a contract in fact lies in the intent and existence of the contract, which amounts to certain conditional rules governing the future behavior of A and B (see Peirce 1997, 1.475 [ca. 1896]), where this is not reducible to the component dyads (signatures), but amounts to a bringing together of these two dyads into a relationship binding for the future.

To stress another aspect of the difference between Second and Third, Peirce sometimes used the example of a jurisdiction's law enforcement. Here, an uninterpreted feeling of fear might be First, and the law court's injunctions and judgments would involve Thirdness. But once "I feel the sheriff's hand on my shoulder, I shall begin to have a sense of actuality" (Peirce 1997, 1.24 [1931–1958])—here we have Secondness. Whereas action confronting one with another is Secondness, Thirdness is to be understood as involving rule-governed conduct, predictions on future behavior, and habit-formation legislating over potential actions and interactions. As such, the lawfulness of any phenomenon exhibiting Thirdness has a decidedly *general* flavor.

Peirce would come to think of Thirdness as present in any case of generality (universality). Strictly speaking, both Firstness and Thirdness involve elements of generality, yet of a different sort: Firstness involves a latent potentiality, a sort of vague generality, the generality of qualitative immediacy and the indeterminacy involved in this. Peirce calls the generality of Firstness *negative generality*. The generality of Thirdness, on the other hand, is one of *necessity*, which is why Peirce made Thirdness the domain of law of nature or rule—this generality of Thirdness is called *positive generality*.

While there is a great deal more that could be said of all this, I will just also note that it is curious that, for Peirce, the sort of generality of Thirdness was, fundamentally, "nothing but a rudimentary form of true continuity," forming a basis for the most demanding conception of continuity, whose articulation he struggled with for most of his life. (See also Peirce's remark that continuity "represents Thirdness almost to perfection" [Peirce 1997, 1.337]. Zalamea [2012] has a number of fascinating discussions of Peirce's evolving understanding of continuity, which the reader is encouraged to consult.)

While, of course, there is no expectation or desire here of trying to map the categories of Peirce's classification onto the mathematics we have been developing, I think the connections between *generality* (of which universality is an extreme form) and *mediation*, as intimated by Peirce, provide an interesting perspective from which to view the category-theoretic formulation of universality in terms of this special mediator or gatekeeper property. It is interesting, as well, that in making this connection more exact, continuity (of the katholic sort) also reemerges in a decisive way. At another level, I think it is also useful to consider how, for a given category, it does often appear that the objects of a category are something like Firsts, while the morphisms are plausibly something like Seconds—where there is even an element of *givenness* to these ("these are just the objects, the units; and these are just how they happen to relate or act on one another"). Universal constructions—limits and colimits—for their part, do seem to fall into something like Thirdness: they cannot be reduced to a single object or isolated morphism relating two objects, but would seem to essentially involve an act of mediation relating or factoring any other morphisms or patterns to others within the category. And it is with such constructions, perhaps more than anywhere else, that the "law-governedness" of a category often emerges.

Altogether, with the category-theoretic notions of (co)limits, we are given a more precise way of thinking about universality as a form of *mediation*—an *optimal* mediation. With the category-theoretic notion of (co)continuous functors, built out of the material of such universal constructions, we are given a more precise way of thinking about continuity in terms of a process or transformation that preserves the special role played by those optimal mediators or gatekeepers.

Sheaf Theory through Examples

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Citation:

Sheaf Theory through Examples
By: Daniel Rosiak
DOI: 10.7551/mitpress/12581.001.0001
ISBN (electronic): 9780262370424
Publisher: The MIT Press
Published: 2022

The open access edition of this book was made possible by generous funding and support from Arcadia – a charitable fund of Lisbet Rausing and Peter Baldwin, and MIT Press Direct to Open

The MIT Press

The open access edition of this book was made possible by generous funding from Arcadia—a charitable fund of Lisbet Rausing and Peter Baldwin.

The MIT Press would like to thank the anonymous peer reviewers who provided comments on drafts of this book. The generous work of academic experts is essential for establishing the authority and quality of our publications. We acknowledge with gratitude the contributions of these otherwise uncredited readers.

This book was set in LaTeX by the author.

Library of Congress Cataloging-in-Publication Data

Names: Rosiak, Daniel, author.
Title: Sheaf theory through examples / Daniel Rosiak.
Description: Cambridge, Massachusetts : The MIT Press, [2022] | Includes bibliographical references and index.
Identifiers: LCCN 2021058949 | ISBN 9780262542159 (paperback)
Subjects: LCSH: Sheaf theory.
Classification: LCC QA612.36 .R67 2022 | DDC 514/.224–dc23/eng20220521
LC record available at https://lccn.loc.gov/2021058949