This is a portion of the eBook at doi:10.7551/mitpress/12581.001.0001

6 There's a Yoneda Lemma for That

In which we cover what is perhaps the most important idea in category theory—the Yoneda results—by first focusing on what these results look like "in the miniature" (in order theory), then covering representability, and finally moving into the results in their full generality and reflecting on the associated "Yoneda philosophy."

Before continuing to develop the story of sheaves, the next two chapters take a step back to complete the account of category theoretic fundamentals. This chapter is devoted to what is perhaps the most important idea in category theory, the Yoneda results. Chapter 7 turns to consideration of adjunctions, and is focused on developing these ideas through a number of examples. In the coming sections of the present chapter, we will motivate the main ideas through a simplified special case, its analogue for posets (in fancier language, its "**2**-enriched" analogue). This motivation requires that one first understand *enrichment*, the introduction of which also gives us a chance to refine our understanding of categories in general.

6.1 First, Enrichment!

Not all categories were created equal. For instance, for certain categories, there may be a natural way of combining elements of the category, that is, of making use of an operation that takes two elements and "adds" or "multiplies" them together. Not all categories admit such a thing. Those that do are called *symmetric monoidal*.

Definition 138 A *symmetric monoidal structure* on a category $\mathcal{V}$ consists of the following data:

1. a bifunctor[79] $-\otimes - : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the *monoidal product*;
2. a unit object $I \in \mathrm{Ob}(\mathcal{V})$, called the *monoidal unit*,

subject to the following specified natural isomorphisms:

$$v \otimes w \cong_\gamma w \otimes v \qquad u \otimes (v \otimes w) \cong_\alpha (u \otimes v) \otimes w \qquad I \otimes v \cong_\lambda v \cong_\rho v \otimes I$$

that witness symmetry, associativity, and unit conditions on the monoidal product. There are then standard "coherence conditions" that these natural transformations are expected to obey.

79. A *bifunctor* is just a functor whose domain is the product of two categories.

A category equipped with such a symmetric monoidal structure is then called a *symmetric monoidal category*, denoted, for example, $(\mathcal{V}, \otimes, I)$. A *monoidal category* is similarly defined, except one leaves out the symmetry natural isomorphism displayed above on the far left. If the natural isomorphisms involving associativity and the unit are replaced by *equalities*, then the monoidal structure is said to be *strict*.

This is defined on categories in general, but an especially simple special case comes from restricting the definition to preorders (as categories).

Definition 139 A *symmetric monoidal structure* on a preorder $(X, \leq)$ consists of

- an element $I \in X$ called the monoidal unit, and
- a function $\otimes : X \times X \to X$, called the monoidal product.

These must further satisfy the following, for all $x_1, x_2, y_1, y_2, x, y, z \in X$, where we use infix notation, that is, $\otimes(x_1, x_2)$ is written $x_1 \otimes x_2$:

- *monotonicity*: if $x_1 \leq y_1$ and $x_2 \leq y_2$, then $x_1 \otimes x_2 \leq y_1 \otimes y_2$;
- *unitality*: $I \otimes x = x$ and $x \otimes I = x$;
- *associativity*: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$;
- *symmetry*: $x \otimes y = y \otimes x$.

Then a preorder equipped with a symmetric monoidal structure, $(X, \leq, I, \otimes)$, is called a *symmetric monoidal preorder*.

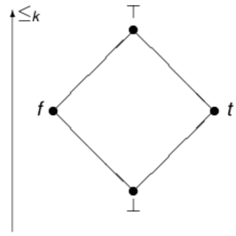
Monoidal units may be given by, for example, $0, 1$, false, true, $\{*\}$, etc. Monoidal "products" include the likes of $\otimes$, $+$, $*$, $\wedge$, $\vee$, $\times$, and so on.

Example 140 The simplest nontrivial preorder is $\mathbf{2} = \{0 \xrightarrow{\leq} 1\}$. Alternatively, you might think of this as $\mathbf{2} = \{\mathit{false}, \mathit{true}\}$ with the single nontrivial arrow $\mathit{false} \leq \mathit{true}$. There are two different symmetric monoidal structures on it. To consider one of these: let the monoidal unit be *true* and the monoidal product be $\wedge$ (AND), leaving us with a monoidal preorder $(\mathbf{2}, \leq, \mathit{true}, \wedge)$.[80]

Example 141 For a set S, the powerset $\mathbb{P}(S)$ of all subsets of S, equipped with the natural order $A \leq B$ given by subset relation $A \subseteq B$, in fact has a symmetric monoidal structure on it: $(\mathbb{P}(S), \leq, S, \cap)$ is a symmetric monoidal preorder.

In particular, taking S a two-element set, this is isomorphic to $(A_4, \leq_k, B, \otimes)$, Belnap's four-valued "knowledge lattice" (or "approximation lattice") $A_4 = (\{\bot, t, f, \top\}, \leq_k)$, often used by relevance and paraconsistent logicians, where the values are the various subsets of $\{\mathit{true}, \mathit{false}\}$ (i.e., $\{t, f\}$). Here, $\top$ (or sometimes B) is "*both* true and false"; $\bot$ is "*neither* true nor false"; $\otimes$ is a "consensus" connective corresponding to meet; and $(A_4, \leq_k)$ is the (complete) lattice corresponding to an ordering on epistemic states ("how much information/knowledge").

80. Fong and Spivak (2020) sensibly calls this **Bool**, but we will just stick with calling it **2**, after its carrier preorder. The reader who desires a more in-depth treatment of enrichment, or who is intrigued by any of these matters, will surely enjoy the recent Fong and Spivak (2020). Readers with a higher tolerance for abstraction might also find Kelly (2005) useful.



This structure accordingly has four "truth values": the classical ones (t and f); a truth value $\perp$ that intuitively captures the notion of a *lack of information* ("neither t nor f"); and a truth value $\top$ that can be deployed to represent *contradictions* or *inconsistency* ("*both t* and f"). The underlying partial order of the lattice has t and f as its intermediate truth values, $\perp$ as the $\leq_k$-minimal element, and $\top$ as the $\leq_k$-maximal element. Overall, the partial order $(\{\perp, t, f, \top\}, \leq_k)$ of the lattice is often regarded as serving to rank the "amount of knowledge or information," where $\leq$ captures the notion of "approximates the information in"—that is, if $x \leq_k y$, then y gives us *at least as much* information as x (possibly more). A move up in the lattice represents an increase in the amount of information, with $\otimes$ taking the uppermost element below both x and y.[81]

Example 142 Let $[0, \infty]$ be the set of nonnegative real quantities, together with ∞. Consider the preorder $([0, \infty], \geq)$, with the natural order $\geq$, for example, $\pi \geq 0.8$, $14.\overline{33} \geq 11$, and of course $\infty \geq x$ for all $x \in [0, \infty]$. There is a symmetric monoidal structure here, with monoidal unit 0 and monoidal product + (where in particular $x + \infty = \infty$ for any $x \in [0, \infty]$). After Fong and Spivak (2020), we can call this symmetric monoidal preorder $\mathbf{Cost} := ([0, \infty], \geq, 0, +)$, since we think of the elements of $[0, \infty]$ as costs.

In the standard definitions of a category that we have seen thus far in this book, the hom-sets are *sets*, that is, objects of the category **Set**. On this approach, with such categories, that the hom-sets are specifically sets effectively means that the task or question of getting from (or relating) one object to another has a *set* of approaches or answers or names. But what if we generalized this story and let the hom-sets of a category come from some category other than **Set**?

Symmetric monoidal categories are important, in large part, because of something we can *do* with them: we can *enrich* an (arbitrary) category in them! What does that mean? Fong and Spivak (2020) suggest a very nice intuitive way of thinking of this: Enriching in, say, a monoidal preorder $\mathcal{V} = (V, \leq, I, \otimes)$ just means "letting $\mathcal{V}$ structure the question of the relations or paths between the objects of the underlying category." In this general context, enriching in different monoidal categories often recovers (while generalizing) important entities in math. For instance, it emerges that categories "enriched in **Cost**," or **Cost**-categories, provide a powerful generalization of the notion of metric space.

Definition 143 Let $\mathcal{V} = (V, \leq, I, \otimes)$ be a symmetric monoidal preorder. A $\mathcal{V}$-category $\mathcal{X}$ consists of

- specification of a set $\mathrm{Ob}(\mathcal{X})$, elements of which are objects,

81. For more on this lattice, see Belnap (1992).

- for every two objects x, y, specification of an element $\mathcal{X}(x, y) \in V$, called the hom-object,

and where these satisfy the two properties

- for every object $x \in \mathrm{Ob}(\mathcal{X})$, we have $I \leq \mathcal{X}(x, x)$, and
- for every three objects $x, y, z \in \mathrm{Ob}(\mathcal{X})$, we have $\mathcal{X}(x, y) \otimes \mathcal{X}(y, z) \leq \mathcal{X}(x, z)$.

In this case, we call $\mathcal{V}$ the *base of the enrichment* for $\mathcal{X}$, or just say $\mathcal{X}$ is *enriched in* $\mathcal{V}$.

