APPENDIX A

Proof of Theorem 4.1

In the situation considered in both van Fraassen's and our dynamic Dutch book example in chapter 4, we had a set of mutually exclusive and jointly exhaustive hypotheses $H_1, \ldots, H_n$ such that, for all i, $\Pr(H_i) > 0$. We further had evidence statements $E_1, \ldots, E_m$, which were independent conditional on any of the hypotheses. We compared two methods for calculating the probabilities of conjunctions of those evidence statements, the special multiplication rule

$$\text{(SMR)} \qquad \Pr(E_1 \wedge \cdots \wedge E_n) \;=\; \sum_{i=1}^{m} \prod_{j=1}^{n} \Pr(H_i) \Pr(E_j \mid H_i),$$

and the more general method

$$\text{(GM)} \quad \Pr(E_1 \wedge \cdots \wedge E_n) \;=$$
$$\sum_{i=1}^{m} \Pr(H_i) \Pr(E_1 \mid H_i) \;\times\; \prod_{j=2}^{n} \sum_{i=1}^{m} \Pr(H_i \mid E_1 \wedge \cdots \wedge E_{j-1}) \Pr(E_j \mid H_i).$$

The claim to be proven then was that

Theorem 4.1 *SMR and GM are equivalent methods for calculating* $\Pr(E_1 \wedge \cdots \wedge E_k)$*, with* $1 < k \leqslant m$*, if and only if conditional probabilities are calculated according to Bayes's theorem.*

Proof: Sufficiency is easy; we prove necessity. For simplicity, we consider only the case where we have two hypotheses H_1 and H_2 that are assumed to be jointly exhaustive and mutually exclusive, and two outcome events E_1 and E_2 that are independent in the previous sense; the generalization is straightforward.

If we calculate conditional probabilities not according to Bayes's theorem but according to the formula representing some probabilistic updating rule at variance with Bayes's rule (such as EXPL, or Good's rule, or Popper's), then for all $i : 1 \leqslant i \leqslant n$ and $k : 1 \leqslant k \leqslant m$ there is a g such that

$$\Pr(H_i \mid E_k) = \frac{\Pr(H_i) \times \Pr(E_k \mid H_i)}{\sum_{j=1}^{n} \Pr(H_j) \times \Pr(E_k \mid H_j)} + g(H_i, E_k),$$

where $g(H_i, E_k) = -\sum_{j \neq i} g(H_j, E_k)$, and it is not the case that $g(H_i, E_k) = 0$ for all i, k (for then our updating rule would no longer be at variance with Bayes's rule). Suppose, toward a contradiction, that SMR and GM are equivalent. Then in the case that we consider, the following equations must hold:

$$\begin{aligned}
&\Pr(E_1 \mid H_1) \Pr(E_2 \mid H_1) \Pr(H_1) + \Pr(E_1 \mid H_2) \Pr(E_2 \mid H_2) \Pr(H_2) = \\
&= \Big[\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)\Big] \times \\
&\quad \Big[\Pr(H_1|E_1) \Pr(E_2 \mid H_1) + \Pr(H_2 \mid E_1) \Pr(E_2 \mid H_2)\Big] \\
&= \Big[\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)\Big] \times \\
&\quad \Bigg[\left(\frac{\Pr(H_1) \Pr(E_1 \mid H_1)}{\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)} + g(H_1, E_1)\right) \times \Pr(E_2 \mid H_1) + \\
&\quad \left(\frac{\Pr(H_2) \Pr(E_1 \mid H_2)}{\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)} + g(H_2, E_1)\right) \times \Pr(E_2|H_2)\Bigg] \\
&= \left[\frac{\Pr(H_1) \Pr(E_1 \mid H_1) \Pr(E_2 \mid H_1)}{\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)} + g(H_1 \mid E_1) \Pr(E_2 \mid H_1)\right] + \\
&\quad \left[\frac{\Pr(H_2) \Pr(E_1 \mid H_2) \Pr(E_2 \mid H_2)}{\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)} + g(H_2 \mid E_1) \Pr(E_2 \mid H_2)\right] \\
&= \Pr(H_1) \Pr(E_1 \mid H_1) \Pr(E_2 \mid H_1) + \\
&\quad \Big[g(H_1, E_1) \Pr(E_2 \mid H_1) \Big(\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)\Big)\Big] + \\
&\quad \Pr(H_2) \Pr(E_1 \mid H_2) \Pr(E_2 \mid H_2) + \\
&\quad \Big[g(H_2, E_1) \Pr(E_2 \mid H_2) \Big(\Pr(H_1) \Pr(E_1 \mid H_1) + \Pr(H_2) \Pr(E_1 \mid H_2)\Big)\Big].
\end{aligned}$$

Because (1) $g(H_1, E_1) = -g(H_2, E_1)$, and (2) $\Pr(E_2 \mid H_1) \neq \Pr(E_2 \mid H_2)$, it must be that $g(H_1, E_1) = g(H_2, E_1) = 0$. That means that Bayes's rule and the non-Bayesian rule do not disagree over $\Pr_{E_1}(H_i)$. However, we can interchange E_1 and E_2 and then it follows by parity of reasoning that they cannot disagree over $\Pr_{E_2}(H_i)$ either. This contradicts our assumption that the rules are at variance with each other. Consequently, SMR and GM are not equivalent. ■