APPENDIX B

# *Proof of Theorem 5.1*

Recall that the weights of a VS rule (as defined in ch. 5) are all positive and add up to 1, and that they are said to reflect truthlikeness in a minimally adequate sense precisely if hypotheses are assigned weights as a function of their distance from the truth, with hypotheses farther from the truth being assigned larger weights than hypotheses closer to the truth.

**Theorem 5.1** *Every VS rule whose weights reflect truthlikeness in a minimally adequate sense is improper.*

*Proof:* Without loss of generality, consider a hypothesis partition of three hypotheses, $H_1$, $H_2$, and $H_3$. Then, for an arbitrary VS rule $\mathcal{V}$ and a given person's probability assignment $\mathbf{p} = (p_1, p_2, p_3)$ to the aforementioned hypotheses, with $p_i$ the probability assigned to $H_i$, this person's expected $\mathcal{V}$ score for a probability assignment $\mathbf{p}^*$ to the same hypotheses is given by the function

$$\begin{aligned}\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] \;=\; & p_1\big(w_{11}(1-p_1^*)^2 + w_{21}(p_2^*)^2 + w_{31}(p_3^*)^2\big) \\ & + p_2\big(w_{12}(p_1^*)^2 + w_{22}(1-p_2^*)^2 + w_{32}(p_3^*)^2\big) \\ & + p_3\big(w_{13}(p_1^*)^2 + w_{23}(p_2^*)^2 + w_{33}(1-p_3^*)^2\big).\end{aligned}$$

Also without loss of generality, assume that the hypotheses are ordered by their distances from each other, with $H_2$ being equally far from $H_1$ and $H_3$, and $H_1$ and $H_3$ being twice as far from each other as they are from $H_2$. Then $w_{11} = w_{33}$, $w_{21} = w_{23}$, $w_{31} = w_{13}$, and $w_{12} = w_{32}$, so that we can simplify notation by defining $w_1 := w_{11} = w_{33}$; $w_2 := w_{21} = w_{23}$; $w_3 := w_{31} = w_{13}$; $w_4 := w_{12} = w_{32}$; and $w_5 := w_{22}$. For $\mathcal{V}$ to be proper, it must hold that $\arg\min_{\mathbf{p}^*} \mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] = \mathbf{p}$, for any distribution $\mathbf{p}$ on $\{H_1, H_2, H_3\}$. To see whether this holds, we use the method of Lagrange multipliers. Specifically, where $f(\mathbf{p}^*) = p_1^* + p_2^* + p_3^*$, we must find values for $p_1^*, p_2^*, p_3^*$, and $\lambda$ such that $\nabla\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] = \lambda\nabla f(\mathbf{p}^*)$ and $f(\mathbf{p}^*) = 1$. Calculating the first-order partial derivatives of $\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)]$, we find

$$\begin{aligned}(\partial/\partial p_1^*)\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] &= -2w_1p_1(1-p_1^*) + 2w_3p_3p_1^* + 2w_4p_2p_1^*;\\ (\partial/\partial p_2^*)\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] &= -2w_5p_2(1-p_2^*) + 2w_2p_1p_2^* + 2w_2p_3p_2^*;\\ (\partial/\partial p_3^*)\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] &= -2w_1p_3(1-p_3^*) + 2w_3p_1p_3^* + 2w_4p_2p_3^*.\end{aligned}$$

Because $\nabla f(\mathbf{p}^*) = \mathbf{1}$, we have $(\partial/\partial p_i^*)\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] = \lambda$ for all $i \leqslant 3$. So in particular, expanding the partial derivatives in $p_1^*$ and $p_3^*$ and dividing both by 2, we have

$$-w_1p_1 + w_1p_1p_1^* + w_3p_3p_1^* + w_4p_2p_1^* = -w_1p_3 + w_1p_3p_3^* + w_3p_1p_3^* + w_4p_2p_3^*,$$

and hence

$$w_1p_1p_1^* + w_3p_3p_1^* + w_4p_2p_1^* - w_1p_3p_3^* - w_3p_1p_3^* - w_4p_2p_3^* - w_1p_1 + w_1p_3 = 0.$$

Suppose that $\mathcal{V}$ is proper, so that $\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)]$ reaches its minimum if $p_1 = p_1^*$, $p_2 = p_2^*$, and $p_3 = p_3^*$. Then there must be values for the $w_i$ such that

$$w_1{p_1}^2 + w_3p_3p_1 + w_4p_2p_1 - w_1{p_3}^2 - w_3p_1p_3 - w_4p_2p_3 - w_1p_1 + w_1p_3 = 0.$$

However, factoring the left-hand side yields

$$(p_1 - p_3)(-w_1 + w_1p_1 + w_4p_2 + w_1p_3).$$

This equals 0 if and only if either (1) $p_1 = p_3$ or (2) $w_1 = w_4$, where the latter follows from the fact that the condition that the right-hand factor equals 0 can be rewritten $w_1(1 - p_1 - p_3) = w_4p_2$, in conjunction with the fact that the $p_i$ sum to 1. Because, as said, for $\mathcal{V}$ to be proper, it must hold for all $\mathbf{p}$ that $\arg\min_{\mathbf{p}^*}\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)] = \mathbf{p}$, we may pick a $\mathbf{p}$ such that $p_1 \neq p_3$, thereby violating condition 1. Regarding condition 2, note that whichever precise values the $w_i$ assume, $w_1$ must be smaller than 1/3 (given that it is assigned to the supposed truth) and $w_4$ must be greater than 1/3 (given that it is assigned to the two hypotheses supposed false). Consequently, on the supposition that $\mathcal{V}$ is proper, we can minimize $\mathbb{E}_{\mathbf{p}}[\mathcal{V}(\mathbf{p}^*)]$ subject to the given constraint precisely if the truthlikeness weights assigned by the rule do *not* reflect truthlikeness in a minimally adequate sense. By assumption, the weights do reflect truthlikeness in a minimally adequate sense. Given that we made no further assumptions about $\mathcal{V}$, it follows that every VS rule is improper if it assigns truthlikeness weights in a minimally adequate fashion. ■

*Remark:* This proof proceeds by constructing a specific counterexample involving three hypotheses that are assumed to stand in specific relations of truthlikeness to each other. To see that this assumption does not undermine the generality of the proof, we note that the said relations are perfectly possible according to all modern measures of truthlikeness. (See p. 140 for references.) As a matter of fact, one can think of the example presented in chapter 5 concerning the possible grades (A, B, or C) a given student may receive as instantiating exactly the relations of truthlikeness that are assumed to hold in the counterexample. It is also to be noted, however, that not *all* known measures of truthlikeness will suffice for the purposes of the proof. Most famously, Pavel Tichý (1974) discovered that on Popper's (1963) measure all false theories are equally far from the truth, contrary to what Popper had hoped to achieve with his measure.