Example 144 What happens if we enrich in **Cost** $= ([0, \infty], \geq, 0, +)$? Following the definition: a **Cost**-category $\mathcal{X}$ consists of

1. a collection $\mathrm{Ob}(\mathcal{X})$, and
2. for every $x, y \in \mathrm{Ob}(\mathcal{X})$ an element $\mathcal{X}(x, y) \in [0, \infty]$.

The idea here is that $\mathrm{Ob}(\mathcal{X})$ provides the "points," while $\mathcal{X}(x, y) \in [0, \infty]$ plays the role of supplying the "distances." Still just following the definition, the properties of a category enriched in **Cost** are given by:

- $0 \geq \mathcal{X}(x, x)$ for all $x \in \mathrm{Ob}(\mathcal{X})$, and
- $\mathcal{X}(x, y) + \mathcal{X}(y, z) \geq \mathcal{X}(x, z)$ for all $x, y, z \in \mathrm{Ob}(\mathcal{X})$.

Note that since $\mathcal{X}(x, x) \in [0, \infty]$, the property $0 \geq \mathcal{X}(x, x)$ implies that $\mathcal{X}(x, x) = 0$. So this is in fact equivalent to the first condition $d(x, x) = 0$ describing a metric. And the second condition here is clearly the usual triangle inequality! We have thus defined, with the notion of a **Cost**-category, an extended (Lawvere) metric space.

Recall the usual definition of a metric space from definition 73 (chapter 4). If we instead take a function $d : X \times X \to [0, \infty] = \mathbb{R}_{\geq 0} \cup \{\infty\}$, then we have an *extended metric space*. From the categorical viewpoint, the generalized construction of a **Cost**-category recovers the notion of a metric space, while already suggesting that the usual definition's conditions

- (2) if $d(x, y) = 0$, then $x = y$;
- (3) $d(x, y) = d(y, x)$

are somehow not as natural or primitive as the other two conditions (triangle inequality and that points are at "zero distance" from themselves). Indeed, there are contexts in which (2) is not satisfied, yet we would still like to have a metric. Also, requiring (3) or symmetry prevents us from regarding a number of constructions we would like to regard as metrics as legitimate metrics, so relaxing this condition also seems desirable.

Example 145 Now take the symmetric monoidal preorder $\mathbf{2} = (\{\mathit{false}, \mathit{true}\}, \leq, \mathit{true}, \wedge)$. Enriching in **2** recovers the notion of a preorder, since for any $x, y \in \mathcal{P}$, with $\mathcal{P}$ a preorder, there is either 0 ("false") or 1 ("true") arrow from x to y. Accordingly, the "homs" here will be objects of **2**, not **Set**. More formally, a **2**-category[82] consists of

- a specification of a set of objects
- for every x, y, an element $\mathcal{X}(x, y) \in \mathbf{2}$

where this data satisfies

82. Not to be confused with the notion of a 2-category, mentioned in remark 49 (chapter 2).

1. for every element $x \in \text{Ob}(\mathcal{X})$, $true \xrightarrow{\leq} \mathcal{X}(x,x)$, so $\mathcal{X}(x,x) = true$
2. for every x, y, z, $\mathcal{X}(x,y) \wedge \mathcal{X}(y,z) \xrightarrow{\leq} \mathcal{X}(x,z)$

The first condition above just amounts to reflexivity and the second to transitivity, used to define a preorder; understanding $\mathcal{X}(x,y) = true$ to just mean that $x \leq y$, clearly this just recovers the notion of a preorder. Thus, the theory of **2**-enriched categories just recovers precisely the theory of ordered sets and the monotone maps between them.

Example 146 Returning to A_4, we can understand t as "told True," f as "told False," $\perp$ as "told nothing" (i.e., *neither* told True nor told False), $\top$ as "*both* told True and told False." $\perp$ is at the bottom of the lattice as it gives no information at all, while $\top$ is at the top since it gives "too much" (or inconsistent) information.

When we enrich in A_4, the resulting A_4-category $\mathcal{X}$ will describe, for any two objects x, y of $\mathcal{X}$, all the (true, false, null, inconsistent) information that has been received or inputed (perhaps from several independent sources) about whether you can "get from" x to y.

Enriching in A_4 implies that the issue of passing from x to y is structured by how much information/knowledge we (or some system, like a computer, prepared to receive and reason about inconsistent information) might have about the question. For instance, "I have been told that 'yes' ('no') one can (cannot) pass from x to y"; or "I have been told both that you can and that you cannot pass from x to y"; or "I have not been told anything about whether or not you can pass from x to y."

In the next few sections, we will make use of this notion of enrichment to build towards a particularly simple presentation of the abstract Yoneda results.

6.2 Downsets and Yoneda in the Miniature

Given a poset $\mathcal{P} = (P, \leq)$, we have seen how we can regard $\mathcal{P}$ as a category. The following notion will be of use to us.

Definition 147 Let $\mathcal{P}$ be a poset, and $A \subseteq \mathcal{P}$ a subset. Then, we call the subset A a *downset* if for each $p \in A$ and $q \in \mathcal{P}$, we have that $p \in A$ and $q \leq p$ implies that $q \in A$. Dually (i.e., reversing all the arrows), a subset $U \subseteq \mathcal{P}$ is an *upper set* (or *up-set*) provided: if $p \in U$ and $p \leq q$, then $q \in U$.

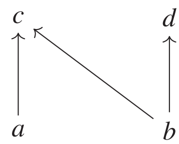
We can further define, for each element $p \in \mathcal{P}$, the downset generated by p—called its *principal downset*, denoted D_p (or just $\downarrow p$)—as

$$\downarrow p := \{q \in \mathcal{P} \mid q \leq p\}.$$

Dually, as one might expect, we can also define, for each point p, its *principal upper set* U_p (or just $\uparrow p$) as

$$\uparrow p := \{q \in \mathcal{P} \mid p \leq q\}.$$

For instance, consider the following poset $\mathcal{P}$, built from $P = \{a, b, c, d\}$, and given by $a \leq c$, $b \leq c$, $b \leq d$, and the obvious identity (reflexivity) $x \leq x$ for all $x \in P$. The data of this poset is perhaps more helpfully displayed in the picture:

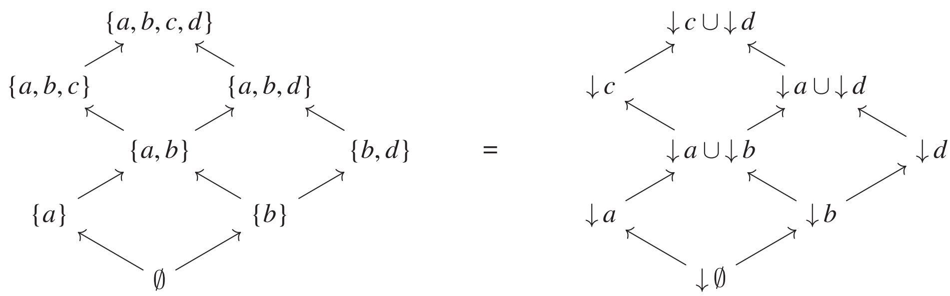


Exercise 18 *Is* $\{a,b,c\}$ *a downset? How about* $\{a,b\}$*? And* $\{a,c,d\}$*?*

Solution Yes, $\{a,b,c\}$ is a downset; same with $\{a,b\}$. But $N=\{a,c,d\}$ is not a downset; for, in particular $d\in N$, yet considering $b\in P$, since $b\leq d$, we should have that $b\in N$, in order for N to be a downset. Yet $b\notin N$.

In general, we denote by $\mathcal{D}(\mathcal{P})$ the collection of all downsets of the poset $\mathcal{P}$. Observe that $\mathcal{D}(\mathcal{P})$ has a natural order on it—namely, $U\leq V$ provided U contained in V. Then, $(\mathcal{D}(\mathcal{P}),\subseteq)$ is itself an order under inclusion, one that we will sometimes just denote by $\mathcal{D}(\mathcal{P})$, or **Down**$(\mathcal{P})$ when we want to emphasize that we are regarding this order as a *category*.[83] This resulting poset consisting of the collection of all downsets of $\mathcal{P}$, ordered by inclusion, is sometimes called the *downset completion*.

The following diagram displays the information of all the downsets of our given $\mathcal{P}$, ordered by inclusion:



There are a couple of valuable general observations to note at this point, which can be illustrated via this particular example. The first observation will allow us to construe downsets in terms of monotone maps (functors) from $\mathcal{P}^{op}$ to the order **2**.

First, consider that any given element A of $\mathcal{D}(\mathcal{P})$ represents something like a "choice" of elements from the underlying set P, with the further requirement that, as a downset, whenever $x\in A$, then any $y\in P$ such that $y\leq x$ in $\mathcal{P}$ is also in A. But this requirement is the same as saying that for any x,y such that $y\leq x$ in $\mathcal{P}$, if we have that "it is *true* that $x\in A$," then we must also have that "it is *true* that $y\in A$." And this is just to say that

$$y\leq x \text{ implies } \phi(y)\geq\phi(x),$$

where ϕ is an antitone map from the order $\mathcal{P}$ to order **2**; or, equivalently, it is a monotone map from the opposite order $\mathcal{P}^{op}$ to **2**. Such maps are themselves ordered under the pointwise inclusion ordering. If we designate such a poset of monotone maps, ordered by

83. Dually, we write $\mathcal{U}(\mathcal{P})$ for the collection of all the upper sets of $\mathcal{P}$; this also has a natural order on it—namely, $U\leq V$ provided U is contained in V, making $(\mathcal{U}(\mathcal{P}),\subseteq)$ an order as well.

inclusion, by $\mathbf{2}^{\mathcal{P}^{op}}$ or Monot$(\mathcal{P}^{op}, \mathbf{2})$, then we can see that there is a map between the orders

$$\mathcal{D}(\mathcal{P}) \to \text{Monot}(\mathcal{P}^{op}, \mathbf{2})$$
$$D \mapsto \phi_D,$$

where ϕ_D acts as the characteristic (or indicator) function, mapping to 1 on D and 0 elsewhere. In other words, given a downset D of $\mathcal{P}$, we define $\phi_D : \mathcal{P}^{op} \to \mathbf{2}$ by setting $\phi_D(x) = 1$ precisely when $x \in D$ (i.e., assigns it to the characteristic function of D). Conversely, given a monotone map in Monot$(\mathcal{P}^{op}, \mathbf{2})$, we can send this to the inverse image $\phi^{-1}(1) \in \mathcal{D}(\mathcal{P})$, recovering a unique downset (you can verify for yourself that the subset $\phi^{-1}(1)$ is a downset). In order theory, in general, a map $F : P \to Q$, where P and Q are posets, is said to be an *order-embedding* provided

$$x \leq y \text{ in } P \text{ iff } F(x) \leq F(y) \text{ in } Q,$$

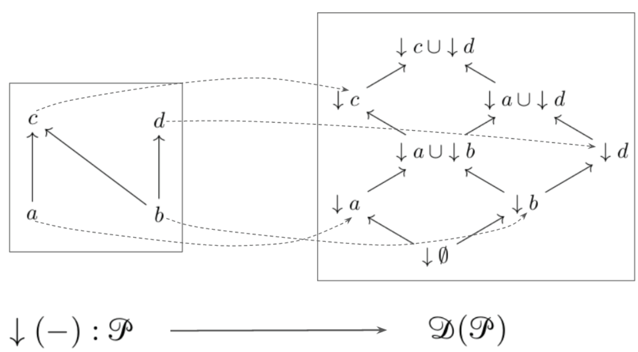
and then such an order-embedding yields an *order-isomorphism* between P and Q. But since, in our case, $A \subseteq B$ iff $\phi_A \leq \phi_B$, altogether we have thus described an order-embedding, giving us an order-isomorphism

$$\mathcal{D}(\mathcal{P}) \cong \text{Monot}(\mathcal{P}^{op}, \mathbf{2}).$$

Notice how, included among the maps $\mathcal{P}^{op} \to \mathbf{2}$ are those ϕ_p for any given element $p \in \mathcal{P}$. These send $q \mapsto 1$ iff $p \leq q$ in $\mathcal{P}$ (or, equivalently, but perhaps more clearly, iff $q \leq p$ in $\mathcal{P}^{op}$, which is our domain), for such a map generally acts as the indicator function of the set of all $x \leq y$, that is, the principal downset $\downarrow y$ of y.

Observe that the principal downset $\downarrow p := \{q \in P : q \leq p\}$ is itself a downset (i.e., it will belong to $\mathcal{D}(\mathcal{P})$), *for any p* in our set $\mathcal{P}$. (This follows from the transitivity of $\mathcal{P}$.) The principal downsets are actually rather special objects among the downsets. To see this, consider again the diagram of the downsets of our particular $\mathcal{P}$. You can see that every object in $\mathcal{D}(\mathcal{P})$ is a principal downset or a union of principal downsets, and that the principal downsets $\downarrow x$ run through all $x \in \mathcal{P}$. Observe also that for $A \subseteq \mathcal{P}$, $\downarrow A$ will be the *smallest* downset that contains A, and moreover $A = \downarrow A$ iff A is a downset. We will return to these facts, and make better sense of them, shortly.

For now, we need to realize that $\downarrow$ can actually be regarded as a monotone map from $\mathcal{P}$ to $\mathcal{D}(\mathcal{P})$, which in our particular case may be pictured as



That this $\downarrow$ really just defines a monotone map—that is, $x \leq y$ implies $\downarrow x \subseteq \downarrow y$—is easy to see in the general case as well. For, let $x \leq y$ in $\mathcal{P}$. We want $\downarrow x \subseteq \downarrow y$ in $\mathcal{D}(\mathcal{P})$. Take any

$x' \in \downarrow x$. Then we must have that $x' \leq x$. By transitivity of the order, $x' \leq x$ and the assumed $x \leq y$ yield $x' \leq y$. Thus, $x' \in \downarrow y$. Altogether, this shows that $\downarrow x \subseteq \downarrow y$.

We have the other direction as well, namely $\downarrow x \subseteq \downarrow y$ in $\mathscr{D}(\mathscr{P})$ implies $x \leq y$ in $\mathscr{P}$. Altogether, then, we actually have another *order-embedding*, embedding any poset into its downset completion:

$$\downarrow(-) : \mathscr{P} \to \mathscr{D}(\mathscr{P}) \quad p \mapsto \downarrow p.$$

This is all part of a much bigger story, so the main results are set aside for emphasis.

Proposition 148 (*Yoneda lemma for posets*) Given $\mathscr{P}$ a poset, $x \in \mathscr{P}$, and $A \in \mathscr{D}(\mathscr{P})$, then

$$x \in A \text{ iff } \downarrow x \subseteq A.$$

Proof. ($\Rightarrow$) Let $x \in A$. Then all $y \leq x$ is also in A, as A is a downset, and in particular $x \in \downarrow x$. Thus, $\downarrow x \subseteq A$.

($\Leftarrow$) Take $y \in \downarrow x$. But then $y \leq x$, and so $y \in A$ since A is a downset. □

Applying the Yoneda lemma to two principal downsets, we have that $y \leq z$ iff for all x[84]

$$x \leq y \Rightarrow x \leq z.$$

The most important corollary, or application, of the lemma is the following:

Proposition 149 (*Yoneda embedding for posets*) This $\downarrow$ defines an order-embedding (embedding any poset into its downset completion):

$$\downarrow(-) : \mathscr{P} \to \mathscr{D}(\mathscr{P}) \quad p \mapsto \downarrow p.$$

Proof. Let $x \leq y$. Then $x \in \downarrow y$ (conversely, if $x \in \downarrow y$, clearly we must have $x \leq y$). Applying the previous lemma (taking $A = \downarrow y$), $x \in \downarrow y$ holds precisely when $\downarrow x \subseteq \downarrow y$. On the other hand, the converse holds as well, that is, $\downarrow x \subseteq \downarrow y$ implies $x \in \downarrow y$, which implies $x \leq y$. Altogether then,

$$x \leq y \text{ iff } (\downarrow x) \subseteq (\downarrow y).$$

□

The Yoneda results thus assure us, in a slogan, that

To know everything "below" an element is just to know that element.

Let us start to generalize this story. Given a poset $\mathscr{P}$, by considering $\mathscr{P}$ as a category, we might have looked at presheaves $\mathscr{P}^{op} \to \mathbf{Set}$. But since preorders are the same thing as **2**-enriched categories, supposing we want not *sets* for our hom-sets, but rather "truth values," it is natural instead to consider **2**-enriched "presheaves" on $\mathscr{P}$. Instead of arbitrary set-valued data, then, such a **2**-presheaf assigns to each $x \in \mathscr{P}$ a *truth-value* in **2**; and this will just recover the monotone maps (functors) $\mathscr{P}^{op} \to \mathbf{2}$ (or, equivalently, an antitone map, or contravariant functor (presheaf), from $\mathscr{P}$ to **2**).

We saw how such a monotone map (from the opposite order) effectively acts as the characteristic (or indicator) function ϕ_A of a downset $A \subseteq P$, forming part of the important

84. Readers familiar with real analysis may recognize in this the construction of *Dedekind cuts*!

order-isomorphism

$$\mathcal{D}(\mathcal{P}) \cong \text{Monot}(\mathcal{P}^{op}, \mathbf{2}).$$

We know, moreover, how to convert any poset into a category. Thus, renaming $\mathcal{D}(\mathcal{P}) :=$ $\mathbf{Down}(\mathcal{P})$ and $\text{Monot}(\mathcal{P}^{op}, \mathbf{2}) := \mathbf{Monot}(\mathcal{P}^{op}, \mathbf{2})$ to emphasize that we are now dealing with categories, the above actually describes

$$\mathbf{2\text{-}PreSh}(\mathcal{P}) := \mathbf{Monot}(\mathcal{P}^{op}, \mathbf{2}) \cong \mathbf{Down}(\mathcal{P}).$$

In the order setting, via the principal downsets, we were able to construct an embedding $\mathcal{P} \to \mathbf{Down}(\mathcal{P})$. Similarly, but in much greater generality, we will see that there is an embedding $\mathbf{C} \to \mathbf{PreSh}(\mathbf{C})$, taking a general category $\mathbf{C}$ to its category of presheaves. Before defining the Yoneda lemma and embedding in the general case, we need to take a step back for a moment and discuss *representability*.

To motivate this, consider that in a poset, regarded as a category, a principal downset on an element $p \in \mathcal{P}$ is just all the arrows into p, where these amount to all elements that p "looks down on" (or all elements that "look up to" p). This identification of "arrows into p" and "*elements* below p" can be made, since a poset is precisely a category for which there is *at most one* arrow $q \to p$ for any p, q, allowing us to *identify* $\text{Hom}_{\mathcal{P}}(q, p)$ with the element q. In other words,

$$\text{Hom}_{\mathcal{P}}(-, p) = \downarrow p.$$

Before cashing in on the power of this statement, we will discuss the "representability" operative here by considering a sort of miniaturized version of this phenomenon.

6.3 Representability Simplified

Consider a general map

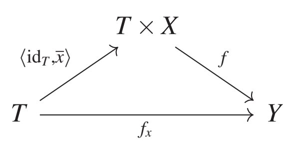
$$T \times X \to Y$$

that has a product for its domain (you can think of this as involving sets and functions for now). We cannot typically expect to be able to reduce this to a specification of what is happening on T and X separately, as the interaction of the two factors of the product is essentially involved in supplying the values of the mapping itself. Thus, in considering a map that has for domain a product (all three objects different, for the most general case),

$$T \times X \xrightarrow{f} Y,$$

we can ask (nontrivial) questions about the relations between any of the separate objects involved in the product and the codomain object.

Observe that if we use the terminal object 1 to pick out "points" of X, via $1 \xrightarrow{x} X$, any such point will give rise, via f, to the map f_x



where $\bar{x}$ is defined as the composite constant map $T \to 1 \xrightarrow{x} X$. Thus,

$$f_x(t) = f(t, x)$$

for all t. In this way, we are now regarding the map f as an X-parameterized family of maps $T \to Y$, one for each of the points of X. In this setting, one possible question would be to consider, for a pair of sets T, Y, whether there is a set X *large enough* for its points to supply, via the maps $f(–, x)$, *all* maps $T \to Y$. As a very simple illustration of this, in the simple case of sets described by their cardinality (number of elements), if T is a set with four elements and Y a set with three elements, then X would need to have

$$3^4 = 81$$

elements, for that is the number of maps $T \to Y$.

This is in fact part of a much more general story—one that gets at representability—than the above discussion might suggest. For, we do not need to restrict attention to sets and their cardinal number properties or even make this a matter of *size*. For a function $g : T \to Y$, whenever there is at least one $1 \xrightarrow{x_0} X$ such that

$$g(–) = f(–, x_0),$$

that is, for all $t \in T$,

$$g(t) = f(t, x_0),$$

we say that g is *representable* by x_0, or f*-represented by* x_0. This might be seen as a toy instance of the much more general notion of representable functors. Recall the hom-functor $\mathrm{Hom}_{\mathbf{C}}(–, c)$ (and its dual $\mathrm{Hom}_{\mathbf{C}}(c, –)$), where this may be given by

$$\mathrm{Hom}_{\mathbf{C}}(–, –) : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}.$$

For any categories $\mathbf{C}, \mathbf{D}, \mathbf{E}$, an isomorphism can be demonstrated between

$$\mathbf{E}^{\mathbf{C} \times \mathbf{D}} \cong (\mathbf{E}^{\mathbf{D}})^{\mathbf{C}},$$

allowing us to move freely between functors $\mathbf{C} \times \mathbf{D} \to \mathbf{E}$ and $\mathbf{C} \to \mathbf{E}^{\mathbf{D}}$. Thus, if we fix one of the variables of this $\mathrm{Hom}_{\mathbf{C}}$, then we get the important *representable functors*:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(a, –) : \mathbf{C} &\to \mathbf{Set} \\ b &\mapsto \mathrm{Hom}_{\mathbf{C}}(a, b) \\ f : b \to c &\mapsto \mathrm{Hom}_{\mathbf{C}}(a, f) : \mathrm{Hom}_{\mathbf{C}}(a, b) \to \mathrm{Hom}_{\mathbf{C}}(a, c) \\ &\qquad g \mapsto f \circ g, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}}(–, a) : \mathbf{C}^{op} &\to \mathbf{Set} \\ b &\mapsto \mathrm{Hom}_{\mathbf{C}}(b, a) \\ f : b \to c &\mapsto \mathrm{Hom}_{\mathbf{C}}(f, a) : \mathrm{Hom}_{\mathbf{C}}(c, a) \to \mathrm{Hom}_{\mathbf{C}}(b, a) \\ &\qquad h \mapsto h \circ f. \end{aligned}$$

But this is just to describe the Yoneda-embedding functors taking, for instance in the contravariant case,

$$\begin{aligned} \mathbf{C} &\to \mathbf{Set}^{\mathbf{C}^{op}} \\ x &\mapsto \mathrm{Hom}_{\mathbf{C}}(–, x). \end{aligned}$$

Functors (presheaves) of this form are then said to be *representable*. More formally,

Definition 150 For a locally small category $\mathbf{C}$, we say that a functor $F: \mathbf{C} \to \mathbf{Set}$ is a *representable functor* if there exists an object $c \in \mathbf{C}$ (sometimes called the *representing object*) together with a natural isomorphism $\mathrm{Hom}_{\mathbf{C}}(c, -) \cong F$; or, equivalently, one speaks of a *representation* for a (covariant) functor F as an object $c \in \mathbf{C}$ together with a specified natural isomorphism $\mathrm{Hom}_{\mathbf{C}}(c, -) \cong F$.[85]

If F is a contravariant functor, then the desired natural isomorphism is given between $\mathrm{Hom}_{\mathbf{C}}(-, c) \cong F$.

In the covariant case, the representable functor can be thought of, intuitively, as encoding how a category "is seen" or "is acted on" by a certain object; in the contravariant case, how the category "sees" or "acts on" the chosen object. For instance, in the category of topological spaces **Top**, if we regard all the maps from **1** (the one-point space) to a space X, this just produces the points of X, that is, "**1** sees points."[86]

It is worth lingering a bit with this notion of representability. It might be useful to mention, moreover, that most functors (valued in **Set**) are *not* representable. If you were to pick a functor randomly, the odds are it would not be representable. Thus, examples of *non*representable functors abound; but perhaps a few concrete nonexamples are in order.

Example 151 The *covariant* powerset functor $\mathbb{P}: \mathbf{Set} \to \mathbf{Set}$ is not representable. This functor $\mathbb{P}$ is such that $\mathbb{P}(X)$ is just the power set of X, and for any function $X \to Y$, the map $\mathbb{P}(X) \to \mathbb{P}(Y)$ takes $A \subseteq \mathbb{P}(X)$ to the image under f, that is, to $f(A)$.

To see that it is not representable, suppose we have a representing object X, that is, $X \in \mathbf{Set}$ represents $\mathbb{P}$. Then, in particular, we will need that

$$|\mathrm{Hom}_{\mathbf{Set}}(X, -)| = |\mathbb{P}(-)|$$

for all sets, that is,

$$|\mathrm{Hom}_{\mathbf{Set}}(X, Y)| = |\mathbb{P}(Y)|$$

for all $Y \in \mathbf{Set}$. But then we can take $Y = \{*\}$, a singleton set. For any nontrivial X, there can be only one map to the singleton set, so $|\mathrm{Hom}_{\mathbf{Set}}(X, Y)| = 1$. Yet the powerset of a singleton set is, of course, of cardinality 2. Thus

$$|\mathrm{Hom}_{\mathbf{Set}}(X, Y)| \neq |\mathbb{P}(Y)|$$

for our given $Y = \{*\}$. This contradiction tells us that there can be no such representing object X in **Set** for the covariant powerset functor.

On the other hand, the contravariant powerset functor (presheaf) is representable![87] Specifically, $\mathbb{P}$ (contravariant now) is representable by the two-element set $2 = \{0, 1\}$, so that for each set Y, we have the isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}(Y, 2) \cong \mathbb{P}(Y)$$
$$f \mapsto f^{-1}(\{1\}).$$

85. Ordinarily, one requires that the domain $\mathbf{C}$ of a representable functor be locally small so that the hom-functors $\mathrm{Hom}_{\mathbf{C}}(c, -)$ and $\mathrm{Hom}_{\mathbf{C}}(-, c)$ are valued in the category of sets.

86. This example is lifted from Leinster (2014).

87. In general, it seems to be easier to find representables among *contravariant* functors.

Effectively, this says that 2 is a set that contains a universal subset $\{1\}$ that pulls back to any other subset, via the characteristic function of that subset.

The great importance of representable functors is in part due to the fact that representable functors can encode a universal property of its representing object. For instance, a category **C** will have an *initial object* precisely when the constant functor $* : \mathbf{C} \to \mathbf{Set}$ is representable, that is, an object $c \in \mathbf{C}$ will be initial iff the hom-functor Y^c is naturally isomorphic to the constant functor sending every object to the singleton set. Dually, an object $c \in \mathbf{C}$ will be *terminal* iff the functor Y_c is naturally isomorphic to the constant functor $* : \mathbf{C}^{op} \to \mathbf{Set}$. Put otherwise: an object $c \in \mathbf{C}$ is *initial* if, for all objects $d \in \mathbf{C}$, there exists a unique morphism $c \to d$; while an object $c \in \mathbf{C}$ is *terminal* if, for all objects $d \in \mathbf{C}$, there exists a unique morphism $d \to c$.

The absence of such universal properties can be used, as we effectively did in dealing with the (covariant) powerset functor above, to show that a candidate nonrepresentable functor is in fact not representable. The general idea here—which method you might use to convince yourself of the nonrepresentability of the functors described in the coming examples—is to (1) assume the functor is representable; (2) consider a possible "universal" element for the functor; and then (3) produce a contradiction by showing that this element cannot actually have the special universal property that it needs to have.

Recall that in **Set**, every one-element (singleton) set is a terminal object (a special object with a special universality property). Thus, in our discussion of the covariant powerset functor, another way of saying that

$$|\mathrm{Hom}_{\mathbf{Set}}(X, \{*\})| \neq |\mathbb{P}(\{*\})|$$

would accordingly have been to say that $\mathbb{P}$ does not preserve the terminal object.

Example 152 The (covariant) functor $\mathbf{Group} \to \mathbf{Set}$ that takes a group to its set of subgroups is not representable.

Example 153 The (covariant) functor $\mathbf{Rng} \to \mathbf{Set}$ that takes a (nonunital) ring R to its set of squares, that is, $\{r^2 : r \in R\}$, is not representable.

It turns out that all universal properties themselves can be captured by the fact that certain data defines an initial or terminal object in an appropriate category, specifically the category of elements of the representable functor, a fact that can be rather useful (but that we simply record, without proof, before giving examples).

Proposition 154 A covariant (contravariant) set-valued functor is representable iff its category of elements has an initial (respectively, terminal) object.

Example 155 Applied to a poset $\mathcal{P}$, consider $\mathcal{P}$ as a category. For an arbitrary element $p \in \mathcal{P}$, first check that the slice category $\mathcal{P}/p$ (also denoted $(\mathcal{P} \downarrow p)$)[88] is just the principal downset generated by p; dually, the co-slice category $(p/\mathcal{P})$ (also denoted $(p \downarrow \mathcal{P})$) is the principal upper set of p. Recall that we can construct the category of elements in terms

88. I hope this latter notation is not too confusing in this context, given that we are also talking about principal downsets!

of the slice category. Thus, a **2**-presheaf (i.e., downset $A \subseteq P$) is representable iff it has a greatest element.

Example 156 Recall the discussion of graph coloring and the functor *nColor*, first described in example 37 (chapter 2). Recall also that a complete graph is just a graph in which every pair of distinct vertices is connected by a unique edge. In graph theory, by K_n we mean the complete graph on n nodes. In terms of graph colorings, it should be obvious that this graph will be the graph with the fewest vertices and edges that needs at least n colors to be colored. This fact actually gets at a universal property of the graph K_n. Leaving the details to the reader, we indicate that the functor *nColor* is represented by K_n. This basically says that if we want to know about the set $nColor(G)$ of n-colorings of a graph G, we can just look at the set of graph homomorphisms from G to the complete graph K_n on n vertices. To appreciate this, notice how, given a morphism $f : G \to K_n$, the condition (by definition of being a graph homomorphism) that $f(e)$ be in the edge-set of K_n for any edge e in the edge-set of G will force $f(x) \neq f(y)$ whenever x and y are adjacent, recapturing the notion of an n-coloring of G.

Incidentally, the reader might wish to ponder which concept from graph theory is captured by morphisms going in the other direction, that is, by homomorphisms $f : K_n \to G$.

In a moment, we will see the most important category-theoretic result, which morally shows how an object is defined completely by its functorial (relational) properties. But as representability often seems to baffle the newcomer, the next (optional) section offers an elaboration on the phenomenon of (non)representability. The reader eager to press on to the main Yoneda results can skip ahead a few pages.

6.4 More on Representability, Fixed Points, and a Paradox

Above, when we were thinking of general morphisms $T \times X \to Y$ as a family of morphisms $T \to Y$ parameterized or indexed by the elements of X—in which setting we were considering a sort of "miniature" version of representability—this was effectively to look at arbitrary maps

$$\hat{f} : X \to Y^T,$$

which led to the question of when (and which) X can "parameterize" all the maps from T itself to some Y. This is effectively the same as asking when such $\hat{f}$ are surjective. In particular, though, we can ask this for $X = T$, so that we are considering

$$\hat{f} : T \to Y^T$$

or

$$f : T \times T \to Y,$$

which are effectively "Y-valued" *relations* or *predicates* on T (or Y-*attributes* of type T). Via the object (function) Y^T, Y-valued predicates can be thought of as "talking about" T. A special circumstance would be where *all* the ways of "talking about itself" can be said by T itself! This is captured by the surjectivity of the map, that is, when *every* element $f : T \to Y$ of Y^T is representable in T. The next result concerns when this can occur.[89]

89. The interested reader may wish to consult Lawvere (2006) and Lawvere and Schanuel (2009) for more extensive discussion of this and the following theorem.

Theorem 157 (*Lawvere's Fixed-Point Theorem*) If

$$\hat{f} : X \to Y^X$$

is surjective (i.e., every $g : X \to Y$ is representable, in the above sense), then Y will have the *fixed-point property*, that is, *every* endomap $\tau : Y \to Y$ has at least one *fixed point*, where this of course means some $y \in Y$ such that

$$\tau(y) = y.$$

Proof. Consider $p : X \to Y$, an arbitrary "predicate" (i.e., element of Y^X). Since any endomap $\alpha : Y \to Y$ just "shuffles around" the elements of Y, we can define p as the composite of the diagonal map, the function f (got from $\hat{f}$ via the standard exponential conversion), and an endomap,

$$\begin{array}{ccc} X \times X & \xrightarrow{f} & Y \\ {\scriptstyle\delta}\uparrow & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow[p]{} & Y. \end{array}$$

By assumption, moreover, there will be an $x \in X$ that *represents* p. Thus,

$$p(x) = \alpha(f(\delta(x))) = \alpha(f(x, x)) = \alpha(p(x)),$$

making $p(x)$ a fixed point of α. □

Notice that Y has the fixed-point property provided *every* endofunction on Y has a fixed point. But any set with more than one element clearly has an endofunction on it that does not have a fixed point (hint: the simplest example is a two-point set, where the points are "true" and "false"; then, an endomap without fixed points is given by the familiar negation map); thus, no set with more than one element will have the fixed-point property. In order to appreciate the importance of the theorem in **Set**, we can present the theorem in another light, namely via the contrapositive.

Theorem 158 (*Cantor's Theorem*) If Y has at least one endomap τ that has no fixed points (i.e., for all $y \in Y, \tau(y) \neq y$), then for every object X and for every

$$X \xrightarrow{\hat{f}} Y^X$$

$\hat{f}$ is not surjective.

In other words, $\hat{f}$ not being surjective means that for every attempt $\phi : X \times X \to Y$ to parameterize maps $X \to Y$ by the points of X, there must be at least one map $g : X \to Y$ that gets left out, that is, it is not representable by ϕ (meaning, does not occur as $\phi(-, x)$ for any point x in X).

Proof. Again, define g as the composite of the diagonal map, the function f (got from $\hat{f}$ via the exponential conversion), and an endomap,

$$\begin{array}{ccc} X \times X & \xrightarrow{f} & Y \\ {\scriptstyle\delta}\uparrow & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

In other words,

$$g(x) = \alpha(f(x,x)).$$

Then, for all $x \in X$,

$$g(-) \neq f(-,x)$$

as functions of one variable. For, if we *did* have $g(-) = f(-,x_0)$ for some $x_0 \in X$, then by evaluation at x_0,

$$f(x_0,x_0) = g(x_0) = \alpha(f(x_0,x_0)),$$

where the leftmost equality follows from g being representable, and the second equality is by definition. But then, α has a fixed point. This is a contradiction. □

Cantor's famous result that there is no surjective map from a set to its powerset

$$X \to 2^X$$

is a special case of the above.

It is best, though, to see how this special result is part of something more general. Given our way of thinking about maps Y^X as providing a particular way (or name for how) X "speaks about" or describes itself, the generalized version of the above can be regarded as saying that, provided the truth-values or properties of X are nontrivial, there will be no way that the elements of object X can "talk about" themselves (in the sense of talking about their own truthfulness or their own properties). The result appeals to an observation concerning the fundamental limitations in how an object X can address its own properties. Many apparent paradoxes of the past seem to play off this. For instance, the Liar paradox was an ancient way of exhibiting the trouble one can get into when natural languages attempt to construct self-referential statements that speak about their own truthfulness—if one permits this, it seems one must open the door to certain inconsistencies in natural language. Russell's famous paradox was basically a simplified version of something Cantor himself already found, one that did not involve the notion of *size*: namely that if we take T as the set of *all* sets, then by Cantor's theorem, there is a set larger than T, namely the powerset of T, yet T is assumed to contain all sets, so we are saying that T contains a subset that is larger than itself. Gödel's famous incompleteness results revealed limitations in formal systems and provability statements within those systems. Brandenburger-Keisler's paradox (a sort of two-person or interactive version of Russell's paradox) concerns the description of a belief situation in which "Ann believes that Bob believes that Ann believes that Bob believes something false about Ann." The paradox is: does Ann believe that Bob has a false belief about Ann? This suggests that not every description of beliefs can be "represented." There are a variety of other results[90] that one could enumerate as further examples of what are arguably all variations on the same theme:

Letting things address their own properties, without limitations, can lead to problems.

The phenomenon of (non)representability is really at the core of such problems. The following (apparently paradoxical) example is mostly meant to get the reader thinking more about some of the subtleties in issues of representability.

90. See Yanofsky (2003) for more; Abramsky (2014) is also of interest, in this connection.

Example 159 The issue underlying the following example sometimes goes under the name of “Grelling’s paradox.”[91] Consider the set of all English words. Some of these words *describe themselves*, while others (most) do not. Adjectives, perhaps more than any other type of word, are used to describe things. So let us restrict attention to the set of adjectives, which we may denote *Adj*. Certain adjectives *describe themselves*, while others (most) do not. Those that describe themselves are said to be *autological* (or *homological*). For instance, the following adjectives are homological: “English” (is English!); “polysyllabic” (is polysyllabic); “Hellenic” (is of Greek origin); “unhyphenated.” Those adjectives, by contrast, that *do not describe themselves* are said to be *heterological*. For instance, the following are heterological: “Spanish” (not a Spanish word!); “misspelled” (is spelled correctly!); “long” (is hardly long); “monosyllabic”; “hyphenated.”

It seems plausible that all adjectives will be either homological or heterological. However, consider the adjective “heterological.” Is it heterological? Suppose it is not. Then, it might naturally be assumed, it will be homological. So it describes itself. Thus “heterological” (which says that it does not describe itself) must be heterological after all. So if “heterological” is not heterological, then it is heterological. On the other hand, then, we suppose that the answer to the question is affirmative, that is, that “heterological” is heterological. Then “heterological,” being heterological, does not describe itself. But this implies that it is *not* heterological after all—since “heterological” says that it is of the sort that does not describe itself, and we just said that “heterological” does not describe itself, so it is not described by the description “does not describe itself”!

We might formalize this seemingly paradoxical situation by first considering that we are dealing with a function

$$f : Adj \times Adj \to 2$$

defined on all adjectives a_1, a_2 by

$$f(a_1, a_2) = \begin{cases} 1 & \text{if } a_2 \text{ describes } a_1 \\ 0 & \text{if } a_2 \text{ does not describe } a_1. \end{cases}$$

Then we know there is a predicate (a map $Adj \to 2$) that can be defined on *Adj* that is not representable by any element of *Adj*. We get this by applying the fixed-point theorem, with α the negation map $\neg : 2 \to 2$, setting $\alpha(0) = 1$ and $\alpha(1) = 0$. More explicitly, using the idea from before,

$$\begin{array}{ccc} Adj \times Adj & \xrightarrow{f} & 2 \\ {\scriptstyle\delta}\uparrow & & \downarrow{\scriptstyle\alpha} \\ Adj & \xrightarrow[g]{} & 2, \end{array}$$

we know how to construct g as a (nonrepresentable) function naming a particular property of adjectives, namely as the characteristic function of a subset of adjectives that cannot be described by any adjectives. In particular, the adjective “heterological” will be in this subset. In terms of the above, that g is such a characteristic function just says that we must have that

$$g(-) \neq f(-, a)$$

91. Yanofsky (2003) has a very nice discussion of this and a number of other such “paradoxes.”

for all adjectives a, since if there *were* an adjective a_0 that satisfied $g(-)=f(-,a_0)$, evaluating at a_0 would give

$$f(a_0,a_0)=g(a_0)=\alpha(f(a_0,a_0)),$$

the first equality from the (assumed) representability of g and the second by definition of g. But this is certainly false, due to the nature of the map α.

Observe that the hypothetical

$$f(a_0,a_0)=g(a_0)=\alpha(f(a_0,a_0)),$$

which yields a contradiction (whether we choose $f(a_0,a_0)=1$, when "a_0 describes itself," or $f(a_0,a_0)=0$, when "a_0 does not describe itself"), makes precise exactly the "paradox" described at the beginning.

Altogether, it is perhaps more telling to consider such a situation in terms of the non-representability of the g given above, where this just means that the property of "not being described by" an adjective (which applies to "heterological" in particular) is *not representable*, for there is no adjective that might represent itself via f.

6.5 Yoneda in the General

Let us now make good use of the notions of representability and the model of the Yoneda results for posets. In the special case of posets, we saw that we can identify a principal downset $\downarrow p$ with a representable functor $\text{Hom}_{\mathcal{P}}(-,p)$. For any element $p\in\mathcal{P}$, there will be a representable **2**-presheaf (think: it is *represented by p*)

$$\phi_p:\mathcal{P}^{op}\to\mathbf{2}$$

that takes $q\mapsto 1$ iff $p\leq q$. In this way, the representable presheaves act as the "characteristic maps" of the principal downsets of $\mathcal{P}$, and the **2**-enriched version of the Yoneda embedding taking each $p\mapsto\phi_p$ is the same as the inclusion of the elements of the poset into its downsets (which is, in turn, the same as considering the **2**-enriched presheaves on $\mathcal{P}$)

$$\mathcal{P}\hookrightarrow\mathbf{Down}(\mathcal{P})\simeq\mathbf{2\text{-}PreSh}(\mathcal{P}).$$

The Yoneda results in the case of categories more generally, that is, in the **Set**-enriched setting, are effectively a far-reaching generalization of this idea, and supply perhaps the most important and well-utilized results in category theory.

Proposition 160 (*Yoneda lemma*) For any functor $F:\mathbf{C}\to\mathbf{Set}$, where **C** is a locally small category, and for any object $c\in\mathbf{C}$, the natural transformations $Y^c\Rightarrow F$ are in bijection with elements of the set $F(c)$, that is,[92]

$$\text{Nat}(Y^c,F)\cong F(c). \tag{6.1}$$

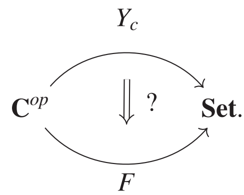
Moreover, this correspondence is natural in both F and c. In the contravariant case, that is, for $F:\mathbf{C}^{op}\to\mathbf{Set}$, things are as above, except we have

$$\text{Nat}(Y_c,F)\cong F(c). \tag{6.2}$$

92. Recall that by Y^c we just mean $\text{Hom}_{\mathbf{C}}(c,-)$, while Y_c is used for $\text{Hom}_{\mathbf{C}}(-,c)$.

We are not going to prove this (it is a good exercise to actually attempt to prove this yourself!), but instead will unpack it and then discuss its significance at a more general level. We will confine attention to the contravariant version in what follows (but dual statements can be made for the covariant version).

The idea is that for a fixed category $\mathbf{C}$, given an object $c \in \mathbf{C}$ and a (contravariant) functor $F : \mathbf{C}^{op} \to \mathbf{Set}$, we know that the object c gives rise to another special (representable) functor $Y_c : \mathbf{C}^{op} \to \mathbf{Set}$. A very natural question to ask, then, is about the maps $Y_c \Rightarrow F$,



The functors we are comparing both live in $\mathbf{Set}^{\mathbf{C}^{op}}$, so the collection of maps from Y_c to F are just the natural transformations that belong to $\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(Y_c, F)$. But what is this set? Notice that from the input data F and c we were given ("given an object c and a functor F"), we could have also constructed the set $F(c)$, by simply applying F on the given object c. The Yoneda lemma just assures us that these two sets are the same! Moreover, all the generality of natural transformations is encoded in the particular case of identity maps (used in the proof of the lemma).

The "naturality" in F mentioned in the lemma just means that, given any $\upsilon : F \to G$, the following diagram commutes:[93]

$$\begin{array}{ccc} \mathrm{Hom}(Y_c, F) & \xrightarrow{\cong} & F(c) \\ {\scriptstyle Hom(Y_c,\upsilon)}\downarrow & & \downarrow{\scriptstyle \upsilon_c} \\ \mathrm{Hom}(Y_c, G) & \xrightarrow[\cong]{} & G(c). \end{array}$$

On the other hand, naturality in c means that, given any $h : c \to c'$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(Y_c, F) & \xrightarrow{\cong} & F(c) \\ {\scriptstyle Hom(Y_h,F)}\uparrow & & \uparrow{\scriptstyle F(h)} \\ \mathrm{Hom}(Y_{c'}, F) & \xrightarrow[\cong]{} & F(c'). \end{array}$$

The most significant application of the Yoneda lemma is given by the *Yoneda embedding*, which tells us that any (locally small) $\mathbf{C}$ will be isomorphic to the full subcategory of $\mathbf{Set}^{\mathbf{C}^{op}}$ spanned by the contravariant representable functors, while $\mathbf{C}^{op}$ will be isomorphic to the full subcategory of $\mathbf{Set}^{\mathbf{C}}$ spanned by the covariant representable functors. We have seen that for each $c \in \mathbf{C}$, we have the covariant functor Y^c going from $\mathbf{C}$ to $\mathbf{Set}$ and the contravariant functor Y_c going from $\mathbf{C}^{op}$ to $\mathbf{Set}$. If we let this functor vary over all the objects of $\mathbf{C}$, the resulting functors can be gathered together into the (for example, covariant) functor $Y^{\bullet} : \mathbf{C}^{op} \to \mathrm{Hom}(\mathbf{C}, \mathbf{Set})$. Dually, we have the contravariant functor Y_c going from $\mathbf{C}^{op}$ to

93. Note: all the "Homs" are $\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}$.

Set, and collecting these functors together as we let c vary will give a functor $Y_{\bullet} : \mathbf{C} \to \mathrm{Hom}(\mathbf{C}^{op}, \mathbf{Set})$.[94]

Definition 161 The *Yoneda embedding* of **C**, a locally small category, supplies functors

$$\mathbf{C} \xhookrightarrow{\ \mathbf{y}\ } \mathbf{Set}^{\mathbf{C}^{op}} \qquad\qquad \mathbf{C}^{op} \xhookrightarrow{\ \mathbf{y}\ } \mathbf{Set}^{\mathbf{C}}$$

$$\begin{array}{ccc} c & \longmapsto & Hom(-,c) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ d & \longmapsto & Hom(-,d) \end{array} \qquad\qquad \begin{array}{ccc} c & \longmapsto & Hom(c,-) \\ {\scriptstyle f}\big\downarrow & & \big\uparrow \\ d & \longmapsto & Hom(d,-) \end{array}$$

defining full and faithful embeddings.[95]

The Yoneda embedding **y** gives us a representation of **C** in a category of set-valued functors and natural transformations. An important consequence of the embedding is that any pair of isomorphic objects $a \cong b$ in **C** are representably isomorphic, that is, $Y^a \cong Y^b$. The Yoneda lemma supplies the converse, namely if either the (co- or contravariant) functors represented by a and b are naturally isomorphic, then a and b will be isomorphic; so in particular, if a and b represent the same functor, then $a \cong b$. In many cases, it will be easier or more revealing to give such an arrow $Y^a \to Y^b$ or $Y_a \to Y_b$ than to supply $a \to b$, for the category $\mathbf{Set}^{\mathbf{C}^{op}}$ in general has more structure than does **C**—namely, it is complete, cocomplete, and "Cartesian closed" (basically, any morphism defined on a product of two objects can be identified with a morphism defined on one of the factors). Thus we can use the more advanced tools and universal properties (like the existence of limits) that come with the presheaf category, and be sure that an arrow of the form $Y_a \to Y_b$, for instance, comes from a unique $a \to b$ even if **C** on its own may not allow the advanced constructions. Analogously, representing a rational number in terms of downward (upward) closed sets under the standard ordering results in a Dedekind cut, and altogether this embeds the rationals into the reals, allowing for solutions to more equations. Passing from a category **C** to its presheaf category can also be regarded as adjoining colimits (think generalized sums) to **C**, and doing so in the most "free" way.[96] In general, in passing to the presheaf category, many nonrepresentable presheaves will show up as well. However, the representables have a very special role to play.

Before concluding our discussion of these matters, let us record a final result. We have seen how an object is defined completely by its functorial (relational) properties. The

94. It is not unusual to rename these functors, as we do in the following definition, with a lowercase (bold) **y** in both cases, leaving the appropriate variance to context.

95. In general, a functor $F : \mathbf{C} \to \mathbf{D}$ induces, for every pair of objects c, c' in **C**, a function on the hom-sets—that is, $F_{c,c'} : \mathrm{Hom}_{\mathbf{C}}(c, c') \to \mathrm{Hom}_{\mathbf{D}}(F(c), F(c'))$. The functor F is called *faithful* provided this function $F_{c,c'}$ is injective for every c, c' in **C**, and called *full* provided $F_{c,c'}$ is surjective for every c, c' in **C**. A full and faithful functor is thus bijective on hom-sets. An *embedding* in the categorical sense is a faithful functor that is also injective on objects (up to isomorphism)—that is, if $F(c) \cong F(c')$, then $c \cong c'$. An embedding in this sense reveals the domain category to be a subcategory of the codomain category. A full and faithful functor that is injective on objects thus gives us a *full embedding*, identifying the domain category as a full subcategory of the codomain category. The proof of the main result can be found in any text on category theory.

96. This is a powerful and general idea, but the reader who desires a more concrete way of thinking about the previous statement, might consider the unions (*colimits*) that showed up in the downset poset, after we embedded $\mathcal{P}$ into $\mathcal{D}(\mathcal{P})$, where these were not present in $\mathcal{P}$ itself.

next proposition tells us that even if a functorial definition does not correspond to an object—that is, if the particular functor is not representable—it is still "built out of" the representables (in particular, it is the colimit of a diagram of representables).

Proposition 162 Every object P in the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$ (i.e., every contravariant functor on $\mathbf{C}$) is a colimit of a diagram of representable objects, in a canonical way, that is,

$$P \cong \operatorname{colim}\left(\int P \xrightarrow{\pi_P} \mathbf{C} \xrightarrow{\mathbf{y}} \mathbf{Set}^{\mathbf{C}^{op}}\right),$$

where π is the projection functor and $\mathbf{y}$ is the Yoneda embedding.

This proposition states that given a functor $P : \mathbf{C}^{op} \to \mathbf{Set}$, there will be a canonical way of constructing an indexing category $\mathbf{J}$ and a corresponding diagram $A : \mathbf{J} \to \mathbf{C}$ of shape $\mathbf{J}$ such that P is isomorphic to the colimit of A composed with the Yoneda embedding. The indexing category that serves to prove the proposition is the category of elements of P.[97]

Specializing to posets gives another way to think about this: taking joins (unions) of downsets is the same thing as taking colimits in $\mathbf{Down}(\mathscr{P})$ regarded as a category; so any element in $\mathbf{Down}(\mathscr{P})$ can be regarded as the colimit (join) of representable functors (principal downsets).

6.6 Philosophical Pass: Yoneda and Relationality

Occasionally, an idea is powerful enough that it seems, almost effortlessly, to transcend its local and native context of application and speak articulately to many other contexts. The idea underlying the Yoneda results is like this. In the most general sense, it might be regarded as saying something like

> *To understand an object it suffices to understand all its relationships with other things.*

Or, to say it in another way,

> *If you want to know whether two objects A and B are the same, just look at whether all the ways of probing A with other things (or, dually, probing other things with A) is the same as all the ways of probing B with other things (or, dually, probing other things with B).*

The previous two slogans are an attempt to convey what is sometimes called the "Yoneda philosophy." It tells us, fundamentally, that if you want to understand what something *is*, there's no need to chase after some "object *in itself*"; instead, just consider all the ways the (candidate) object transforms, perturbs, and constrains other things (or, dually, all the ways it is transformed, perturbed, and constrained by others)—this will tell you what it is!

This idea that an object is determined by its totality of behaviors in relation to other entities capable of affecting it (or being affected by it), in addition to appealing to many of our intuitions, is an idea that one can find versions of throughout a number of different

97. A proof of this fact can be found in Riehl (2016). That *every presheaf is a colimit of representable presheaves* is closely related to another construction, namely the *Cauchy completion* (or *Karoubi envelope*) of a category, in which the fact that representable presheaves are *continuous* in a precise sense is exploited. The main idea here is that while we have the powerful Yoneda (full and faithful) embedding sending a category $\mathbf{C}$ to the category of presheaves $\mathbf{Set}^{\mathbf{C}^{op}}$, in general, a category $\mathbf{C}$ cannot be recovered from $\mathbf{Set}^{\mathbf{C}^{op}}$, so a natural question to ask is how or to what extent, given $\mathbf{Set}^{\mathbf{C}^{op}}$, it can be said to determine $\mathbf{C}$. Basically, if a category $\mathbf{C}$ (or $\mathbf{C}^{op}$) can be shown to be Cauchy complete, then not only can it be recovered (up to equivalence) from the presheaf category (or covariant functor category of variable sets $\mathbf{Set}^{\mathbf{C}}$), but it can be shown to *generate* the original presheaf (variable set) category.

contexts, for instance in the seventeenth-century philosopher Spinoza's idea that what a body *is* (its "essence") is inseparable from all the ways that the body can affect (causally influence) and be affected (causally influenced) by *other bodies*. In this general approach, what an object *is* can be entirely encapsulated by regarding *all at once* (generically) all of its interrelations or possible interactions with the other objects of its world. In the covariant case, we do this by regarding its ways of affecting other things; dually, in the contravariant case, by its particular ways of being affected by the other objects that inhabit its world. To use another metaphor: if you want to understand if two "destination points" are the same, just inspect whether, ranging over all the addresses with which they can communicate, the networks of routes connecting them to the addresses are the same. For a given object c, the representable functor just captures, all at once, the most generic and universal picture of that object, supplying a placeholder for each of the possible attributes of that object (and, following Spivak [2014], then Yoneda's lemma can be thought of as saying that to specify an actual object of type c, it suffices to fill in all the placeholders for every attribute found in the generic thing of type c).

If one regards $\mathrm{Hom}_{\mathbf{C}}(-,A)(U) = \mathrm{Hom}_{\mathbf{C}}(U,A)$ as telling us about "A viewed from the perspective of U," then the fact (from the Yoneda embedding) that

$$\mathrm{Hom}_{\mathbf{C}}(-,A) \cong \mathrm{Hom}_{\mathbf{C}}(-,B) \text{ iff } A \cong B \text{ iff } \mathrm{Hom}_{\mathbf{C}}(B,-) \cong \mathrm{Hom}_{\mathbf{C}}(A,-)$$

can be glossed as saying that two objects A and B will be the same precisely when they "look the same" from all perspectives U.[98] This paradigm seems especially natural in many contexts, beyond mathematics. It seems especially appropriate to an adequate description of *learning*, wherein an object comes to be known and recognized through probing it with other things, varying the perspectives on it. For instance, suppose you are tasked with having a robot learn how to identify objects that it has never been exposed to before, without relying on much training data or manual supervision, with the aim of having it come away with an ability to correctly discriminate between (what you naively take to be) different objects and readily recognize other instances of objects of the same type in other contexts.

You might place the robot in a room with a number of other objects, say, a ceramic cup, some red rubber balls, a steel cable, a small plant, a plastic bottle filled with water, a worm, and a thermostat. If the robot cannot interact with the objects in any way, and the observable interactions and changes that would unfold without its intervention are rather uneventful or slow to unfold, it is not clear how the robot could learn anything at all. On the other hand, suppose you have enabled the robot to inspect, manipulate, or otherwise instigate or probe the objects in a number of ways, and observe the outcomes. At first, these action attempts might be more or less randomized. The robot might simply locally perform simple action sequences or gestures such as

Grasp, Release, Put, Pull, Push, Rotate, Twist, Throw, Squeeze, Bend, Stack, See, Hear, Locate.

98. It is not uncommon to hear such interpretations, namely that we can retrieve the object itself via all the "perspectives on it." In case there are any unscrupulous listeners ready to confuse this with a kind of relativism, the relationism of Yoneda has nothing to do with this. Thankfully, such misunderstandings can be blocked by Yoneda itself. For the (untenable) relativist interpretation would need to assume, among other questionable assumptions, that there is an object (namely of the type "human being") that can represent *any* functor whatsoever, that thereby itself mediates all possible exchanges between objects. But Yoneda tells us no such thing.

It may grasp a rubber ball, twist the plastic water bottle, attempt to bend the steel cable, see something move (without its having performed any other action that might explain this motion) or hear it wriggling. Such actions can provide the robot with much information about the objects populating the room, and sometimes even the mere successful implementation of a certain isolated action can confirm additional information, such as about the location ("within reach radius") of an object-candidate that is engaged by *Grasp*. And once the robot has a decent working sense of some possible object-candidates, it can use one to probe others and learn even more. It may *Push* on a number of objects, with little to no effect, and then push on (what we know to be) the thermostat, altering the room's temperature, on which change it may observe different effects throughout the room (e.g., the water evaporates, the plant withers, the worm moves more, other things remain unchanged in certain relevant respects). In this way, our robot goes around the room and probes "possible object" regions in different ways, and observes the effects of these variations, giving them their own name. Via such probings of the objects of the room, and composite action-sequences, such as *Grasp, then Release, then Hear*, it seems that our robot will have a chance at learning "what is what."

Later, if we take our robot and place it in a new room, one that has in it just (what we know to be) a red balloon and a (similarly shaped) blue rubber ball; and if, in the previous room, all that the robot came to know of (what we know to be) the red rubber ball was the visual information (gathered through *See*) of its color and shape; then it may assume that the red balloon *is* the object it knew as the rubber ball (which will go under the name of a mapping from some A_i to color data), while it may assume the blue rubber ball is some entirely new thing. On the other hand, if our robot *had* probed the red rubber ball in more ways—say, having subjected it to *Bounce*, *Twist*, *Throw*, *Hear*, *Grasp*—you can be sure that it would take much less for it to come to recognize that the blue rubber ball was something like the object it knew in the previous room, while the red balloon was something very different.

The idea of Yoneda is that we can be assured that if the robot wants to learn whether some object A is the same thing as object B, it will suffice for it learn whether

$$\mathrm{Hom}_{\mathbf{C}}(-, A) \cong \mathrm{Hom}_{\mathbf{C}}(-, B)$$

or, dually,

$$\mathrm{Hom}_{\mathbf{C}}(B, -) \cong \mathrm{Hom}_{\mathbf{C}}(A, -).$$

In terms of the discussion above, this means having the robot explore whether

> all the ways of probing A with objects of its environment amount to the same as all the ways of probing B with objects of its environment.

This is a fascinating idea, philosophically, and one that we think has much in its favor even beyond the narrower context of mathematics. In Japanese, the word for human being, *ningen*, is made up of two characters, the first of which means something like a human or person, while the second is a representation of the doors of a gate and means something like "betweenness," so that the literal meaning of the word "human" is "the relation between persons." This—rather than the narrative of atomistic individualism, that often ignores or glosses over the immense load of relational constraints and determinations that come with differential obligations, pressures, and opportunities—seems more attuned to the Yoneda way of thinking.

The fundamental intuition behind the Yoneda philosophy, then, is that to know or access an object it suffices to know or access how it can be transformed by different objects, or how other objects transform into it. More exactly, Yoneda tells us that if there is a natural way of passing an object c's vision of its world (or how it is seen by its world) on to a functor F on that same category, then to recover this vision it suffices to ask F how it acts on c. While, mathematically speaking, the usefulness of the lemma often boils down to the fact that we are able to reduce the computation of natural transformations (which can be unwieldy) to the simple evaluation of a (set-valued) functor on an object, in a sense the full philosophical significance of the lemma points in the other direction. Given a category and an object in that category, rather than regard the object "on its own" (moreover, treating the entire category in a detached manner, as delimiting the outer boundaries of our consideration), via Yoneda we can regard that object as entirely characterized by its perspective or action on its world (or its world's perspective or action on it), and moreover place the category in which it lives in the wider category of all presheaves or sets varying over that category. Via Yoneda, we can perform this sort of passage from the detached consideration of a given object to the consideration of all its interrelations with the other objects of its world *for every object of a given category*. In doing so, we can think of ourselves as taking an entire category **C** that previously was itself being regarded in a detached manner, and placing it in the more "continuous" (in a loose sense) context of the category of all the presheaves over **C**. The category of presheaves over **C** into which **C** is embedded not only has certain desirable properties that the original category may lack, like possessing all categorical limits, but it can be understood (in both intuitive and in various technical ways) as providing the continuous counterpart to the detached consideration of the original category.

This is a section of doi:10.7551/mitpress/12581.001.0001

Sheaf Theory through Examples

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Citation:

Sheaf Theory through Examples
By: Daniel Rosiak
DOI: 10.7551/mitpress/12581.001.0001
ISBN (electronic): 9780262370424
Publisher: The MIT Press
Published: 2022

The open access edition of this book was made possible by generous funding and support from Arcadia – a charitable fund of Lisbet Rausing and Peter Baldwin, and MIT Press Direct to Open

The MIT Press

The open access edition of this book was made possible by generous funding from Arcadia—a charitable fund of Lisbet Rausing and Peter Baldwin.

ARCADIA
A charitable fund of Lisbet Rausing and Peter Baldwin

The MIT Press would like to thank the anonymous peer reviewers who provided comments on drafts of this book. The generous work of academic experts is essential for establishing the authority and quality of our publications. We acknowledge with gratitude the contributions of these otherwise uncredited readers.

This book was set in LaTeX by the author.

Library of Congress Cataloging-in-Publication Data

Names: Rosiak, Daniel, author.
Title: Sheaf theory through examples / Daniel Rosiak.
Description: Cambridge, Massachusetts : The MIT Press, [2022] | Includes bibliographical references and index.
Identifiers: LCCN 2021058949 | ISBN 9780262542159 (paperback)
Subjects: LCSH: Sheaf theory.
Classification: LCC QA612.36 .R67 2022 | DDC 514/.224–dc23/eng20220521
LC record available at https://lccn.loc.gov/2021058949